MATH 304

Assignment 2
February 14, 2020

Name ______________________

For full credit justify your answers.

Problem 1. Let $n \in \mathbb{N}$. In this problem we consider the reverse identity matrix, which is sometimes called anti-identity matrix. The reverse identity matrix, denoted by Z_n, is the matrix in which the columns of the identity matrix I_n are written in reversed order. The last column of the identity matrix is the first column of the reverse identity matrix and so on:

$$Z_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ Z_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ Z_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \ Z_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \cdots, \ Z_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}.$$

Sometimes the reverse identity matrix Z_n is described as the $n \times n$ square matrix with ones on the diagonal going from the lower left corner to the upper right corner and all the other entries are zero. A rigorous definition of Z_n is as follows: For $k \in \{1, \ldots, n\}$ denote be $I_{n,k}$ the k-th column of I_n and by $Z_{n,k}$ the k-th column of Z_n. Then $Z_{n,k} = I_{n,n-k+1}$ for all $k \in \{1, \ldots, n\}$.

Prove that the reverse identity matrix Z_n is diagonalizable. Find an invertible matrix P_n and a diagonal matrix D_n such that $Z_n P_n = P_n D_n$.

Problem 2. In this problem we consider three kinds of $n \times n$ matrices:

$$L_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}, \quad U_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad M_n = \begin{bmatrix} n & n-1 & \cdots & 2 & 1 \\ n-1 & n-1 & \cdots & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \cdots & 2 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

(a) There is a simple relationship among matrices L_n, U_n and M_n. Discover it.
(b) For each matrix above calculate its determinant.
(c) Calculate the inverse of each given matrix.

Hint: It might help to consider special cases:

$$M_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \ M_3 = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \ M_4 = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \ M_5 = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \ M_6 = \begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Problem 3. Let m and n be positive integers. Let A be an arbitrary $n \times m$ matrix. There are 12 vector spaces related to the matrix A that are of interest. Those are

$$\begin{array}{lll} \operatorname{Col} A & \operatorname{Row} A & \operatorname{Nul} A \\ (\operatorname{Col} A)^{\perp} & (\operatorname{Row} A)^{\perp} & (\operatorname{Nul} A)^{\perp} \\ \operatorname{Col}(A^{\top}) & \operatorname{Row}(A^{\top}) & \operatorname{Nul}(A^{\top}) \\ \big(\operatorname{Col}(A^{\top})\big)^{\perp} & \big(\operatorname{Row}(A^{\top})\big)^{\perp} & \big(\operatorname{Nul}(A^{\top})\big)^{\perp} \end{array}$$

First identify each of the 12 subspaces as a subspace of $\mathbb{R}^n$ or $\mathbb{R}^m$. Then identify the subspaces that are equal. What is the largest number of genuinely distinct subspaces that can occur in the above list? Give an example. What is the smallest number of genuinely distinct subspaces that can occur in the above list? Give an example.

Problem 4. On October 4 and October 7 I posted about the following matrix

$$A = \begin{bmatrix} 1 & 3 & 2 & 2 & 2 \\ 2 & 0 & -2 & 1 & 1 \\ 2 & 1 & -1 & 1 & 2 \\ 1 & 4 & 3 & 2 & 3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows from magic properties of RREF that the sets

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

are exceptionally important.

(a) Explain why are the sets $\mathcal{B}$ and $\mathcal{C}$ exceptionally important and explain your claim.

(b) Consider the linear transformation

$$T : \operatorname{Row} A \to \operatorname{Col} A \quad \text{defined by} \quad T\mathbf{x} = A\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \operatorname{Row} A.$$

Find the matrix of T relative to the natural bases of Row and $\operatorname{Col} A$ which appear elsewhere in this problem.

(c) Prove that the transformation T is a bijection, that is T is one-to-one and onto.

Problem 5. Let m and n be positive integers. Let A be an $n \times m$ matrix. Give a complete proof that $\operatorname{Col}(A^\top) = \operatorname{Col}(A^\top A)$.

Problem 6. Let A be an $n \times m$ matrix with linearly independent columns. Let $\mathbf{y}$ be a vector in $\mathbb{R}^n$. Prove that

$$\operatorname{Proj}_{\operatorname{Col} A} \mathbf{y} = A\big(A^\top A\big)^{-1} A^\top \mathbf{y}.$$

In other words, prove that the vector $A\big(A^\top A\big)^{-1} A^\top \mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto $\operatorname{Col}(A)$.

Problem 7. Let n be an integer greater than 1 and let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$. Let $A = I_n - 2\mathbf{u}\mathbf{u}^\top$, where I is the $n \times n$ identity matrix.

(a) Prove that the matrix A is symmetric.
(b) Prove that the matrix A is orthogonal.
(c) Find all the eigenvalues and the eigenvectors of A.
(d) Find an orthogonal diagonalization of A.
(e) Calculate the sum of the diagonal entries of A. Calculate the sum of the eigenvalues of A, counting each eigenvalue as many as times as its multiplicity. This is just one example, but I would like to encourage you to state a conjecture from what you calculated: State your conjecture clearly.
(f) Verify your conjecture on the matrices in Problem 1. Choose another matrix from the textbook or some other source and verify your conjecture.
(g) Prove your conjecture for an arbitrary real 2×2 matrix.