CHAPTER 2

The Set $\mathbb{R}$ of Real Numbers

All concepts that we will study in this course have their roots in the set of real numbers. We assume that you are familiar with some basic properties of the real numbers $\mathbb{R}$ and of the subsets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ of $\mathbb{R}$. However, in order to clarify exactly what we need to know about $\mathbb{R}$, we set down its basic properties (called axioms) and some of their consequences.

2.1. Axioms of a field

The following are the basic properties (axioms) of $\mathbb{R}$ that relate to addition and multiplication in $\mathbb{R}$:

Axiom 1 (A0). If $a, b \in \mathbb{R}$, then the sum $a + b$ is uniquely defined element in $\mathbb{R}$. That is, there exists a function $+$ (called "plus") defined on $\mathbb{R} \times \mathbb{R}$ and with the values in $\mathbb{R}$.

Axiom 2 (A1). $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{R}$.

Axiom 3 (A2). $a + b = b + a$ for all $a, b \in \mathbb{R}$.

Axiom 4 (A3). There exists an element 0 in $\mathbb{R}$ such that $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$.

Axiom 5 (A4). If $a \in \mathbb{R}$, then the equation $a + x = 0$ has a solution $-a \in \mathbb{R}$.

Axiom 6 (M0). If $a, b \in \mathbb{R}$, then the product $a \cdot b$ (usually denoted by ab) is uniquely defined number in $\mathbb{R}$. That is, there exists a function $\cdot$ (called "times") defined on $\mathbb{R} \times \mathbb{R}$ and with the values in $\mathbb{R}$.

Axiom 7 (M1). $a(bc) = (ab)c$ for all $a, b, c \in \mathbb{R}$.

Axiom 8 (M2). $ab = ba$ for all $a, b \in \mathbb{R}$.

Axiom 9 (M3). There exists an element 1 in $\mathbb{R}$ such that $1 \neq 0$ and $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$.

Axiom 10 (M4). If $a \in \mathbb{R}$ and $a \neq 0$, then the equation $a \cdot x = 1$ has a solution $a^{-1} = \dfrac{1}{a}$ in $\mathbb{R}$.

Axiom 11 (DL). $a(b + c) = ab + ac$ for all $a, b, c \in \mathbb{R}$.

Remark 2.1.1. Notice that the only specific real numbers mentioned in the axioms are 0 and 1. You can verify that the set $\{0, 1\}$ with the functions $+$ and $\cdot$ defined by

$$0 + 0 = 1 + 1 = 0, \ 0 + 1 = 1 + 0 = 1 \quad \text{and} \quad 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \ 1 \cdot 1 = 1.$$

satisfies all Axioms 1 through 11. Hence, we need more axioms to describe the set of real numbers.

Axioms A1 and M1 are called *associative laws* and Axioms A2 and M2 are *commutative laws*. Axiom DL is the *distributive law*; this law justifies "factorization" and "multiplying out" in algebra. A triple of a set, "plus" and "times" functions which satisfies Axioms 1 through 11 is called a *field*. The basic algebraic properties of $\mathbb{R}$ can be proved solely on the basis of the field axioms. We illustrate this claim by the following exercise.

EXERCISE 2.1.2. Let $a, b, c \in \mathbb{R}$. Prove the following statements.

(a) If $a + c = b + c$, then $a = b$.
(b) $a \cdot 0 = 0$ for all $a \in \mathbb{R}$.
(c) $-a = a$ if and only if $a = 0$.
(d) $-(-a) = a$ for all $a \in \mathbb{R}$.
(e) $(-a)b = -(ab)$ for all $a, b \in \mathbb{R}$.
(f) $(-a)(-b) = ab$ for all $a, b \in \mathbb{R}$.
(g) If $a \neq 0$, then $\left(a^{-1}\right)^{-1} = a$.
(h) If $ac = bc$ and $c \neq 0$, then $a = b$.
(i) $ab = 0$ if and only if $a = 0$ or $b = 0$.
(j) If $a \neq 0$ and $b \neq 0$, then $(ab)^{-1} = a^{-1}b^{-1}$.

REMARK 2.1.3. We will prove (a), (e) and a part of (i) below. Others you can do as exercise. One of the statements in Exercise 2.1.2 cannot be proved using Axioms 2 through 11. To prove that particular property we will need results from Section 2.2.

SOLUTION. (a) Assume that $a+c = b+c$. By Axiom 1 adding any number x to both sides of the equality, leads to $(a+c)+x = (b+c)+x$. It follows from Axiom 2 that $a+(c+x) = b+(c+x)$. By Axiom 5 there exists an element $-c \in \mathbb{R}$ such that $c + (-c) = 0$. Choose $x = -c$. Then $a = a + 0 = a + (c + (-c)) = b + (c + (-c)) = b + 0 = b$.

(e) Let $a, b \in \mathbb{R}$. Then, by Axiom 5 there exists $-a \in \mathbb{R}$ such that $a+(-a) = 0$. By Axiom 6 it follows that $\bigl(a + (-a)\bigr)b = 0 \cdot b$. By Axiom 11 and part (b) of this exercise, it follows that $ab + (-a)b = 0$. Since $ab \in \mathbb{R}$, by Axiom 5 there exists $-(ab) \in \mathbb{R}$ such that $ab + (-(ab)) = 0$. Using Axiom 3 we conclude that $(-a)b+ab = -(ab)+ab$. By part (a) of this proof we conclude that $(-a)b = -(ab)$.

We prove "only if" part of (i). That is, we prove the implication:

$$ab = 0 \quad \text{implies} \quad a = 0 \quad \text{or} \quad b = 0. \tag{2.1.1}$$

Assume that $ab = 0$. Consider two cases: Case 1: $a = 0$ and Case 2: $a \neq 0$.
Case 1. In this case the implication (2.1.1) is true and there is nothing to prove.
Case 2. Since in this case we assume that $a \neq 0$, by Axiom 10 there exists an element $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$. multiplying both sides of $ab = 0$ by a^{-1} we get $(ab)a^{-1} = 0 \cdot a^{-1}$. Therefore $b = b \cdot 1 = b(aa^{-1}) = (ba)a^{-1} = (ab)a^{-1} = 0 \cdot a^{-1} = 0$. □

REMARK 2.1.4. Let $a, b \in \mathbb{R}$. Instead of $a + (-b)$ we write $a - b$ and we write $\dfrac{a}{b}$ or a/b instead of ab^{-1}.

2.2. Axioms of order in a field

The set $\mathbb{R}$ also has an order structure $<$ satisfying the following axioms.

Axiom 12 (O1). Given any $a, b \in \mathbb{R}$, exactly one of the following three statements is true: $a < b$, $a = b$, or $b < a$.

Axiom 13 (O2). Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

Axiom 14 (O3). Given any $a, b, c \in \mathbb{R}$, if $a < b$ then $a + c < b + c$.

Axiom 15 (O4). Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $0 < c$, then $ac < bc$.

Axiom O2 is called the *transitive law*. A field with an order satisfying Axioms O1 through O4 is called an *ordered field*.

The notation $a \leq b$ stands for the statement: $a < b$ or $a = b$.

Definition 2.2.1. A number $x \in \mathbb{R}$ is *positive* if $x > 0$. A number $x \in \mathbb{R}$ is *negative* if $x < 0$.

Exercise 2.2.2. Prove the following statements for $a, b, c \in \mathbb{R}$.

(a) If $a < b$ then $-b < -a$.
(b) If $a < b$ and $c < 0$, then $b\,c < a\,c$.
(c) Assume $a > 0$ and $b \neq 0$. Prove that $b > 0$ if and only if $ab > 0$.
(d) If $a \neq 0$, then $0 < aa$.
(e) $0 < 1$.
(f) If $a > 0$, then $\dfrac{1}{a} > 0$.
(g) If $0 < a < b$, then $0 < \dfrac{1}{b} < \dfrac{1}{a}$.

Solution. (a) Assume $a < b$. By Axiom 14 we have $a + (-b) < b + (-b)$. Thus $(-b) + a < 0$. Using Axiom 14 again, we conclude that $\big((-b) + a\big) + (-a) < 0 + (-a)$, and consequently $-b < -a$.

Do (b) as an exercise.

Now we prove (c). Assume $a > 0$ and $b \neq 0$. This assumption is used throughout this part of the proof. Since $a \neq 0$ and $b \neq 0$, by Exercise 2.1.2 (i) it follows that $ab \neq 0$. The implication: "If $b > 0$, then $ab > 0$." is a special case of Axiom 15. Next we deal with the implication "If $ab > 0$, then $b > 0$." It turns out that the contrapositive is easier to prove. The negation of $b > 0$ is $b \leq 0$. But, it is assumed that $b \neq 0$. Thus, with this assumption, the negation of $b > 0$ is $b < 0$. Similarly, the negation of $ab > 0$ is $ab < 0$. Hence the contrapositive of "If $ab > 0$, then $b > 0$." is "If $b < 0$, then $ab < 0$." The last implication follows directly from part (b). This completes the proof of (c).

(d) Consider two different cases: $a > 0$ and $a < 0$. If $a > 0$, then (c) implies that $a^2 >$. If $a < 0$, then, by (a), $-0 < -a$, and since $-0 = 0$ we have $-a > 0$. By the first part of this proof, we conclude that $(-a)(-a) > 0$. By part (f) of Exercise 2.1.2 we have $(-a)(-a) = a\,a$. Therefore $aa > 0$ for all $a \neq 0$.

Do (e) as an exercise.

To prove (f) we assume $a > 0$. By Axiom 10, $a\,\dfrac{1}{a} = 1$. By Axiom 9, $1 \neq 0$. Hence, $a\,\dfrac{1}{a} \neq 0$. By Exercise 2.1.2 (i) $\dfrac{1}{a} \neq 0$ and by (e) $1 > 0$. Now we can apply the "if" part of (c). (Take $b = 1/a$ in (c).) We conclude that $a\,\dfrac{1}{a} = 1 > 0$ implies $\dfrac{1}{a} > 0$. This proves (f).

Do (g) as an exercise. $\square$

DEFINITION 2.2.3. We define the following eight numbers

$$2 = 1+1, \qquad 3 = 2+1, \qquad 4 = 3+1, \qquad 5 = 4+1,$$
$$6 = 5+1, \qquad 7 = 6+1, \qquad 8 = 7+1, \qquad 9 = 8+1.$$

The numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are called *digits*.

In the preceding definition we implied that the digits are distinct numbers. The next exercise justifies this claim.

EXERCISE 2.2.4. Prove the inequalities:

$$0 < 1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$$

The following three exercises deal with squares of real numbers. As usual, for $a \in \mathbb{R}$, a product aa is called a *square* and it is denoted by a^2.

EXERCISE 2.2.5. Let $a \in \mathbb{R}$. Prove that the equation $x^2 = a$, has at most two solutions in $\mathbb{R}$.

SOLUTION. Consider the set

$$S = \bigl\{x \in \mathbb{R} : x^2 = a\bigr\}.$$

If $S = \emptyset$, then the statement is true. Now assume that $S \neq \emptyset$ and let $b \in S$. From $b \in S$, we deduce that $b \in \mathbb{R}$ and $b^2 = a$. Since $b \in \mathbb{R}$, $-b \in \mathbb{R}$. Next we will prove

$$S = \bigl\{b, -b\bigr\}. \tag{2.2.1}$$

Let $c \in S$. Then $c^2 = a$, and therefore $c^2 = b^2$. Consequently, $c^2 - b^2 = 0$. Using Axioms 2 through 11 and properties in Exercise 2.1.2 we can prove that $(c-b)(c+b) = c^2 - b^2$. Therefore $(c-b)(c+b) = c^2 - b^2 = 0$. Exercise 2.1.2 (i) implies that $c - b = 0$ or $c + b = 0$. Thus $c = b$ or $c = -b$. This proves

$$S \subset \{b, -b\}. \tag{2.2.2}$$

Next we prove $\{b, -b\} \subset S$. By assumption $b \in S$. Since $(-b)^2 = b^2$, we have $(-b)^2 = a$. Hence $-b \in S$. Therefore

$$\{b, -b\} \subset S. \tag{2.2.3}$$

Relations (2.2.2) and (2.2.3) imply equality (2.2.1). Since the set $\{b, -b\}$ has at most two elements the statement is proved. $\square$

EXERCISE 2.2.6. Let $0 \leq x, y$. Prove that $x < y$ if and only if $x^2 < y^2$.

EXERCISE 2.2.7. If $\alpha > 1$ and $\alpha > x^2$, then $\alpha > x$.

EXERCISE 2.2.8. Let $a, b, c, d \in \mathbb{R}$.

(i) Prove or disprove the statement: If $a < b$ and $c < d$, then $a - c < b - d$.
(ii) If you disproved the statement in (i), change the assumptions about c and d to make a correct statement. Prove your new statement.

THE PROPERTIES OF REAL NUMBERS PROVED IN THIS AND THE PREVIOUS SECTION ARE ESSENTIAL. MANY OF THEM ARE TRULY ELEMENTARY (ALTHOUGH SOMETIMES HARD TO PROVE) AND YOU CAN (AND I WILL) USE SUCH PROPERTIES IN PROOFS WITHOUT ANY JUSTIFICATION. BUT, WHEN YOU ARE USING MORE SUBTLE PROPERTIES (LIKE ONES IN EXERCISES 2.2.6, 2.2.7, OR 2.2.8) YOU SHOULD STATE EXPLICITLY WHICH PROPERTY YOU ARE USING AND EXPLAIN INFORMALLY WHY IT IS TRUE.

2.3. Intervals

EXERCISE 2.3.1. Let a and b be real numbers such that $a < b$. Prove that there exists $c \in \mathbb{R}$ such that $a < c < b$.

The preceding exercise justifies the following definition.

DEFINITION 2.3.2. Let a and b be real numbers such that $a < b$. We will use the following notation and terminology:

$$\begin{aligned}
[a,b] &:= \big\{x \in \mathbb{R} \,:\, a \leq x \leq b\big\} && \text{is called} \quad \textit{a closed interval},\\
(a,b) &:= \big\{x \in \mathbb{R} \,:\, a < x < b\big\} && \text{is called} \quad \textit{an open interval},\\
[a,b) &:= \big\{x \in \mathbb{R} \,:\, a \leq x < b\big\} && \text{is called} \quad \textit{a half-open interval},\\
(a,b] &:= \big\{x \in \mathbb{R} \,:\, a < x \leq b\big\} && \text{is called} \quad \textit{a half-open interval}.
\end{aligned}$$

We also define four types of unbounded intervals:

$$\begin{aligned}
[a,+\infty) &:= \big\{x \in \mathbb{R} \,:\, a \leq x\big\} && \text{is called} \quad \textit{a closed unbounded interval},\\
(a,+\infty) &:= \big\{x \in \mathbb{R} \,:\, a < x\big\} && \text{is called} \quad \textit{an open unbounded interval}\\
(-\infty,b] &:= \big\{x \in \mathbb{R} \,:\, x \leq b\big\} && \text{is called} \quad \textit{an unbounded closed interval},\\
(-\infty,b) &:= \big\{x \in \mathbb{R} \,:\, x < b\big\} && \text{is called} \quad \textit{an unbounded open interval},
\end{aligned}$$

Geometric illustrations of these intervals are given below.

Figure 1: A closed interval

Figure 2: An open interval

Figure 3: A half-open interval

Figure 4: A half-open interval

Figure 5: A closed infinite interval

Figure 6: An open infinite interval

Figure 7: An infinite closed interval

Figure 8: An infinite open interval

REMARK 2.3.3. The infinity symbols $-\infty$ and $+\infty$ are used to indicate that the set is unbounded in the negative $(-\infty)$ or positive $(+\infty)$ direction of the real number line. The symbols $-\infty$ and $+\infty$ are just symbols; they are <u>not</u> real numbers. Therefore we always exclude them as endpoints by using parentheses.

We conclude this section with few exercises about families of intervals.

EXERCISE 2.3.4. Let $a \in \mathbb{R}$. Prove that

$$\bigcap \big\{(a-u, a+u) \,:\, u > 0\big\} = \{a\}.$$

EXERCISE 2.3.5. Let $a, b \in \mathbb{R}$ and $a < b$. Prove that
$$\bigcap \big\{ (a, b+u) \, : \, u > 0 \big\} = (a, b].$$

EXERCISE 2.3.6. Let $a, b \in \mathbb{R}$ and $a < b$. Prove that
$$\bigcap \Big\{ (a-u, b+u) \, : \, u > 0 \Big\} = [a, b].$$

SOLUTION. Denote by A the intersection in the equality and assume $x \in A$. Then, by the definition of intersection, $x \in (a-u, b+u)$ for all $u > 0$. By the definition of an open interval, $a-u < x$ and $x < b+u$ for all $u > 0$. Hence, $a-x < u$ and $x-b < u$ for all $u > 0$. Consequently, $a-x \notin (0, +\infty)$ and $x-b \notin (0, +\infty)$. Therefore, $a-x \in (-\infty, 0]$ and $x-b \in (-\infty, 0]$, that is, $a \leq x$ and $x \leq b$. By the definition of a closed interval $x \in [a, b]$. This proves $A \subset [a, b]$.

Now assume that $x \in [a, b]$. Then, $a-x \leq 0$ and $x-b \leq 0$. Let $u > 0$ be arbitrary. By the transitivity of the order in $\mathbb{R}$, $a-x \leq u$ and $x-b \leq u$ for all $u > 0$. Hence, $a-u \leq x$ and $x \leq b+u$ for all $u > 0$. Consequently, $x \in (a-u, b+u)$ for all $u > 0$. Therefore, $x \in A$. This proves $[a, b] \subset A$.

Since we proved both $A \subset [a, b]$ and $[a, b] \subset A$, the equality $A = [a, b]$ is proved. □

EXERCISE 2.3.7. Let $a, b \in \mathbb{R}$ and $a < b$. Prove that
$$\bigcup \Big\{ [a+u, b) \, : \, 0 < u < b-a \Big\} = (a, b).$$

EXERCISE 2.3.8. Let $a, b \in \mathbb{R}$ and $a < b$. Prove that
$$\bigcup \Big\{ [a+u, b-u] \, : \, 0 < u < \frac{b-a}{2} \Big\} = (a, b).$$

2.4. Bounded sets. Minimum and Maximum

DEFINITION 2.4.1. Let A be a nonempty subset of $\mathbb{R}$. If there exists $b \in \mathbb{R}$ such that

$$x \leq b \quad \text{for all} \quad x \in A, \tag{2.4.1}$$

then A is said to be *bounded above*. A number b satisfying (2.4.1) is called an *upper bound* of A.

Similarly we define:

DEFINITION 2.4.2. Let A be a nonempty subset of $\mathbb{R}$. If there exists $a \in \mathbb{R}$ such that

$$a \leq x \quad \text{for all} \quad x \in A, \tag{2.4.2}$$

then A is said to be *bounded below*. A number a satisfying (2.4.2) is called a *lower bound* of A.

DEFINITION 2.4.3. A nonempty subset of $\mathbb{R}$ which is both bounded above and bounded below is said to be *bounded*.

EXERCISE 2.4.4. Let A be a nonempty subset of $\mathbb{R}$. Prove that A is bounded if and only if there exists $K > 0$ such that $|x| \leq K$ for all $x \in A$.

EXERCISE 2.4.5. Let A be a nonempty subset of $\mathbb{R}$. Prove that A is bounded if and only if there exist $a, b \in \mathbb{R}$, such that $a < b$ and $A \subset [a, b]$.

Exercise 2.4.6. Prove that $\{x \in \mathbb{R} : x^2 < 2\}$ is a bounded set.

Exercise 2.4.7. Prove that $\left\{ \dfrac{n^{(-1)^n}}{n+1} : n \in \mathbb{N} \right\}$ is a bounded set.

Exercise 2.4.8. Let A and B be bounded above subsets of $\mathbb{R}$. Prove that $A \cup B$ is bounded above.

Next we introduce the definitions of the minimum and the maximum.

Definition 2.4.9. Let A be a nonempty subset of $\mathbb{R}$. A number $a \in \mathbb{R}$ is a *minimum* of A if it has the following two properties:

(i) $a \leq x$ for all $x \in A$; (ii) $a \in A$.

The minimum of A (if it exists) is denoted by $\min A$.

Definition 2.4.10. Let A be a nonempty subset of $\mathbb{R}$. A number $b \in A$ is a *maximum* of A if it has the following two properties:

(i) $x \leq b$ for all $x \in A$; (ii) $b \in A$.

The maximum of A (if it exists) is denoted by $\max A$.

Exercise 2.4.11. Let $x, y \in \mathbb{R}$. Prove that the set $A = \{x, y\}$ has a minimum and a maximum.

Remark 2.4.12. What does it mean for a nonempty subset of $\mathbb{R}$ not to have a minimum? To answer this question we first restate Definition 2.4.9 as follows. A nonempty set A has a minimum if

$$\exists\, a \in A \quad \text{such that} \quad \forall\, x \in A \quad \text{we have} \quad x \geq a. \tag{2.4.3}$$

Next we formulate the negation of the statement (2.4.3):

$$\forall\, a \in A \quad \exists\, x \in A \quad \text{such that} \quad x < a. \tag{2.4.4}$$

Notice that the number x in (2.4.4) depends on a. Sometimes it is useful to emphasize this dependence by writing $x(a)$. A more precise version of the negation is:

$$\forall\, a \in A \quad \exists\, x(a) \in A \quad \text{such that} \quad x(a) < a.$$

Exercise 2.4.13. Prove that the set of all positive numbers does not have a minimum.

Exercise 2.4.14. Give examples of subsets A, B, C of $\mathbb{R}$ such that:

(a) A does not have neither a minimum nor a maximum.
(b) B has a minimum but not a maximum.
(c) C has a minimum and a maximum.

2.5. Three functions: the unit step, the sign and the absolute value

There are only two specific numbers mentioned in Axioms 2 through 15. These are 0 and 1. The number -1 is implicitly mentioned in Axiom 4. Therefore the following two functions are of interest.

Definition 2.5.1. We define the following two functions:

$$\operatorname{us}(x) := \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases} \qquad \text{and} \qquad \operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

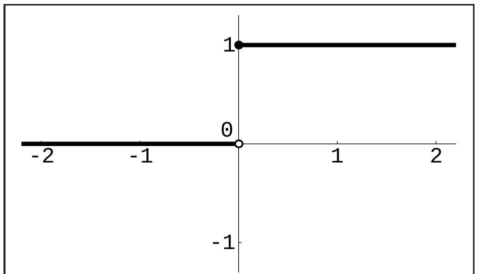


Figure 9: The unit step function

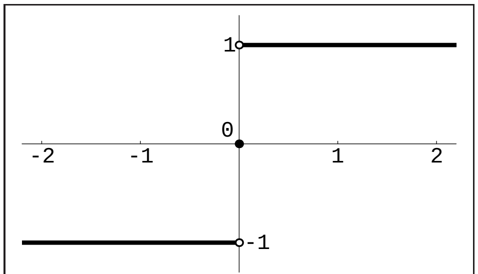


Figure 10: The sign function

The first function is called the *unit step* (or the *Heaviside step*) function. The second one is called the *sign* function. The definition and the notation for the sign function are standard. However, some authors define the value of the unit step function at 0 to be 1/2. Also, the notation for the unit step function is not standardized; H is often used instead of us. I decided to use two letter notation since it is more in the spirit of sgn and other familiar functions $\sin, \cos, \ln, \exp, \ldots$. Although these two functions are not part of the standard calculus course, I hope that you will agree that they are very simple.

EXERCISE 2.5.2. Prove the identity: $\operatorname{sgn}(x) = \operatorname{us}(x) - \operatorname{us}(-x)$.

EXERCISE 2.5.3. Prove the identity: $\operatorname{us}(x) = 1 - \big(\operatorname{sgn}(x) - 1\big)\big(\operatorname{sgn}(x)\big)/2$.

EXERCISE 2.5.4. Let $x, y \in \mathbb{R}$. Prove the following equalities:

$$\begin{aligned} \max\{x, y\} &= x + (y - x)\ \operatorname{us}(y - x), \\ \min\{x, y\} &= y + (x - y)\ \operatorname{us}(y - x). \end{aligned}$$

DEFINITION 2.5.5. The *absolute value* function is defined as

$$\operatorname{abs}(x) = x \operatorname{sgn}(x) \quad \big(\forall x \in \mathbb{R}\big).$$

We will also use the standard notation $\operatorname{abs}(x) = |x|$. The number $|x|$ is called the *absolute value* of the number x.

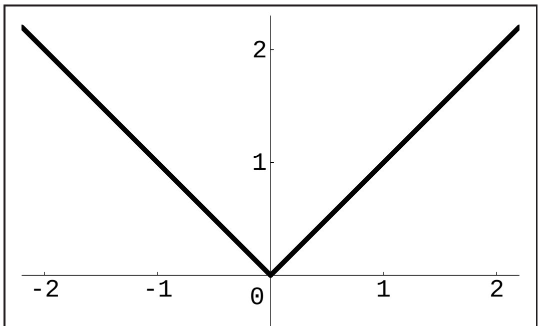


Figure 11: The absolute value function

In the plots above we used a geometric representation of real numbers as points on a straight line. Such representation is obtained by choosing a point on a line to represent 0 and another point to represent 1. Then, every real number corresponds to a point on the line (called the number line), and every point on the number line corresponds to a real number. This geometric representation is often very useful in doing the problems.

Geometrically, the absolute value of a represents the distance between 0 and a, or, generally $|a - b|$ is the *distance* between a and b on the number line.

The basic properties of the absolute value are given in the exercises below. All of the exercises can be proved by considering all possible cases for the numbers involved. This is not difficult when an exercise involves only one number. It gets harder when an exercise involves two or more numbers. Proofs that avoid cases are more elegant and easier to comprehend. Therefore you should always seek such proofs; see Exercise 2.5.9.

EXERCISE 2.5.6. Prove the following identities.

(a) $|x| = \max\big\{x, -x\big\} \quad \big(\forall\, x \in \mathbb{R}\big)$.

(b) $|x| = x\,\big(2\,\mathrm{us}(x) - 1\big) \quad \big(\forall\, x \in \mathbb{R}\big)$.

EXERCISE 2.5.7. Prove the following statements.

(i) $|a| \geq 0$ for all $a \in \mathbb{R}$.

(ii) $|-a| = |a|$ for all $a \in \mathbb{R}$.

(iii) $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.

EXERCISE 2.5.8. Let $x, a \in \mathbb{R}$ and $a \geq 0$. Prove the following equivalences.

(a) $|x| \leq a$ if and only if $-a \leq x$ and $x \leq a$.

(b) $|x| \geq a$ if and only if $x \leq -a$ or $x \geq a$.

EXERCISE 2.5.9. For all $a, b \in \mathbb{R}$ we have

$$|a + b| \leq |a| + |b|.$$

SOLUTION. By Exercise 2.5.6 (a), $a \leq |a|$ and $b \leq |b|$. Therefore, $a + b \leq |a| + |b|$. Similarly, $-a \leq |a|$ and $-b \leq |b|$. Therefore, $-a - b \leq |a| + |b|$. Since $-a-b = -(a+b)$, we have $-(a+b) \leq |a|+|b|$. Hence, we proved both $a+b \leq |a|+|b|$ and $-(a + b) \leq |a| + |b|$. Therefore,

$$\max\{a + b, -(a + b)\} \leq |a| + |b|. \qquad \square$$

EXERCISE 2.5.10. Find specific $a, b \in \mathbb{R}$ such that $|a + b| = |a| + |b|$. Next, formulate a general statement by completing the following equivalence

$$|a + b| = |a| + |b| \quad \text{if and only if} \quad \boxed{}\,.$$

Prove your statement.

EXERCISE 2.5.11. Formulate a general statement by completing the following equivalence

$$|a + b| < |a| + |b| \quad \text{if and only if} \quad \boxed{}\,.$$

Prove your statement.

EXERCISE 2.5.12. Let $x, y, z \in \mathbb{R}$. Interpret the numbers $|x - y|$, $|y - z|$ and $|x - z|$ as distances and discover an inequality that they must satisfy. (It might help to think of x, y and z as towns on I-5.) Prove your inequality.

EXERCISE 2.5.13. For all $a, b \in \mathbb{R}$ we have

$$\big||a| - |b|\big| \leq |a - b|.$$

The inequalities in Exercises 2.5.9, 2.5.12 and 2.5.13 are often called with one name, the *triangle inequality*.

EXERCISE 2.5.14. Let $x, a \in \mathbb{R}$. If $|x - a| \leq 1$, then $|x| \leq 1 + |a|$.

EXERCISE 2.5.15. Let $x, a \in \mathbb{R}$. If $|x-a| \leq 1$, then $|x+a| \leq 1+2|a|$.

EXERCISE 2.5.16. Let $x, a, u \in \mathbb{R}$ and let $u > 0$. If $|x-a| < u$ and $|x-a| \leq 1$, then $\left|x^2 - a^2\right| < u(1+2|a|)$.

EXERCISE 2.5.17. Let $x, a \in \mathbb{R}$ and let $a \neq 0$. If $|x-a| < \dfrac{|a|}{2}$, then $|x| > \dfrac{|a|}{2}$.

EXERCISE 2.5.18. Let $a \in \mathbb{R}$ and $\epsilon > 0$. Then

$$\bigl\{x \in \mathbb{R} : |x-a| < \epsilon\bigr\} = (a-\epsilon, a+\epsilon).$$

2.6. The set $\mathbb{N}$ of natural numbers

We mentioned natural numbers and integers informally in the course of our discussion of the fundamental properties of $\mathbb{R}$. Notice again that the only numbers that are specifically mentioned in Axioms 1 through 15 are 0 and 1. But, in Section 2.2 Exercise 2.2.4 we proved that there are other numbers in $\mathbb{R}$, and we defined the numbers $2, 3, 4, 5, 6, 7, 8, 9$. The reason that we stopped at 9 is the fact that the number $9+1$ plays a special role in our culture. We could continue this process further, but it would not lead to a rigorous definition of the set of natural numbers. Therefore we chose a different route.

Consider the following two properties of a subset S of $\mathbb{R}$:

$$1 \in S, \tag{2.6.1}$$

$$n \in S \quad \Rightarrow \quad n+1 \in S. \tag{2.6.2}$$

There are many subsets of $\mathbb{R}$ that have these two properties. For example one such set is the set of positive real numbers, that is the open infinite interval,

$$(0, +\infty).$$

Another such set is the closed infinite interval

$$[1, +\infty),$$

and also the union

$$\{1\} \cup [2, +\infty).$$

There are many such sets. Next we form the family of <u>all</u> subsets of $\mathbb{R}$ with the properties (2.6.1) and (2.6.2):

$$\mathcal{N} := \Bigl\{S \subset \mathbb{R} \, : \, 1 \in S \ \text{ and } \ n \in S \ \Rightarrow \ n+1 \in S\Bigr\}$$

Intuitively, the set of natural numbers is the smallest set in $\mathcal{N}$.

DEFINITION 2.6.1. We define $\mathbb{N}$ to be the intersection of the family $\mathcal{N}$:

$$\mathbb{N} := \bigcap \bigl\{S \subset \mathbb{R} \, : \, S \in \mathcal{N}\bigr\}.$$

That is, $k \in \mathbb{N}$ if and only if $k \in S$ for all $S \in \mathcal{N}$. The elements of the set $\mathbb{N}$ are called *natural numbers.*

With this definition and Axioms 1 through 15 we should be able to prove all familiar properties of natural numbers.

THEOREM 2.6.2. (N 1) $1 \in \mathbb{N}$.
(N 2) *The formula* $\sigma(n) = n+1$ *defines a function* $\sigma : \mathbb{N} \to \mathbb{N}$.
(N 3) *If* $\sigma(m) = \sigma(n)$, *then* $n = m$; *that is* σ *is one-to-one.*
(N 4) *For all* $n \in \mathbb{N}$, $\sigma(n) \neq 1$.

(N 5) *If $K \subset \mathbb{N}$ has the following two properties*

$$1 \in K,$$
$$\big(\forall n \in \mathbb{N}\big) \quad n \in K \quad \Rightarrow \quad n+1 \in K,$$

then $K = \mathbb{N}$.

Proof. Since $1 \in S$ for all $S \in \mathcal{N}$, we have $1 \in \mathbb{N}$. This proves (N 1). To prove (N 2), let $n \in \mathbb{N}$ be arbitrary. Then $n \in S$ for all $S \in \mathcal{N}$. Since (2.6.2) holds for each $S \in \mathcal{N}$, we conclude that $n+1 \in S$ for all $S \in \mathcal{N}$. Hence $n+1 \in \mathbb{N}$ for all $n \in \mathbb{N}$. Property (N 3) follows from Exercise 2.1.2 (a). To prove (N 5) assume that $K \subset \mathbb{N}$ and K has properties (2.6.1) and (2.6.2). Then $K \in \mathcal{N}$. Consequently, $\mathbb{N} = \bigcap\{S : S \in \mathcal{N}\} \subset K$. Thus, $K = \mathbb{N}$. $\square$

Remark 2.6.3. The five properties of $\mathbb{N}$ proved in Theorem 2.6.2 are known as Peano's axioms. Italian mathematician Giuseppe Peano (1858-1932) used these five properties for axiomatic foundation of natural numbers. All other familiar properties of the natural numbers can be proved using these axioms. The theory of natural numbers developed from Peano's axioms is called Peano's arithmetic.

An important consequence of the property (N 5) in Theorem 2.6.2 is the *Principle of Mathematical Induction*. It is stated and proved in the next theorem. This principle is the main tool in dealing with statements involving natural numbers.

Theorem 2.6.4. *Let $P(n)$, $n \in \mathbb{N}$, be a family of statements such that*

(I) *$P(1)$ is true,*
(II) *For all $n \in \mathbb{N}$, $P(n)$ implies $P(n+1)$.*

Then the statement $P(n)$ is true for each $n \in \mathbb{N}$.

Proof. Consider the set

$$S = \big\{ n \in \mathbb{N} : P(n) \;\text{ is true } \big\}.$$

By (I), $1 \in S$. By (II), for all $n \in \mathbb{N}$, if $n \in S$, then $n+1 \in S$. Hence, S has both properties from Theorem 2.6.2 (5). Consequently, $S = \mathbb{N}$. This means that for all $n \in \mathbb{N}$ the statement $P(n)$ is true. $\square$

Remark 2.6.5. The step (II) of the mathematical induction requires you to reach the conclusion that $P(n+1)$ is true by using the assumption that $P(n)$ is true, i.e., you have to prove the implication $P(n) \Rightarrow P(n+1)$ for all $n \in \mathbb{N}$.

The following theorem can be proved using the properties from Theorem 2.6.2 and the principle of mathematical induction.

Theorem 2.6.6. (i) $1 = \min \mathbb{N}$*; that is,* $1 \in \mathbb{N}$ *and* $1 \leq n$ *for all* $n \in \mathbb{N}$.

(ii) *For every $n \in \mathbb{N}\backslash\{1\}$, we have $n-1 \in \mathbb{N}$.*
(iii) *For all $m, n \in \mathbb{N}$, we have $m+n \in \mathbb{N}$.*
(iv) *For all $m, n \in \mathbb{N}$ we have $mn \in \mathbb{N}$.*
(v) *For all $m, n \in \mathbb{N}$ such that $m < n$, we have $n-m \in \mathbb{N}$.*
(vi) *If $m, n \in \mathbb{N}$ and $m < n$, then $m+1 \leq n$.*

Proof. (i) As we mentioned before the closed infinite interval $[1, +\infty)$ belongs to the family $\mathcal{N}$. Therefore $\mathbb{N} \subset [1, +\infty)$. Therefore $n \geq 1$ for all $n \in \mathbb{N}$. Since $1 \in \mathbb{N}$ was proved in Theorem 2.6.2, (i) is proved.

(ii) Consider the following set $S = \{1\} \cup \{m \in \mathbb{N} : m - 1 \in \mathbb{N}\}$. Clearly $S \subset \mathbb{N}$ and $1 \in S$. Notice also that $2 \in S$, since $2 - 1 = 1 \in \mathbb{N}$. Next we will prove

$$n \in S \Rightarrow n + 1 \in S. \tag{2.6.3}$$

Assume $n \in S$. We distinguish two cases: $n = 1$ and $n \in \{m \in \mathbb{N} : m - 1 \in \mathbb{N}\}$. If $n = 1$, then $n + 1 = 2 \in S$. Hence (2.6.3) holds in this case. If $n \in \{m \in \mathbb{N} : m - 1 \in \mathbb{N}\}$, then $n \in \mathbb{N}$ and $n - 1 \in \mathbb{N}$. By Theorem 2.6.2 (1), $n + 1 \in \mathbb{N}$ and, obviously, $(n + 1) - 1 = n \in \mathbb{N}$. Therefore $n + 1 \in \{m \in \mathbb{N} : m - 1 \in \mathbb{N}\}$. Hence $n + 1 \in S$. Thus (2.6.3) holds. Now, by Theorem 2.6.2 (5), $S = \mathbb{N}$. This proves $\mathbb{N} \backslash \{1\} = \{m \in \mathbb{N} : m - 1 \in \mathbb{N}\}$.

Remaining properties are proved similarly. □

The Principle of Mathematical Induction is also used to define functions on $\mathbb{N}$. The process described in the next proposition is called the *Principle of Inductive Definition.*

PROPOSITION 2.6.7. *If a function f has the following two properties*

(I) $f(1)$ *is defined,*

(II) $(\forall\, n \in \mathbb{N}) \quad f(n+1) \quad$ *is defined in terms of* $\quad f(1), \dots, f(n)$,

then f is defined on $\mathbb{N}$.

PROOF. Denote the domain of f by D. Let $k \in \mathbb{N}$ and set the statement $P(k)$ to be: $1, \dots, k \in D$. Clearly $P(1)$ is true by (I). Now, let $n \in \mathbb{N}$ be arbitrary and assume that $P(n)$ is true. That is assume that $1, \dots, n \in D$. By (II) $f(n+1)$ is defined in terms of $f(1), \dots, f(n)$. Since by the inductive hypothesis all $f(1), \dots, f(n)$ are defined, we conclude that $f(n+1)$ is defined. Thus $n+1 \in D$. Since we assume that $1, \dots, n \in D$, we have proved that $1, \dots, n, n+1 \in D$. Hence $P(n+1)$ is proved. By the Principle of Mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$. Therefore $1, \dots, n \in D$ for all $n \in \mathbb{N}$. Consequently, $n \in D$ for all $n \in \mathbb{N}$. □

A definition of a function with properties (I) and (II) in Proposition 2.6.7 is called *recursive* or *inductive* definition.

DEFINITION 2.6.8. A function whose domain equals $\mathbb{N}$ and whose range is in $\mathbb{R}$ is called a *sequence in* $\mathbb{R}$.

REMARK 2.6.9. Traditionally, if $f : A \to B$ is a function and if $x \in A$, then the value of f at x is denoted by $f(x)$. In addition to this traditional notation, for a sequence $f : \mathbb{N} \to \mathbb{R}$ we will often write f_n instead of $f(n)$, $n \in \mathbb{N}$. When convenient we will use both notations for the same sequence. The reason for this is purely typographical. For example if $n = \dfrac{m(m+1)}{2} + 1$, then it is awkward to write $f_{\frac{m(m+1)}{2}+1}$. In such a case, the expression $f\big(\frac{m(m+1)}{2} + 1\big)$ is preferable since it is easier to read and understand.

2.7. Examples and Exercises related to $\mathbb{N}$

The following two examples deal with two familiar functions: the factorial and the power function. Let $n \in \mathbb{N}$. The factorial is informally "defined" as

$$n! = 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n.$$

Let $a \in \mathbb{R}$. The n-th power of a is informally expressed as

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a \cdot a}_{n \text{ times}}.$$

Next we give the rigorous definitions of the factorial and the power function as examples of recursive definitions.

EXAMPLE 2.7.1. The function $f : \mathbb{N} \to \mathbb{N}$ defined by

(i) $f(1) = 1$,
(ii) $\big(\forall\, n \in \mathbb{N}\big)$ $f(n+1) = (n+1)\, f(n)$,

is called the *factorial*.

The standard notation for the factorial is $f(n) = n!$. The definition of factorial is extended to 0 by setting $0! = 1$.

EXAMPLE 2.7.2. Let $a \in \mathbb{R}$. Define the function $g : \mathbb{N} \to \mathbb{R}$ by

(i) $g(1) = a$,
(ii) $\big(\forall\, n \in \mathbb{N}\big)$ $g(n+1) = a\, g(n)$.

The standard notation for the function g is $g(n) = a^n$. The expression a^n is called the n-th *power* of a. For $a \neq 0$, the definition of the power is extended to 0 by setting $a^0 = 1$. The expression 0^0 is not defined.

EXERCISE 2.7.3. Let $a, b \in \mathbb{R}$ be such that $a, b \geq 0$. Let $n \in \mathbb{N}$. Prove that $a < b$ if and only if $a^n < b^n$.

Use the Principle of Mathematical Induction to do the following exercises.

EXERCISE 2.7.4. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

(i) $f(1) = 1$,
(ii) $\big(\forall\, n \in \mathbb{N}\big)$ $f(n+1) = f(n) + (2n+1)$.

Evaluate the values $f(2), f(3), f(4), f(5)$. Based on the numbers that you get, guess a simple formula for $f(n)$ and prove it.

EXERCISE 2.7.5. Consider the function $T : \mathbb{N} \to \mathbb{N}$ defined by

(i) $T(1) = 1$,
(ii) $\big(\forall\, n \in \mathbb{N}\big)$ $T(n+1) = T(n) + (n+1)$.

Evaluate the values $T(2), T(3), T(4), T(5), T(6)$. Based on these numbers guess a simple formula for $T(n)$ in terms of n and prove it.

REMARK 2.7.6. The numbers $T(n)$, $n \in \mathbb{N}$, are called *triangular numbers*. For $n \in \mathbb{N}$, the triangular number

$$T(n) = 1 + 2 + \cdots + (n-1) + n$$

is the additive analog of the factorial (see Example 2.7.1)

$$n! = 1 \cdot 2 \cdot \,\cdots\, \cdot (n-1) \cdot n.$$

For completeness we set $T(0) = 0$.

EXERCISE 2.7.7. Let $a, x \in \mathbb{R}$. Consider the function $g : \mathbb{N} \to \mathbb{R}$ defined by

(i) $g(1) = a$,
(ii) $\big(\forall\, n \in \mathbb{N}\big)$ $g(n+1) = g(n) + a\, x^n$.

Another way of writing $g(n)$ is

$$g(n) = \sum_{k=0}^{n-1} a\, x^k.$$

Informally this sum is sometimes written as

$$g(n) = a + a\,x + \cdots + a\,x^{n-1}.$$

This sum is called the *geometric sum*.

Prove that

$$g(n) = \begin{cases} a\,\dfrac{1-x^n}{1-x} & \text{if } \ x \neq 1, \\[2ex] n\,a & \text{if } \ x = 1. \end{cases}$$

Exercise 2.7.8 (Bernoulli's inequality). Let $n \in \mathbb{N}$ and $x > -1$. Then

$$(1+x)^n \geq 1 + n\,x.$$

Exercise 2.7.9. Let $n \in \mathbb{N}$ and let $x \in \mathbb{R}$ be such that $0 \leq x \leq 1$. Then

$$(1+x)^n \leq 1 + \bigl(2^n - 1\bigr)x.$$

Exercise 2.7.10 (Binomial theorem). Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\, y^k.$$

Here $\dbinom{n}{k}$ denotes the *binomial coefficient* which is defined by

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!}, \qquad n \in \mathbb{N}, \quad k = 0, 1, \ldots, n.$$

The most important property of binomial coefficients is given by the following equality

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}, \qquad n \in \mathbb{N}, \quad k = 1, \ldots, n.$$

This formula is proved by using the definition of the binomial coefficients and the rules for adding fractions.

2.8. Finite sets, infinite sets, countable sets

One of the most important applications of the natural numbers is counting. The following special subsets of $\mathbb{N}$ are used for counting

$$[\![1, n]\!]_{\mathbb{N}} := \bigl\{k \in \mathbb{N} : k \leq n\bigr\}, \quad n \in \mathbb{N}.$$

Since this notation will be used often in this section, we will, for simplicity of notation drop the subscript $\mathbb{N}$ in $[\![1, n]\!]_{\mathbb{N}}$ and simply write $[\![1, n]\!]$.

Exercise 2.8.1. Let $m \in \mathbb{N}$. If $n \in \mathbb{N}$ and there exists a bijection $f : [\![1, m]\!] \to [\![1, n]\!]$, then $n = m$.

SOLUTION. We will prove the claim by Mathematical Induction with respect to m. For $m = 1$ the statement reads: "If $n \in \mathbb{N}$ and there exists a bijection $f : [\![1, 1]\!] \to [\![1, n]\!]$, then $n = 1$". To prove this statement, assume that $f : [\![1, 1]\!] \to [\![1, n]\!]$ is a bijection. Since f is onto, there exist $j, k \in [\![1, 1]\!]$ such that $f(j) = 1$ and $f(k) = n$. Since $j, k \in [\![1, 1]\!] = \{1\}$, we have $j = k = 1$. Hence $f(1) = 1$ and $f(1) = n$. Since f is a function, we must have $n = 1$. Now let $m \in \mathbb{N}$ be arbitrary and assume that the statement: "If $n \in \mathbb{N}$ and there exists a bijection $f : [\![1, m]\!] \to [\![1, n]\!]$, then $n = m$." is true. (This is the inductive assumption.) The final step of the Mathematical Induction is to prove: "If $p \in \mathbb{N}$ and there exists a bijection $g : [\![1, m+1]\!] \to [\![1, p]\!]$, then $p = m+1$." To prove this implication assume that $p \in \mathbb{N}$ and that $g : [\![1, m+1]\!] \to [\![1, p]\!]$ is a bijection. Since $m \in \mathbb{N}$, $m+1 > 1$. Since g is one-to-one, $g(m+1) \neq g(1)$. Hence $\max\{g(1), g(m+1)\} > 1$ and clearly $\max\{g(1), g(m+1)\} \leq p$. Hence $p > 1$. We continue the proof by considering two cases.

Case 1. $g(m+1) = p$. Define $h : [\![1, m]\!] \to [\![1, p-1]\!]$ by $h(k) := g(k), k \in [\![1, m]\!]$. Since g is a bijection, h is a bijection as well. By the inductive assumption $p-1 = m$. Therefore $p = m+1$. Thus the proof is finished in this case.

Case 2. $g(m+1) \in [\![1, p-1]\!]$. Since g is onto, there exists $j \in [\![1, m]\!]$ such that $g(j) = p$. Now define $h : [\![1, m]\!] \to [\![1, p-1]\!]$, by

$$h(k) := \begin{cases} g(k), & \text{if} \quad k \in [\![1, m]\!] \setminus \{j\} \\ g(m+1), & \text{if} \quad k = j. \end{cases}$$

You can verify that h is a bijection. Now, by the inductive assumption $p - 1 = m$. Therefore $p = m+1$. Thus the proof is finished in this case. □

Next we give a formal mathematical definition of the counting process.

DEFINITION 2.8.2. A set A is *finite* if there exists a natural number n and a bijection $f : [\![1, n]\!]_{\mathbb{N}} \to A$. In this case we say that *A has n elements*. We use the notation $\#A$ for the number of elements of A.

REMARK 2.8.3. Notice that there is a possibility for an ambiguity in Definition 2.8.2. There could exist another natural number m and a bijection $g : [\![1, m]\!]_{\mathbb{N}} \to A$. Then, $f^{-1} \circ g : [\![1, m]\!]_{\mathbb{N}} \to [\![1, n]\!]_{\mathbb{N}}$ is a bijection, and, by Exercise 2.8.1, $m = n$. Hence Definition 2.8.2 is not ambiguous.

DEFINITION 2.8.4. A nonempty set which is not finite is said to be *infinite.*

Finite sets are often informally written as $A = \{a_1, a_2, \ldots, a_n\}$. However, this way of writing does not imply that the mapping $k \mapsto a_k$, $k \in [\![1, n]\!]_{\mathbb{N}}$, is a bijection, but it does imply that this mapping is a surjection.

THEOREM 2.8.5. *Let $n \in \mathbb{N}$. If A is a nonempty subset of $[\![1, n]\!]_{\mathbb{N}}$, then A is finite and $\#A = k$ for some $k \in [\![1, n]\!]_{\mathbb{N}}$.*

PROOF. We will prove the statement by Mathematical induction. For $n = 1$ the claim is: If A is a nonempty subset of $\{1\}$, then A is finite and $\#A = 1$. This statement is true, since the only nonempty subset of $\{1\}$ is the set $\{1\}$.

We now state the inductive hypothesis. Let $m \in \mathbb{N}$ be arbitrary. Assume that the statement "If A is a nonempty subset of $[\![1, m]\!]_{\mathbb{N}}$, then A is finite and $\#A = j$ for some $j \in [\![1, m]\!]_{\mathbb{N}}$." is true.

Next we will prove the statement: "If B is a nonempty subset of $[\![1, m+1]\!]_{\mathbb{N}}$, then B is finite and $\#B = k$ for some $k \in [\![1, m+1]\!]_{\mathbb{N}}$."

Assume that B is a nonempty subset of $[\![1, m+1]\!]_{\mathbb{N}}$. We consider the following three cases:

Case 1: $B = [\![1, m+1]\!]_{\mathbb{N}}$,
Case 2: $m+1 \notin B$,
Case 3: $m+1 \in B$ and $p \notin B$ for some $p \in [\![1, m]\!]_{\mathbb{N}}$.

In Case 1 the claim is true, since the set $[\![1, m+1]\!]_{\mathbb{N}}$ is clearly finite and $\#B = m+1$. In Case 2, $B \subset [\![1, m]\!]_{\mathbb{N}}$. Hence, by the inductive hypothesis, B is finite and $\#B = j$ for some $j \in [\![1, m]\!]_{\mathbb{N}}$. Since $[\![1, m]\!]_{\mathbb{N}} \subset [\![1, m+1]\!]_{\mathbb{N}}$, the claim is true in this case as well.

Now consider Case 3. Assume that $m+1 \in B$ and $p \notin B$ for some $p \in [\![1, m]\!]_{\mathbb{N}}$. Define the set B' by $B' = \big(B \setminus \{m+1\}\big) \cup \{p\}$. It is clear that $B' \subset [\![1, m]\!]_{\mathbb{N}}$. Therefore, by the inductive hypothesis, B' is finite and $\#B' = j$ for some $j \in [\![1, m]\!]_{\mathbb{N}}$. It is also clear that the function $f : B' \to B$ defined for $x \in B'$ by

$$f(x) = \begin{cases} x & \text{if } x \neq p, \\ m+1 & \text{if } x = p, \end{cases}$$

is a bijection. Therefore, B is finite and $\#B = \#B' = j$ for some $j \in [\![1, m]\!]_{\mathbb{N}}$. This completes the proof. □

Corollary 2.8.6. *Each nonempty subset of a finite set is finite.*

Proof. Let X be a finite set. Then there exists $n \in \mathbb{N}$ and a bijection

$$f : [\![1, n]\!]_{\mathbb{N}} \to X.$$

Let Y be a nonempty subset of X. Consider the set

$$B = \big\{k \in [\![1, n]\!]_{\mathbb{N}} \,:\, f(k) \in Y\big\}.$$

Since Y is nonempty subset of X, B is a nonempty subset of $[\![1, n]\!]_{\mathbb{N}}$. By Theorem 2.8.5, B is finite and $\#B = k$ for some $k \in [\![1, n]\!]_{\mathbb{N}}$. Hence, there exists a bijection $g : [\![1, k]\!]_{\mathbb{N}} \to B$. It is not difficult to prove that the composition $f \circ g : [\![1, k]\!]_{\mathbb{N}} \to Y$ is a bijection. (Prove this claim as an exercise.) Therefore, Y is finite. □

Exercise 2.8.7. If A and B are finite sets and $A \cap B = \emptyset$, then

$$\#\big(A \cup B\big) = \#A + \#B.$$

Solution. Let A and B be finite sets such that $A \cap B = \emptyset$. Since A is finite there exist $m \in \mathbb{N}$ and a bijection $f : [\![1, m]\!]_{\mathbb{N}} \to A$. Since B is finite there exist $n \in \mathbb{N}$ and a bijection $g : [\![1, n]\!]_{\mathbb{N}} \to B$. Notice that the equivalence

$$x \in [\![1, m+n]\!]_{\mathbb{N}} \setminus [\![1, m]\!]_{\mathbb{N}} \quad \Leftrightarrow \quad x - m \in [\![1, n]\!]_{\mathbb{N}}$$

follows from the basic properties of $\mathbb{N}$. Now we define $h : [\![1, m+n]\!]_{\mathbb{N}} \to A \cup B$ by

$$h(x) = \begin{cases} f(x), & x \in [\![1, m]\!]_{\mathbb{N}}, \\ g(x-m), & x \in [\![1, m+n]\!]_{\mathbb{N}} \setminus [\![1, m]\!]_{\mathbb{N}}. \end{cases}$$

It is not difficult to prove that h is a bijection. Here is the proof. Let $c \in A \cup B$ be arbitrary. Then, $c \in A$ or $c \in B$. If $c \in A$, then, since f is a surjection, there exists $k \in [\![1, m]\!]_{\mathbb{N}}$ such that $f(k) = c$. Then $k \in [\![1, m+n]\!]_{\mathbb{N}}$ and by definition of

h, $h(k) = f(k) = c$. If $c \in B$, then, since g is a surjection, there exists $j \in [\![1, n]\!]_{\mathbb{N}}$ such that $g(j) = c$. Then $j + m \in [\![1, m+n]\!]_{\mathbb{N}} \setminus [\![1, m]\!]_{\mathbb{N}}$ and by definition of h, $h(j + m) = g(j) = c$. Thus, h is a surjection. To prove that h is a bijection, let $x, y \in [\![1, m+n]\!]_{\mathbb{N}}$ and assume $x < y$. Consider the following three cases: Case 1: $y \leq m$, Case 2: $x \leq m < y$, Case 3: $m < x$. In Case 1, since f is an injection, $f(x) \neq f(y)$. Since in this case $h(x) = f(x)$ and $h(y) = f(y)$, we have $h(x) \neq h(y)$. In Case 2, $h(x) \in A$ and $h(y) \in B$. Since $A \cap B = \emptyset$, $h(x) \neq h(y)$. In Case 3 we have $1 \leq x - m < y - m \leq n$. Since g is an injection, $g(x - m) \neq g(y - m)$. Since in this case $h(x) = g(x - m)$ and $h(y) = g(y - m)$, we have $h(x) \neq h(y)$. Thus, in each case, $x, y \in [\![1, m+n]\!]_{\mathbb{N}}$ and $x < y$ imply $h(x) \neq h(y)$. This proves that h is an injection. Thus h is a bijection. □

Exercise 2.8.8. Let $A \subset B$ and assume B is finite. Then A is finite and $\#A \leq \#B$.

Exercise 2.8.9. Let $A \subset B$ and assume B is finite. If A is a proper subset of B, then $\#A < \#B$.

Exercise 2.8.10. Let $A \subset B$ and assume B is finite. If $\#A = \#B$, then $A = B$.

Exercise 2.8.11. If A is a finite set and $f : A \to A$ is an injection, then f is a surjection.

Exercise 2.8.12. If there exists a proper subset $B \subset A$ and a bijection $g : A \to B$, then A is an infinite set. Hint: This statement can be viewed as a partial contrapositive of Exercise 2.8.11. The contrapositive of $P \wedge Q \Rightarrow R$ is $\neg R \Rightarrow \neg P \vee \neg Q$. However, the implication $Q \wedge \neg R \Rightarrow \neg P$, can be considered as a partial contrapositive.

Exercise 2.8.13. If A is a finite subset of $\mathbb{R}$, then A has a minimum and a maximum.

Remark 2.8.14. The importance of Exercise 2.8.13 is twofold. First, it states the most important property of finite sets of real numbers. Second, its contrapositive provides a simple way of proving that a set is infinite: If a nonempty subset of $\mathbb{R}$ does not have a minimum or it does not have a maximum, then it is infinite.

The fact that infinite sets might not have a minimum and/or maximum makes dealing with such sets more difficult. On a positive side an important feature of real numbers (see Section 2.14) is that there is a class of subsets of $\mathbb{R}$ which are guaranteed to have a minimum or a maximum.

Exercise 2.8.15. Prove that $\mathbb{N}$ does not have a maximum.

Exercise 2.8.16. Prove that a nonempty subset of $\mathbb{N}$ is finite if and only if it has a maximum.

Exercise 2.8.17. Prove that the set $\mathbb{N}$ is infinite.

Exercise 2.8.18. Let A be a nonempty subset of $\mathbb{N}$. Then A has a minimum.

Remark 2.8.19. Subsets of $\mathbb{N}$ can be infinite. As I mentioned in Remark 2.8.14 a problem with infinite sets is a possible absence of minimum and maximum. Exercise 2.8.18 tells us that a subset of natural numbers must at least have a minimum. Consequently, infinite subsets of $\mathbb{N}$ are not as bad as infinite subsets of $\mathbb{R}$.

Solution of Exercise 2.8.18. This proof uses the following two facts:

(1) Each finite set has a minimum. (Proved in Exercise 2.8.13.)
(2) For each $n \in \mathbb{N}$ each subset of the set $\{1, 2, \dots, n\} = [\![1, n]\!]_{\mathbb{N}}$ is finite. (Proved in Corollary 2.8.6.)

Since $A \neq \emptyset$, there exists $n \in A$. Consider the set $B = \{x \in A : x \leq n\}$. Then $B \subseteq [\![1, n]\!]_{\mathbb{N}}$. By fact (2) B is finite. Now, by fact (1) B has a minimum; denote it by $m = \min B$. Then m is also the minimum of A. (Here is a proof: If $a \in A$, then either $a \leq n$, or $n < a$. In the first case $a \in B$, and therefore $m \leq a$. If $n < a$, then $m \leq n < a$, and therefore $m \leq a$ for each $a \in A$.) $\square$

Definition 2.8.20. A set A is *countable* if there exists a bijection $f : \mathbb{N} \to A$.

Exercise 2.8.21. Prove that the set of even natural numbers is countable.

Exercise 2.8.22. If S is an infinite subset of $\mathbb{N}$, then S is countable.

Solution. (This is an extended Hint.) Let S be an infinite subset of $\mathbb{N}$. Let $s \in S$ be arbitrary. Then the set $S \cap [\![1, s]\!]$ is finite, since it is a nonempty subset of the finite set $[\![1, s]\!]$. Define the function:

$$f(s) := \#\big(S \cap [\![1, s]\!]\big), \quad s \in S.$$

Clearly $f : S \to \mathbb{N}$. The function f has the following two properties:

(I) If $s, t \in S$ and $s < t$, then $f(s) < f(t)$.
(II) If $s = \min S$, then $f(s) = 1$.
(III) If $s \in S$ and $t = \min\big(S \setminus [\![1, s]\!]\big)$, then $f(t) = f(s) + 1$.

Property (I) follows from Exercise 2.8.8. Property (II) follows from the fact that, $s = \min S$ implies $S \cap [\![1, s]\!] = \{s\}$. Property (III) follows from Exercise 2.8.7.

Property (I) implies that f is an injection. Properties (II) and (III) imply that the range of f, call it T has the following properties: $1 \in T$ and $n \in T \ \Rightarrow n{+}1 \in T$. Since $T \subset \mathbb{N}$, this, by Theorem 2.6.2 (5), implies $T = \mathbb{N}$. Thus f is a surjection. Hence, f is a bijection. $\square$

It will be proved in Sections 2.11 and 2.12 that the set of integers and the set of rational numbers are countable sets.

We conclude this section with a proposition which makes it easier to prove that an infinite set is countable. It states that it is sufficient to construct a surjection of $\mathbb{N}$ onto that set. This will be used to prove that the set of rational numbers is countable.

Proposition 2.8.23. *Let A be an infinite set and let $g : \mathbb{N} \to A$ be a surjection. Then A is countable. That is, there exists a bijection $\phi : \mathbb{N} \to A$.*

Proof. Assume that A is an infinite set and that $g : \mathbb{N} \to A$ is a surjection. Since g is a surjection the set $\big\{k \in \mathbb{N} : g(k) = a\big\}$ is nonempty for each $a \in A$. By Exercise 2.8.18 this set has a minimum. Define the function

$$h(a) := \min\big\{k \in \mathbb{N} : g(k) = a\big\}, \quad a \in A.$$

Clearly $h : A \to \mathbb{N}$. As an exercise the reader can prove that h is one-to-one. Denote by $S \subset$ the range of h. Then h is a bijection between A and S. Since A is infinite, S is also infinite. (Prove this as an exercise.) Therefore, by Exercise 2.8.22, there exists a bijection $f : S \to \mathbb{N}$. Now, $h : A \to S$ is a bijection and $f : S \to \mathbb{N}$. Hence the composition $f \circ h$ is also a bijection. Since $f \circ h : A \to \mathbb{N}$ the proof is complete. $\square$

2.9. More on countable sets

In the following example we give a recursive definition of the sequence indicated by the following table. The **triangular numbers** are in bold face.

n	**1**	2	**3**	4	5	**6**	7	8	9	**10**	11	12	13	14	**15**	16	17	18	19	20	**21**	22
r_n	1	2	2	3	3	3	4	4	4	4	5	5	5	5	5	6	6	6	6	6	6	7

EXAMPLE 2.9.1. Define the function $r : \mathbb{N} \to \mathbb{N}$ by

(i) $r_1 = 1$,

(ii) $\bigl(\forall\, n \in \mathbb{N}\bigr) \quad r_{n+1} = r_n + \operatorname{us}\Bigl(n - \dfrac{r_n(r_n+1)}{2}\Bigr)$.

To understand the recursive formula in (ii) we calculate the first few terms. We already know that $r_1 = 1$. Therefore

$$r_2 = r_1 + \operatorname{us}\Bigl(1 - \frac{r_1(r_1+1)}{2}\Bigr) = 1 + \operatorname{us}\bigl(1-1\bigr) = 1+1 = 2.$$

In a similar way we calculate:

$$\begin{aligned}
r_3 &= r_2 + \operatorname{us}\Bigl(2 - \frac{2(2+1)}{2}\Bigr) = 2 + \operatorname{us}\bigl(2-3\bigr) = 2+0 = 2,\\
r_4 &= r_3 + \operatorname{us}\Bigl(3 - \frac{2(2+1)}{2}\Bigr) = 2 + \operatorname{us}\bigl(3-3\bigr) = 2+1 = 3,\\
r_5 &= r_4 + \operatorname{us}\Bigl(4 - \frac{3(3+1)}{2}\Bigr) = 3 + \operatorname{us}\bigl(4-6\bigr) = 3+0 = 3, \ldots.
\end{aligned}$$

Although the claims of the next three exercises are quite evident from the pattern in the sequence r, we provide formal proofs using the principle of mathematical induction. It would be nice to have shorter formal proofs.

EXERCISE 2.9.2. Let $m, n \in \mathbb{N}$. If $m < n$, then $r_m \leq r_n$.

SOLUTION. Assume $m, n \in \mathbb{N}$ and $m < n$. Set $n - m = k$. Then $k \in \mathbb{N}$. We will prove the statement by induction on k. For $k = 1$, $n = m+1$ and by definition of r

$$r_n = r_{m+1} = r_m + \operatorname{us}\Bigl(m - \frac{r_m(r_m+1)}{2}\Bigr) \geq r_m.$$

Let now $k \in \mathbb{N}$ be arbitrary and assume that $n - m = k$ implies $r_m \leq r_n$. Let $n', m' \in \mathbb{N}$ be such that $n' - m' = k+1$. Then $n' - 1 - m' = k \in \mathbb{N}$. By the inductive assumption $r_{m'} \leq r_{n'-1}$. Since $n' - (n'-1) = 1$, we have $r_{n'-1} \leq r_{n'}$. Therefore, $r_{m'} \leq r_{n'}$. □

EXERCISE 2.9.3. Prove $\dfrac{(r_n-1)r_n}{2} < n \leq \dfrac{r_n(r_n+1)}{2}$ for all $n \in \mathbb{N}$.

SOLUTION. Proof by induction follows. For $n = 1$ the statement is true. Let $n \in \mathbb{N}$ be arbitrary and assume

$$\frac{(r_n-1)r_n}{2} < n \leq \frac{r_n(r_n+1)}{2} \tag{2.9.1}$$

We need to prove

$$\frac{(r_{n+1}-1)r_{n+1}}{2} < n+1 \leq \frac{r_{n+1}(r_{n+1}+1)}{2}. \tag{2.9.2}$$

We distinguish two cases for n in (2.9.1):

$$\text{Case 1.} \qquad n < \frac{r_n(r_n+1)}{2}, \tag{2.9.3}$$

$$\text{Case 2.} \qquad n = \frac{r_n(r_n+1)}{2}. \tag{2.9.4}$$

In Case 1, since $n - \frac{r_n(r_n+1)}{2} < 0$, by definition of r, $r_{n+1} = r_n$. Therefore,

$$\frac{(r_{n+1}-1)r_{n+1}}{2} = \frac{(r_n-1)r_n}{2} < n < n+1 \le \frac{r_n(r_n+1)}{2} = \frac{r_{n+1}(r_{n+1}+1)}{2}.$$

Hence (2.9.2) holds in this case. In Case 2, since $n - \frac{r_n(r_n+1)}{2} = 0$, by definition of r, $r_{n+1} = r_n + 1$. Therefore

$$\begin{aligned}\frac{(r_{n+1}-1)r_{n+1}}{2} = \frac{r_n(r_n+1)}{2} = n &< n+1 \\ &\le n+1+r_n \\ &= \frac{r_n(r_n+1)}{2} + r_n + 1 \\ &= \frac{(r_n+1)(r_n+2)}{2} \\ &= \frac{r_{n+1}(r_{n+1}+1)}{2}.\end{aligned}$$

Thus (2.9.2) is proved in both cases. □

Remark 2.9.4. A different way to state the claim of Exercise 2.9.3 is: Let $k, m \in \mathbb{N}$. If $r_k = m$, then $\dfrac{(m-1)m}{2} < k \le \dfrac{m(m+1)}{2}$.

Exercise 2.9.5. For all $m \in \mathbb{N}$ we have

$$\big\{k \in \mathbb{N} : r_k = m\big\} = \Big\{j \in \mathbb{N} : \frac{(m-1)m}{2} + 1 \le j \le \frac{m(m+1)}{2}\Big\}. \tag{2.9.5}$$

Solution. By the calculations at the beginning of this section and by Exercise 2.9.2, (2.9.5) is true for $m = 1$ and $m = 2$. Let $m \in \mathbb{N}$. It follows from Exercise 2.9.3 and Remark 2.9.4 that

$$\big\{k \in \mathbb{N} : r_k = m\big\} \subset \Big\{j \in \mathbb{N} : \frac{(m-1)m}{2} + 1 \le j \le \frac{m(m+1)}{2}\Big\}. \tag{2.9.6}$$

As a subset of a finite set, $\big\{k \in \mathbb{N} : r_k = m\big\}$ is finite. Set

$$b_m = \max\big\{k \in \mathbb{N} : r_k = m\big\}.$$

By (2.9.6), $b_m \le \dfrac{m(m+1)}{2}$. Clearly, $r(b_m) = m$ and $r(b_m+1) = m+1$. Therefore, by definition of r, $b_m - \dfrac{m(m+1)}{2} \ge 0$, and consequently

$$b_m = \max\big\{k \in \mathbb{N} : r_k = m\big\} = \frac{m(m+1)}{2}.$$

The last equality holds for all $m \in \mathbb{N}$. If $m \ge 2$, then $m - 1 \in \mathbb{N}$, and thus

$$\max\big\{k \in \mathbb{N} : r_k = m-1\big\} = \frac{(m-1)m}{2}.$$

Hence

$$r\Big(\frac{(m-1)m}{2}+1\Big)=(m-1)+1=m.$$

If $\dfrac{(m-1)m}{2}+1\leq j\leq\dfrac{m(m+1)}{2}$, by Exercise 2.9.2 we have

$$m=r\Big(\frac{(m-1)m}{2}+1\Big)\leq r(j)\leq r\Big(\frac{m(m+1)}{2}\Big)=m,$$

that is $r(j)=m$. Therefore

$$\Big\{j\in\mathbb{N}\,:\,\frac{(m-1)m}{2}+1\leq j\leq\frac{m(m+1)}{2}\Big\}\subset\big\{k\in\mathbb{N}\,:\,r_k=m\big\}. \tag{2.9.7}$$

The claim follows from (2.9.6) and (2.9.7). □

Remark 2.9.6. Recall that by Exercise 2.7.5 the triangular numbers are given by $T(n)=\frac{n(n+1)}{2}$ for all $n\in\mathbb{N}$. Hence, the claim of Exercise 2.9.5 can be stated as: For all $m\in\mathbb{N}, m\geq 2$, we have

$$\big\{k\in\mathbb{N}\,:\,r_k=m\big\}=\Big\{j\in\mathbb{N}\,:\,T(m-1)+1\leq j\leq T(m)\Big\}. \tag{2.9.8}$$

In particular,

$$r\big(T(m)\big)=m\qquad\big(\forall\ m\in\mathbb{N}\big). \tag{2.9.9}$$

The following exercise deals with the Cartesian square of the set $\mathbb{N}$; that is the set $\mathbb{N}\times\mathbb{N}$. Recall that this is the set of all ordered pairs of positive integers:

$$\mathbb{N}\times\mathbb{N}:=\big\{(s,t)\,:\,s,t\in\mathbb{N}\big\}.$$

The set $\mathbb{N}\times\mathbb{N}$ is illustrated by the following infinite table:

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \dots\\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \dots\\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \dots\\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \dots\\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \dots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Exercise 2.9.7. Prove that the function $A:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ defined by

$$A(s,t)=\frac{(s+t-2)(s+t-1)}{2}+s,\quad s,t\in\mathbb{N},$$

is a bijection.

A long Hint: Prove that the inverse of A is given by

$$B(n)=\left(n-\frac{(r_n-1)r_n}{2}\,,\,\frac{r_n(r_n+1)}{2}-n+1\right),\quad n\in\mathbb{N}.$$

Here r is the sequence recursively defined in Example 2.9.1. Notice also that by Exercise 2.7.5 the formulas for A and B can be written as

$$A(s,t) = T(s+t-2) + s, \qquad s,t \in \mathbb{N},$$
$$B(n) = \Bigl(n - T\bigl(r_n - 1\bigr)\,,\, T\bigl(r_n\bigr) - n + 1\Bigr), \quad n \in \mathbb{N}.$$

Let $s,t \in \mathbb{N}$. To evaluate $B\bigl(A(s,t)\bigr)$ you will need to evaluate $r\bigl(T(s+t-2)+s\bigr)$ first. For this, use equality (2.9.8) and the following inequalities

$$T(s+t-2) + 1 \leq T(s+t-2) + s \leq T(s+t-2) + s + t - 1 = T(s+t-1),$$

to conclude that

$$r\bigl(T(s+t-2)+s\bigr) = s+t-1.$$

Hence, $r\bigl(A(s,t)\bigr) = s+t-1$. With this identity calculating $B\bigl(A(s,t)\bigr)$ should be easier. This is the end of HINT.

To visualize the action of the function A on $\mathbb{N} \times \mathbb{N}$ we rearrange the table preceding Exercise 2.9.7 in a triangular shape and place the value of A in a circle next to the corresponding ordered pair. As a result we get the following table.

$$\begin{array}{ccccccc}
1\ (1,1) & 2\ (1,2) & 4\ (1,3) & 7\ (1,4) & 11\ (1,5) & 16\ (1,6) & \dots \\
 & 3\ (2,1) & 5\ (2,2) & 8\ (2,3) & 12\ (2,4) & 17\ (2,5) & \dots \\
 & & 6\ (3,1) & 9\ (3,2) & 13\ (3,3) & 18\ (3,4) & \dots \\
 & & & 10\ (4,1) & 14\ (4,2) & 19\ (4,3) & \dots \\
 & & & & 15\ (5,1) & 20\ (5,2) & \dots \\
 & & & & & 21\ (6,1) & \dots \\
 & & & & & & \ddots
\end{array}$$

EXERCISE 2.9.8. Let $\mathcal{A}$ be countable family of sets, that is there exists a bijection $f : \mathbb{N} \to \mathcal{A}$. Assume that each set in $\mathcal{A}$ is countable. Prove that $\bigcup\bigl\{A : A \in \mathcal{A}\bigr\}$ is also countable.

EXERCISE 2.9.9. Prove that the set of all functions $f : \mathbb{N} \to \{0,1\}$ is not countable. (As mentioned in Section 1.6 this set of functions is denoted by $\{0,1\}^{\mathbb{N}}$.)

SOLUTION. To prove the claim we will take an arbitrary function $\Phi : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ and prove that Φ is <u>not</u> a surjection. This will be proved by constructing a specific element $f \in \{0,1\}^{\mathbb{N}}$, that is $f : \mathbb{N} \to \{0,1\}$, such that $f \neq \Phi_n$ for all $n \in \mathbb{N}$. To construct f let us analyze Φ_n, $n \in \mathbb{N}$. Clearly $\Phi_1 : \mathbb{N} \to \{0,1\}$, that is $\Phi_1(n) \in \{0,1\}$ for all $n \in \mathbb{N}$. We can indicate the action of Φ_1 on $\mathbb{N}$ by listing its first 8 values:

$$\Phi_1(1) \qquad \Phi_1(2) \qquad \Phi_1(3) \qquad \Phi_1(4) \qquad \Phi_1(5) \qquad \Phi_1(6) \qquad \Phi_1(7) \quad \dots$$

We can do the same for $\Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6, \ldots$ to get

$$\begin{array}{cccccccc}
\boxed{\Phi_1(1)} & \Phi_1(2) & \Phi_1(3) & \Phi_1(4) & \Phi_1(5) & \Phi_1(6) & \Phi_1(7) & \ldots \\
\Phi_2(1) & \boxed{\Phi_2(2)} & \Phi_2(3) & \Phi_2(4) & \Phi_2(5) & \Phi_2(6) & \Phi_2(7) & \ldots \\
\Phi_3(1) & \Phi_3(2) & \boxed{\Phi_3(3)} & \Phi_3(4) & \Phi_3(5) & \Phi_3(6) & \Phi_3(7) & \ldots \\
\Phi_4(1) & \Phi_4(2) & \Phi_4(3) & \boxed{\Phi_4(4)} & \Phi_4(5) & \Phi_4(6) & \Phi_4(7) & \ldots \\
\Phi_5(1) & \Phi_5(2) & \Phi_5(3) & \Phi_5(4) & \boxed{\Phi_5(5)} & \Phi_5(6) & \Phi_5(7) & \ldots \\
\Phi_6(1) & \Phi_6(2) & \Phi_6(3) & \Phi_6(4) & \Phi_6(5) & \boxed{\Phi_6(6)} & \Phi_6(7) & \ldots \\
\Phi_7(1) & \Phi_7(2) & \Phi_7(3) & \Phi_7(4) & \Phi_7(5) & \Phi_7(6) & \boxed{\Phi_7(7)} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Now we are ready to define $f : \mathbb{N} \to \{0,1\}$ which will differ from each Φ_n, $n \in \mathbb{N}$. Set

$$f(n) := 1 - \Phi_n(n), \qquad n \in \mathbb{N}.$$

Since $\Phi_n(n) \in \{0,1\}$ we have that $f(n) = 1 - \Phi_n(n) \neq \Phi_n(n)$ for all $n \in \mathbb{N}$. Therefore

$$f \neq \Phi_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence $\Phi : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ is not a surjection. Since $\Phi : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ was arbitrary function, we conclude that there does not exist a bijection between $\mathbb{N}$ and $\{0,1\}^{\mathbb{N}}$. □

2.10. The Archimedean property

Is the set $\mathbb{N}$ bounded above? Of course it is not! But, how to prove that? It turns out that it is not possible to prove this with Axioms 1 through 15. Therefore we state this property of $\mathbb{N}$ as a new axiom.

Axiom 16 (AP, Archimedean Property). For every $b \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $n > b$.

Exercise 2.10.1. If $b > 0$, then there exists a natural number n such that $\dfrac{1}{n} < b$.

Exercise 2.10.2. If $c \in \mathbb{R}$ and $-\dfrac{1}{n} < c < \dfrac{1}{n}$ for all $n \in \mathbb{N}$, then $c = 0$.

Exercise 2.10.3. If a and b are positive real numbers, then there exists a natural number n such that $b < na$. This tells us that, even if a is quite small and b quite large, some integer multiple of a will exceed b.

Remark 2.10.4. Note that if we set $b = 1$ we obtain the statement in Exercise 2.10.1 and if we set $a = 1$, we obtain Axiom AP.)

Exercise 2.10.5. If $\alpha, \beta \in \mathbb{R}$. $0 \leq \alpha < \beta$ and $\beta - \alpha > 1$, then there exists $m \in \mathbb{N}$ such that $\alpha < m < \beta$ (that is, there exists $m \in (\alpha, \beta) \cap \mathbb{N}$).

Solution. Consider the set $A = \{k \in \mathbb{N} : \alpha < k\}$. By Axiom 16 the set A is not empty. Clearly $A \subset \mathbb{N}$. By Exercise 2.8.18 the set A has a minimum element. Put $m = \min A$. Now we have to prove that $\alpha < m < \beta$. Since $m \in A$, we have $\alpha < m$. In order to prove that $m < \beta$ we consider the following two cases: Case 1: $m = 1$ and Case 2: $m \in \mathbb{N} \backslash \{1\}$.

Case 1. Assume that $m = 1$. Since $1 < \beta - \alpha < \beta$ we see that $m < \beta$.
Case 2. Assume that $m \in \mathbb{N}\backslash\{1\}$. By Theorem 2.6.6 (ii), $j = m - 1 \in \mathbb{N}$. Clearly $j < m$. Is $j \in A$? NO: j is not in A since $j < m = \min A$. Since $j \in \mathbb{N}$ and $j \notin A$, we have $j \leq \alpha$. Add 1 to both sides of this inequality and we get $m = j+1 \leq \alpha+1 < \beta$. Therefore $m < \beta$. □

Remark 2.10.6. The goal of Exercise 2.10.5 is to prove the existence of a natural number with a certain property. In other words, given α and β we must construct a natural number m with the given property. What are possible tools for this construction? The proof above uses a remarkable idea how to do "constructions" of numbers:

Step 1: Identify a set of candidates for the desired number.
Step 2: The set of candidates is nice enough that it has an extreme element. (In this case it is a minimum.)
Step 3: Where else could our special number be hiding?

Exercise 2.10.7. Let $A \subset \mathbb{N}$ be a nonempty and bounded subset of $\mathbb{N}$. Prove that A is finite.

2.11. Integers

We define an *integer* to be a real number x such that either $x = 0$ or x is a natural number or $-x$ is a natural number. The set of all integers is denoted by $\mathbb{Z}$. Hence

$$\mathbb{Z} = \bigl\{x \in \mathbb{R} : \quad x \in \mathbb{N} \quad \text{or} \quad x = 0 \quad \text{or} \quad -x \in \mathbb{N}\bigr\}.$$

Exercise 2.11.1. Prove that $\mathbb{Z}$ is countable.

Exercise 2.11.2. Prove that $\mathbb{Z} \times \mathbb{N}$ is countable.

Exercise 2.11.3. A nonempty bounded below subset of $\mathbb{Z}$ has a minimum.

Exercise 2.11.4. A nonempty bounded above subset of $\mathbb{Z}$ has a maximum.

Exercise 2.11.5. Let $x \in \mathbb{R}$ be arbitrary. Prove that the set $\bigl\{k \in \mathbb{Z} : k \leq x\bigr\}$ has a maximum.

Exercise 2.11.6. Let $x \in \mathbb{R}$ be arbitrary. Prove that the set $\bigl\{k \in \mathbb{Z} : k \geq x\bigr\}$ has a minimum.

Based on the last two exercises we define the following two functions which relate real numbers to integers.

Definition 2.11.7. The *floor* function is defined by

$$\operatorname{flr}(x) = \lfloor x \rfloor := \max\bigl\{k \in \mathbb{Z} : k \leq x\bigr\}, \qquad x \in \mathbb{R}.$$

The *ceiling* function is defined by

$$\operatorname{clg}(x) = \lceil x \rceil := \min\bigl\{k \in \mathbb{Z} : x \leq k\bigr\}, \qquad x \in \mathbb{R}.$$

Exercise 2.11.8. Prove $x - 1 < \lfloor x \rfloor \leq x$ for all $x \in \mathbb{R}$.

Exercise 2.11.9. Prove $x \leq \lceil x \rceil < x + 1$ for all $x \in \mathbb{R}$.

Exercise 2.11.10. Prove $x \left\lceil \dfrac{1}{x} \right\rceil \geq 1$ for all $x > 0$.

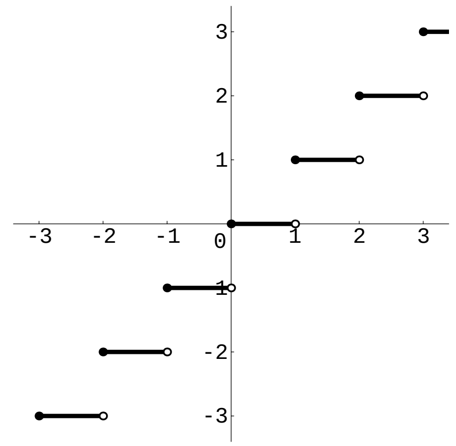


Figure 12: The floor function

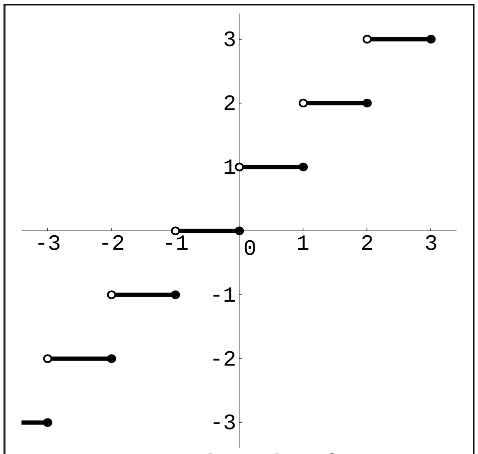


Figure 13: The ceiling function

Exercise 2.11.11. Let $a, b \in \mathbb{R}$ and assume $b-a \geq 1$. Prove $a < \dfrac{\lceil a \rceil + \lfloor b \rfloor}{2} < b$.

Exercise 2.11.12. Let $a, b \in \mathbb{R}$ and assume $a < b$. Prove

$$a < \frac{\left\lceil \left\lceil \dfrac{1}{b-a} \right\rceil a \right\rceil + \left\lfloor \left\lceil \dfrac{1}{b-a} \right\rceil b \right\rfloor}{2 \left\lceil \dfrac{1}{b-a} \right\rceil} < b.$$

2.12. Rational numbers

Now we define a *rational number* to be a real number of the form $m \cdot \frac{1}{n}$ where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. (We write shortly m/n or $\frac{m}{n}$ instead of $m \cdot \frac{1}{n}$.) The set of all rational numbers we denote by $\mathbb{Q}$, that is

$$\mathbb{Q} = \left\{ x \in \mathbb{R} \ : \ \exists\, m \in \mathbb{Z} \text{ and } \exists\, n \in \mathbb{N} \ \text{ such that } \ x = \frac{m}{n} \right\}.$$

We will prove later that $\mathbb{Q}$ is a proper subset of $\mathbb{R}$.

Exercise 2.12.1. If $a, b \in \mathbb{R}$ and $a < b$, then there exists a rational number $r \in \mathbb{Q}$ such that $a < r < b$

Exercise 2.12.2. Let $a, b \in \mathbb{R}$ and $a < b$. Prove that $\{x \in \mathbb{Q} \, : \, a < x < b\}$ is an infinite set.

Exercise 2.12.3. Denote by $\mathbb{Q}_+$ the set of all positive rational numbers. Prove that $\mathbb{Q}_+$ is countable.

Exercise 2.12.4. Prove that there exists a bijection between $\mathbb{Q}_+$ and $\mathbb{Q}$.

Exercise 2.12.5. Prove that the set $\mathbb{Q}$ is countable.

Exercise 2.12.6. Prove that $r^2 \neq 2$ for all $r \in \mathbb{Q}$.

Solution. Now that we have a definition of $\mathbb{Q}$ we can prove that for each $r \in \mathbb{Q}$ there exist $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ which are not both even such that $r = p/q$. Let $r \in \mathbb{Q}$ be arbitrary. Set

$$S = \left\{ n \in \mathbb{N} : \exists\; m \in \mathbb{Z} \ \text{ such that } \ r = \frac{m}{n} \right\}.$$

Since $r \in \mathbb{Q}$ the set S is not empty. Since S is a nonempty subset of $\mathbb{N}$, by Exercise 2.8.18 S has a minimum. Set $q = \min S$. Since $q \in S$, there exists $p \in \mathbb{Z}$ such that $r = p/q$. Next we will prove that p and q are not both even. That is we will prove the following implication: If $q = \min S$, $p \in \mathbb{Z}$, and $r = p/q$, then p and q are not both even.

It is easier to prove a partial contrapositive of the last implication. If $n \in S$, $m \in \mathbb{Z}$, $r = m/n$ and both m and n are even, then n is not a minimum of S. So, assume $n \in S$, $m \in \mathbb{Z}$ are both even and $r = m/n$. Then there exist $k \in \mathbb{Z}$ and $j \in \mathbb{N}$ such that $m = 2k$ and $n = 2j$. Clearly $j < n$. Also,

$$r = \frac{m}{n} = \frac{2\,k}{2\,j} = \frac{k}{j}.$$

Hence $j \in S$ and therefore n is not a minimum of S.

In Chapter 1 we indicated how to prove that $2\,q^2 \neq p^2$. Hence $r^2 \neq 2$ is proved. □

Remark 2.12.7. Since we expect that there is a real number a such that $a^2 = 2$, we see the fact that there is no rational number x such that $x^2 = 2$ as a deficiency of the set of rational numbers. Note that the set $\mathbb{Q}$ of rational numbers satisfies all Axioms 1 through 16. Therefore we can not expect that based on Axioms 1 through 16 we can prove that there exists a real number number α such that $a^2 = 2$. Therefore we need an extra axiom for the set of real numbers; an axiom that will not be satisfied by the set of rational numbers. This is the **Completeness Axiom** which we introduce in Section 2.14.

2.13. A prelude to the Completeness Axiom

In this section we will introduce two sets closely related to finding a positive number a such that $a^2 = 2$. The first set is

$$T = \{x \in \mathbb{R} : x > 0 \ \text{ and } \ x^2 \geq 2\},$$

and the second is simply the complement of T in $\mathbb{R}$:

$$S = \mathbb{R} \backslash T = \{y \in \mathbb{R} : y \leq 0 \ \text{ or } \ y^2 < 2\}.$$

Next we list and prove five properties of the sets S and T that are relevant to finding $a \in \mathbb{R}$ such that $a^2 = 2$.

Property 1. The sets S and T are nonempty sets.

Proof of Property 1: It is clear that $1 \in S$ and $2 \in T$. Therefore Property 1 holds. □

Property 2. $S \cup T = \mathbb{R}$.

Proof of Property 2: This property follows from the definition of S. □

Property 3. For all $s \in S$ and for all $t \in T$, $s < t$.

Proof of Property 3: Let $s \in S$ and $t \in T$. Then by the definition of T, $t > 0$ and $t^2 \geq 2$. There are two possibilities for s: either $s \leq 0$ or, $s > 0$ and $s^2 < 2$. If $s \leq 0$ it is clear that $s < t$. If $s > 0$ and $s^2 < 2$, then $s^2 < t^2$. Now Exercise 2.2.6 implies that $s < t$. Thus in both cases for s we proved that $s < t$. □

Property 4. The set S does not have a maximum.

Proof of Property 4: It is clear that $0 \in S$. Therefore, if $s < 0$, then s is not a maximum of S. Next we consider $s \in S$ and $s \geq 0$. Then $s^2 < 2$ and $2s + 1 > 0$, and therefore

$$0 < \frac{2 - s^2}{2s + 1}.$$

By Exercise 2.10.1, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \frac{2 - s^2}{2s + 1}.$$

Consequently,

$$\frac{2s + 1}{n} < 2 - s^2. \tag{2.13.1}$$

Since $n \geq 1$ we have $n^2 \geq n$ and therefore $1/n^2 \leq 1/n$. This inequality and (2.13.1) yield

$$\left(s + \frac{1}{n}\right)^2 = s^2 + 2s\frac{1}{n} + \frac{1}{n^2} \leq s^2 + 2s\frac{1}{n} + \frac{1}{n} \leq s^2 + \frac{2s + 1}{n} < s^2 + 2 - s^2 = 2.$$

Hence $(s + 1/n) \in S$. Since $s < s + 1/n$, s is not a maximum of S. This proves that S does not have a maximum. □

Property 5. If $x \in T$ and $x^2 > 2$, then x is not a minimum of T.

Proof of Property 5: Let $x \in T$ and $x^2 > 2$. Then $x/2 > 1/x$. Adding $x/2$ to both sides we get $x > x/2 + 1/x$. Put

$$y = \frac{x}{2} + \frac{1}{x}.$$

Then $y > 0$ and $x > y$. Next we calculate y^2:

$$y^2 = \left(\frac{x}{2} + \frac{1}{x}\right)^2 = \frac{x^2}{4} + 1 + \frac{1}{x^2} = 2 + \frac{x^2}{4} - 1 + \frac{1}{x^2} = 2 + \left(\frac{x}{2} - \frac{1}{x}\right)^2 > 2.$$

Thus $y \in T$. Since $y < x$, x is not a minimum of T. □

Based on Properties 1 through 5 the only possibility for the positive number a such that $a^2 = 2$ is that it is a minimum of T. Based on Axioms 1 through 18 we can not prove that T has a minimum. Therefore it is necessary to introduce a new axiom which will assure that T has a minimum. That is the **Completeness axiom**.

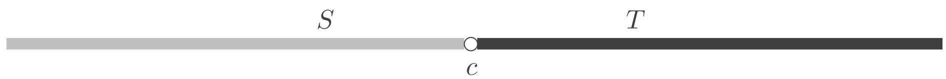
Remark 2.13.1. Property 5 can be proved using the method used to prove Property 4. Similarly, Property 4 can be proved using the method used to prove Property 5 as follows. Let $s \in S$ and $s > 0$. Then $s^2 < 2$ and consequently $s^2 + 2 < 4$. Further, $1 < 4/(s^2 + 2)$, and hence $s < 4s/(s^2 + 2)$. Put $v = 4s/(s^2 + 2)$. Then $s < v$ and

$$v^2 = \left(\frac{4s}{s^2 + 2}\right)^2 = \frac{2^2}{\left(\frac{s}{2} + \frac{1}{s}\right)^2} = \frac{4}{2 + \left(\frac{s}{2} - \frac{1}{s}\right)^2} < 2.$$

2.14. The Completeness Axiom

Axiom 17 (CA, Completeness Axiom). If S and T are subsets of $\mathbb{R}$ such that
(a) $S \neq \emptyset$, $T \neq \emptyset$,
(b) $S \cup T = \mathbb{R}$,
(c) $s < t$ for all $s \in S$ and for all $t \in T$,
then either S has a maximum or T has a minimum.

Visually this corresponds to the picture



Since we perceive the real number line to have no holes, the number c that is marked by the open circle must be either in S or in T. This cannot be proved rigorously. Therefore this property of the real numbers is stated as the Completeness Axiom of $\mathbb{R}$.

Definition 2.14.1. A pair S and T of subsets of $\mathbb{R}$ satisfying the properties (a),(b) and (c) in Axiom CA is called a *Dedekind cut*.

Applying Axiom CA to the sets S and T defined in Section 2.13 we conclude that the set T must have a minimum. (Notice that we proved in Section 2.13 that the set $S = \mathbb{R}\setminus T$ does not have a maximum.) As proved in Section 2.13 this minimum has the property $(\min T)^2 = 2$. In this way (using Axiom 17) we proved the existence of a positive real number x such that $x^2 = 2$. The uniqueness of a positive real number x such that $x^2 = 2$ follows from Exercise 2.2.6. By definition this number is called a *square root of* 2 and it is denoted by $\sqrt{2}$. In the following exercise we prove the existence of a square root for every positive real number.

Exercise 2.14.2. Let $a > 0$. Prove that there exists a unique positive real number α such that $\alpha^2 = a$.

The following definition is justified by the preceding exercise.

Definition 2.14.3. Let $a > 0$. The unique positive real number α such that $\alpha^2 = a$ is called the *square root of* a and it is denoted by $\sqrt{a}$. We also set $\sqrt{0} = 0$. The function $a \mapsto \sqrt{a}$ defined on $[0, +\infty)$ is called the *square root function*.

Theorem 2.14.4 (Cantor's Intersection Theorem). *Assume that for each $n \in \mathbb{N}$ we are given a bounded closed interval $[a_n, b_n] \subset \mathbb{R}$. Assume that for $m, n \in \mathbb{N}$ such that $m \leq n$ we have*

$$\big[a_n, b_n\big] \subset \big[a_m, b_m\big], \quad \text{that is} \quad a_m \leq a_n \leq b_n \leq b_m. \tag{2.14.1}$$

Then

$$\bigcap_{n \in \mathbb{N}} \big[a_n, b_n\big] \neq \emptyset.$$

Proof. Define the following two sets

$$S = \big\{x \in \mathbb{R} \,:\, x \leq b_n \quad (\forall\, n \in \mathbb{N})\big\},$$
$$T = \mathbb{R}\setminus S = \big\{y \in \mathbb{R} \,:\, (\exists\, k \in \mathbb{N}) \;\; y > b_k\big\}.$$

The sets S and T form a Dedekind cut. We justify this clim by proving (a), (b) and (c) in Axiom CA. Since $1 \leq n$ for all $n \in \mathbb{N}$, assumption (2.14.1) implies

$$a_1 \leq a_n \leq b_n \leq b_1 < 1 + b_1.$$

Therefore $a_1 \in S$ and $1 + b_1 \in T$. Hence (a) holds. Statement (b) is obvious. To prove (c), let $s \in S$ and $t \in T$ be arbitrary. Since $t \in T$, there exists $k \in \mathbb{N}$ such that $t > b_k$. Since $s \in S$, $b_k \geq s$. Therefore $t > s$, and (c) is proved.

Next we prove that T does not have a minimum. Let $t \in T$ be arbitrary. Then, there exists $k \in \mathbb{N}$ such that $t > b_k$. Then

$$t > \frac{t + b_k}{2} > b_k,$$

and therefore, $\bigl(t + b_k\bigr)/2 \in T$. Hence, t is not a minimum of T.

By the Completeness Axiom, S must have a maximum. Set $c = \max S$. Next we will prove that

$$a_n \leq c \leq b_n \quad \bigl(\forall\, n \in \mathbb{N}\bigr). \tag{2.14.2}$$

Since $c \in S$ we have

$$c \leq b_n \quad \bigl(\forall\, n \in \mathbb{N}\bigr).$$

This proves the right inequality in (2.14.2). To prove the left inequality in (2.14.2), we will prove the implication

$$c = \max S \quad \Rightarrow \quad a_n \leq c \quad \bigl(\forall\, n \in \mathbb{N}\bigr). \tag{2.14.3}$$

As it is often the case, the contrapositive is easier to prove:

$$\exists\, k \in \mathbb{N} \ \text{ such that } \ a_k > x \quad \Rightarrow \quad x < \max S. \tag{2.14.4}$$

To prove the last implication, let $k \in \mathbb{N}$ be such that $a_k > x$. By (2.14.1), for all $n \in \mathbb{N}$ we have $a_k \leq b_n$. Therefore

$$x < \frac{x + a_k}{2} < a_k \leq b_n \quad \bigl(\forall\, n \in \mathbb{N}\bigr).$$

Thus $\bigl(x + a_k\bigr)/2 \in S$. This implies $x < \max S$ and (2.14.4) is proved. Consequently (2.14.3) holds. Implication (2.14.3) is the proof of the left inequality in (2.14.2). Now (2.14.2) is proved. Clearly (2.14.2) implies

$$c \in \bigcap_{n \in \mathbb{N}} \bigl[a_n, b_n\bigr], \quad \text{that is,} \quad \bigcap_{n \in \mathbb{N}} \bigl[a_n, b_n\bigr] \neq \emptyset.$$

This proves Cantor's Intersection Theorem. □

Exercise 2.14.5. Let $a, b \in \mathbb{R}$ be such that $a < b$. Let $f : \mathbb{N} \to (a, b)$ be a function. Then f is not a surjection.

Proof. This is a sketch of a proof. First notice that by the assumption $a < f(n) < b$ for all $n \in \mathbb{N}$.

Next, for each $n \in \mathbb{N}$ we will define a closed interval $\bigl[a_n, b_n\bigr] \subset (a, b)$ using the principle of inductive definition.

(i) Set $a_1 = \frac{2}{3}\, f(1) + \frac{1}{3}\, b, \quad b_1 = \frac{1}{3}\, f(1) + \frac{2}{3}\, b.$

(ii) For each $n \in \mathbb{N}$, we distinguish the following three cases and in each case we define a_{n+1} and b_{n+1} as follows.

CASE 1: If $f(n+1) \leq a_n$ set

$$a_{n+1} = \frac{1}{2}\,a_n + \frac{1}{2}\,b_n, \qquad b_{n+1} = b_n.$$

CASE 2: If $a_n < f(n+1) < b_n$ set

$$a_{n+1} = \frac{1}{2}\,f(n+1) + \frac{1}{2}\,b_n, \qquad b_{n+1} = b_n.$$

CASE 3: If $f(n+1) \geq b_n$ set

$$a_{n+1} = a_n, \qquad b_{n+1} = \frac{1}{2}\,a_n + \frac{1}{2}\,b_n.$$

The following two properties are not difficult to prove:

(A) The closed intervals $\bigl[a_n, b_n\bigr]$, $n \in \mathbb{N}$, satisfy (2.14.1). Therefore, by Cantor's intersection theorem (Exercise 2.14.4), there exists $c \in (a,b)$ such that $c \in \bigl[a_n, b_n\bigr]$ for all $n \in \mathbb{N}$.

(B) For each $n \in \mathbb{N}$ we have $f(n) \notin \bigl[a_n, b_n\bigr]$.

From (A) and (B) above we conclude that $f(n) \neq c$ for all $n \in \mathbb{N}$ and $c \in (a,b)$. Thus f is not a surjection. □

Another important consequence of the Completeness Axiom is the existence of the surrogates for minimums and maximums which we study in the next section.

2.15. Infimums and supremums

DEFINITION 2.15.1. Let A be a nonempty subset of $\mathbb{R}$. A number $w \in \mathbb{R}$ is a *supremum* (or a *least upper bound*) of A if

(i) w is an upper bound for A, and
(ii) if v is an upper bound for A and $w \neq v$, then $w < v$.

DEFINITION 2.15.2. Let A be a nonempty subset of $\mathbb{R}$. A number $u \in \mathbb{R}$ is an *infimum* (or *greatest lower bound*) of A if

(i) u is a lower bound for A, and
(ii) if v is a lower bound for A and $v \neq u$, then $v < w$.

If u and w are as in Definitions 2.15.1 and 2.15.2, we write

$$u = \sup A \ (= \operatorname{lub} A\ ) \quad \text{ and } \quad w = \inf A\,(= \operatorname{glb} A).$$

EXERCISE 2.15.3. If $(\sup A) \in A$, then $\sup A = \max A$. State and prove the analogous statement for $\inf A$.

EXERCISE 2.15.4. Let A be a nonempty and bounded above subset of $\mathbb{R}$. Prove that the set of all upper bounds of A has a minimum.

The following exercise gives the standard form of the **Completeness Axiom**.

EXERCISE 2.15.5. A nonempty subset of $\mathbb{R}$ that is bounded above has a supremum. In other words, if a set $A \subset \mathbb{R}$ is nonempty and bounded above, then $\sup A$ exists and it is a real number.

EXERCISE 2.15.6. A nonempty and bounded below subset of $\mathbb{R}$ has an infimum.

EXERCISE 2.15.7. Let $A \subset \mathbb{R}$, $A \neq \emptyset$ and A is bounded below. Prove that $a = \inf A$ if and only if

(a) a is a lower bound of A, that is, $a \leq x$, for all $x \in A$;

(b) for each $\epsilon > 0$ there exists $x \in A$ such that $x < a + \epsilon$.

Notice that x in (b) depends on ϵ. Sometimes it is useful to indicate this dependence by writing x_ϵ or $x(\epsilon)$ instead of x.

Exercise 2.15.8. State and prove a characterization of $\sup A$ which is analogous to the characterization of $\inf A$ given in Exercise 2.15.7.

2.16. Exercises

Exercise 2.16.1. Find sup and inf for the sets $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ and $B = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$. Formal proofs are required. (By a formal proof I mean a rigorous mathematical proof of properties (i) and (ii) in Definitions 2.15.1 and 2.15.2.)

Exercise 2.16.2. Find sup and inf for the set $\left\{ \frac{n^{(-1)^n}}{n+1} : n \in \mathbb{N} \right\}$.

Exercise 2.16.3. Let A be a nonempty and bounded above subset of $\mathbb{R}$. If B is a nonempty subset of A, then B is bounded above and $\sup B \leq \sup A$. Formulate the corresponding statement for the infimums.

Exercise 2.16.4. Let A and B be nonempty bounded above subsets of $\mathbb{R}$. Prove

$$\sup(A \cup B) = \max\bigl\{\sup A, \sup B\bigr\}.$$

Exercise 2.16.5. Let A and B be nonempty subsets of $\mathbb{R}$ such that for all $x \in A$ and for all $y \in B$ we have $x \leq y$. Prove that $\sup A \leq \inf B$.

If the condition $x \leq y$ is replaced by the condition $x < y$, can we conclude that $\sup A < \inf B$?

Exercise 2.16.6. Suppose that A and B are nonempty subsets of $\mathbb{R}$ such that for all $x \in A$ and for all $y \in B$ we have $x \leq y$. Prove that $\sup A = \inf B$ if and only if for each $\delta > 0$ there exist $x \in A$ and $y \in B$ such that $x + \delta > y$.

Exercise 2.16.7. Let A be a nonempty and bounded above subset of $\mathbb{R}$, and let F be a finite subset of A. If $(\sup A) \notin A$, then $\sup(A \backslash F) = \sup A$.

State and prove an analogous statement for $\inf A$?

Exercise 2.16.8. Consider the set

$$A = \bigl\{ x \in \mathbb{R} : x > 0 \quad \text{and } x^2 < 2 \bigr\}.$$

Prove that A is nonempty and bounded. Put $\alpha = \sup A$. Prove that $\alpha^2 = 2$.

Note: Do this exercise using only the properties of the supremum. Do not use the existence of $\sqrt{2}$ proved in Exercise 2.14.2.

Exercise 2.16.9. Let $a > 0$. Consider the set

$$A = \bigl\{ x \in \mathbb{R} : x > 0 \quad \text{and} \quad x^2 < a \bigr\}.$$

Prove that A is nonempty and bounded. Put $\alpha = \sup A$. Prove that $\alpha^2 = a$.

Exercise 2.16.10. Let $a > 0$ and $n \in \mathbb{N}$. Consider the set

$$A = \bigl\{ x \in \mathbb{R} : x > 0 \quad \text{and} \quad x^n < a \bigr\}.$$

Prove that A is nonempty and bounded. Put $\alpha = \sup A$. Prove that $\alpha^n = a$.

Exercise 2.16.11. Let $n \in \mathbb{N}$. Prove that the function $f : [0, +\infty) \to [0, +\infty)$ defined by $f(x) = x^n, x \geq 0$, is a bijection.

Definition 2.16.12. The inverse of the bijection $f : [0, +\infty) \to [0, +\infty)$ from Exercise 2.16.11 is called the *n-th root function*. For $x \geq 0$ the value of the n-th root function at x is denoted by $\sqrt[n]{x}$ and it is called the *n-th root of x*.

Exercise 2.16.13. Let A be a nonempty subset of $\mathbb{R}$. Define the difference set A_d of A to be

$$A_d := \{ b - a \,:\, a, b \in A \text{ and } a < b \}$$

If A is infinite and bounded, then $\inf A_d = 0$.

Remark 2.16.14. A partial contraposition of the last exercise is as follows. If A is infinite and $\inf A_d > 0$, then A is not bounded. Since we proved that $\mathbb{N}$ is infinite and clearly $\mathbb{N}_d = \mathbb{N}$ and hence $\inf \mathbb{N}_d = 1$, the contrapositive of Exercise 2.16.13 implies that $\mathbb{N}$ is not bounded. This is exactly what we stated in Axiom 16. Thus, Axiom 16 can be proved using the Completeness Axiom. As such Axiom 16 turns out to be unnecessary. I introduced it since a wanted to study basic properties of the integers and of the rational numbers in detail, before introducing the completeness axiom. Notice that the existence of the floor and the ceiling function and the fact that there are rational numbers in any open interval all depend on the Archimedean property, and via the Archimedean property these properties depend on the Completeness Axiom.

In conclusion, the set $\mathbb{R}$ is completely described by Axioms 1 through 15 and the Completeness Axiom. All claims about real numbers can be proved using these 16 axioms and their consequences. As you probably already noticed in proofs we also use elementary properties of sets and operations with sets.

2.17. Topology of $\mathbb{R}$

The terminology that we introduce in the next definition provides the essential vocabulary of the modern analysis.

Definition 2.17.1. All points in this definition are elements of $\mathbb{R}$ and all sets are subsets of $\mathbb{R}$.

(a) Let $\epsilon > 0$. A *neighborhood* (or ϵ-*neighborhood*) of a point a is the set

$$N(a, \epsilon) = \{ x \in \mathbb{R} : |x - a| < \epsilon \} = (a - \epsilon, a + \epsilon)$$

The number ϵ is called the *radius* of $N(a, \epsilon)$.

(b) A point a is an *accumulation point* of a set E if every neighborhood of a contains a point $x \neq a$ such that $x \in E$. That is, a is an *accumulation point* of the set E if

$$E \bigcap \big(N(a, \epsilon) \setminus \{a\} \big) \neq \emptyset \quad \text{for all} \quad \epsilon > 0.$$

(c) A set E is *closed* if it contains all its accumulation points. That is, E is closed if the following implication holds:

$$x \text{ is an accumulation point of } E \;\Rightarrow\; x \in E.$$

(d) A point a is an *interior point* of the set E if there is a neighborhood of a that is a subset of E. That is, a is an *interior point* of E if there exists $\epsilon > 0$ such that $N(a, \epsilon) \subset E$.

(e) A set E is *open* if every point of E is an interior point of E.

(f) A set E is *compact* if every infinite subset of E has an accumulation point in E.
(g) Let $E \subseteq F$. A set E is *dense* in F if every neighborhood of every point in F contains a point of E.

EXERCISE 2.17.2. Find all accumulation points of the set $\left\{ \dfrac{n^{(-1)^n}}{n+1} \, : \, n \in \mathbb{N} \right\}$. Provide formal proofs.

EXERCISE 2.17.3. Find all accumulation points of $\left\{ \dfrac{4}{n} + \dfrac{n}{4} - \left\lfloor \dfrac{n}{4} \right\rfloor \, : \, n \in \mathbb{N} \right\}$. Provide formal proofs.

EXERCISE 2.17.4. Let $A \subset \mathbb{R}$ be a bounded set. If A does not have a maximum, then $\sup A$ is an accumulation point of A. Is the converse of the last implication true?

EXERCISE 2.17.5. Let $a < b$. Prove that the open interval (a, b) is an open set. Prove that the complement of (a, b), that is the set $\mathbb{R} \setminus (a, b)$ is closed. (HINT: State the contrapositive of the implication in the definition of a closed set. Simplify the contrapositive using the concept of an interior point.)

EXERCISE 2.17.6. Let $a < b$. Prove that the closed interval $[a, b]$ is a closed set. Prove that the complement of $[a, b]$, that is the set $\mathbb{R} \setminus [a, b]$ is open.

EXERCISE 2.17.7. Let $a < b$. Consider the interval $[a, b)$. Is this a closed set? Is it open?

EXERCISE 2.17.8. Is $\mathbb{R}$ a closed set? Is it open?

EXERCISE 2.17.9. Prove that $G \subset \mathbb{R}$ is open if and only if $\mathbb{R} \setminus G$ is closed.

EXERCISE 2.17.10. Let $a < b$. Prove that the closed interval $[a, b]$ is a compact set.

HINT: Use Cantor's intersection theorem. Consider an arbitrary infinite subset E of $[a, b]$. Define a sequence of closed intervals $[a_n, b_n]$, $n \in \mathbb{N}$, such that, for all $n \in \mathbb{N}$,

$$[a_n, b_n] \subset [a, b], \qquad [a_{n+1}, b_{n+1}] \subset [a_n, b_n], \qquad b_n - a_n = (b - a)/2^{n-1},$$

and, most importantly, $[a_n, b_n] \cap E$ is infinite.

DEFINITION 2.17.11. A family $\mathcal{G}$ of open sets is an *open cover* for a set E if

$$E \subset \bigcup \{ G : G \in \mathcal{G} \}.$$

DEFINITION 2.17.12. If every open cover of a set E has a finite subfamily that is also an open cover of E, than we say that E has the *Heine-Borel* property.

EXERCISE 2.17.13. Let $a < b$. Prove that the closed interval $[a, b]$ has the Heine-Borel property.

HINT: Let $\mathcal{G}$ be an arbitrary open cover of $[a, b]$. Consider the set

$$S = \left\{ x \in (a, b] \, : \, \exists \, n \in \mathbb{N} \text{ and } \exists \, G_1, \ldots, G_n \in \mathcal{G} \text{ such that } [a, x] \subset \bigcup_{j=1}^{n} G_j \right\}.$$

2.17.1. The structure of open sets in $\mathbb{R}$.

DEFINITION 2.17.14. A subset $I \subset \mathbb{R}$ is an open interval if one of the following four conditions is satisfied

- $I = \mathbb{R}$.
- There exists $a \in \mathbb{R}$ such that $I = (-\infty, a)$.
- There exists $b \in \mathbb{R}$ such that $I = (b, +\infty)$.
- There exist $a, b \in \mathbb{R}$ such that $a < b$ and $I = (a, b)$.

EXERCISE 2.17.15. Let $\mathcal{I}$ be an infinite family of open mutually disjoint intervals. (Mutually disjoint means that if $I_1, I_2 \in \mathcal{I}$ and $I_1 \neq I_2$, then $I_1 \cap I_2 \neq \emptyset$.) Prove that $\mathcal{I}$ is countable.

EXERCISE 2.17.16. Let G be a nonempty open subset of $\mathbb{R}$. Assume that $\mathbb{R} \setminus G$ is neither bounded above nor below. Prove that for each $x \in G$ there exist $a, b \in \mathbb{R} \setminus G$ such that $a < b$, $x \in (a, b)$ and $(a, b) \subset G$.

EXERCISE 2.17.17. Let G be a nonempty open subset of $\mathbb{R}$. Assume that $\mathbb{R} \setminus G$ is neither bounded above nor below. Prove that there exists a finite or countable family of open mutually disjoint intervals whose union equals G.

EXERCISE 2.17.18. Let G be a nonempty open subset of $\mathbb{R}$. Then there exists a finite or countable family of open mutually disjoint intervals whose union equals G.