CHAPTER 4

Continuous functions

In this chapter I will always denote a non-empty subset of $\mathbb{R}$.

4.1. The ϵ-δ definition of a continuous function

DEFINITION 4.1.1. A function $f : I \to \mathbb{R}$ is *continuous at a point* $x_0 \in I$ if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$x \in \big(x_0 - \delta(\epsilon), x_0 + \delta(\epsilon)\big) \cap I \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon. \tag{4.1.1}$$

The function f is *continuous on* I if it is continuous at each point of I.

Note that the implication in (4.1.1) can be restated as

$$x \in I \ \text{ and } \ |x - x_0| < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon.$$

Next we restate Definition 4.1.1 using the terminology introduced in Section 2.17. For a function $f : I \to \mathbb{R}$ and a subset $A \subset I$ we will use the notation $f(A)$ to denote the set $\big\{y \in \mathbb{R} : \exists\, x \in A \text{ s.t. } f(x) = y\big\} = \big\{f(x) : x \in A\big\}$.

A function $f : I \to \mathbb{R}$ is continuous at a point $x_0 \in I$ if for each neighborhood V of $f(x_0)$ there exists a neighborhood U of x_0 such that

$$f\big(I \cap U\big) \subset V.$$

4.2. Finding $\delta(\epsilon)$ for a given function at a given point

In this and the next section we will prove that some familiar functions are continuous.

A general strategy for proving that a given function f is continuous at a given point x_0 is as follows:

Step 1. Simplify the expression $|f(x) - f(x_0)|$ and try to establish a simple connection with the expression $|x - x_0|$. The simplest connection is to discover positive constants δ_0 and K such that

$$x \in I \ \text{ and } \ x_0 - \delta_0 < x < x_0 + \delta_0 \quad \Rightarrow \quad |f(x) - f(x_0)| \leq K\,|x - x_0|. \tag{4.2.1}$$

Formulate your discovery as a lemma.

Step 2. Let $\epsilon > 0$ be given. Use the result in Step 1 to define your $\delta(\epsilon)$. For example, if (4.2.1) holds, then $\delta(\epsilon) = \min\big\{\epsilon/K, \delta_0\big\}$.

Step 3. Use the definition of $\delta(\epsilon)$ from Step 2 and the lemma from Step 1 to prove implication (4.1.1).

EXAMPLE 4.2.1. We will show that the function $f(x) = x^2$ is continuous at $x_0 = 3$. Here $I = \mathbb{R}$ and we do not need to worry about the domain of f.

Step 1. First simplify

$$\big|f(x) - f(x_0)\big| = \big|x^2 - 3^2\big| = \big|(x+3)(x-3)\big| = \big|x+3\big|\,\big|x-3\big|. \tag{4.2.2}$$

Now we notice that if $2 < x < 4$ we have $\bigl|x+3\bigr| = x+3 \le 7$. Thus (4.2.1) holds with $\delta_0 = 1$ and $K = 7$. We formulate this result as a lemma.

LEMMA. *Let $f(x) = x^2$ and $x_0 = 3$. Then*

$$|x-3| < 1 \quad \Rightarrow \quad \bigl|x^2 - 3^2\bigr| < 7|x-3|. \tag{4.2.3}$$

PROOF. Let $|x-3| < 1$. Then $2 < x < 4$. Therefore $x+3 > 0$ and $|x+3| = x+3 < 7$. By (4.2.2) we now have $\bigl|x^2 - 3^2\bigr| < 7|x-3|$. □

Step 2. Now we define $\delta(\epsilon) = \min\bigl\{\epsilon/7, 1\bigr\}$.
Step 3. It remains to prove (4.1.1). To this end, assume $|x-3| < \min\bigl\{\epsilon/7, 1\bigr\}$. Then $|x-3| < 1$. Therefore, by Lemma we have $\bigl|x^2 - 3^2\bigr| < 7|x-3|$. Since by the assumption $|x-3| < \epsilon/7$, we have $7|x-3| < \epsilon$. Now the inequalities

$$\bigl|x^2 - 3^2\bigr| < 7|x-3| \quad \text{and} \quad 7|x-3| < \epsilon$$

imply that $\bigl|x^2 - 3^2\bigr| < \epsilon$. This proves (4.1.1) and completes the proof that the function $f(x) = x^2$ is continuous at $x_0 = 3$.

EXERCISE 4.2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5x - 8$. Prove that f is continuous at $x_0 = -3$.

EXERCISE 4.2.3. Prove that the reciprocal function $x \mapsto \dfrac{1}{x}$, $x \neq 0$, is continuous at $x_0 = 1/2$.

EXERCISE 4.2.4. Let

$$f(x) = x \left\lfloor \frac{1}{x} \right\rfloor \quad \text{for} \quad x \neq 0 \quad \text{and} \quad f(0) = 1.$$

Prove that the function f is continuous at $x_0 = 0$.

EXERCISE 4.2.5. State carefully what it means for a function f *not* to be continuous at a point x_0 in its domain. (Express this as a formal mathematical statement.)

EXERCISE 4.2.6. Consider the function f defined in Exercise 4.2.4. Find a point x_0 at which the function f is not continuous. Provide a formal proof. Provide a detailed sketch of the graph of f near the point x_0.

EXERCISE 4.2.7. Show that the function of Exercise 4.2.2 is continuous on $\mathbb{R}$.

EXERCISE 4.2.8. Prove that the function $q(x) = 3x^2 + 5$ is continuous at $x = 2$.

EXERCISE 4.2.9. Prove that $q(x) = 3x^2 + 5$ is continuous on $\mathbb{R}$.

4.3. Familiar continuous functions

EXERCISE 4.3.1. Let $m, k \in \mathbb{R}$ and $m \neq 0$. Prove that the linear function $\ell(x) = m\,x + k$ is continuous on $\mathbb{R}$.

EXERCISE 4.3.2. Let $a, b, c \in \mathbb{R}$ and $a \neq 0$. Prove that the quadratic function $q(x) = a\,x^2 + b\,x + c$ is continuous on $\mathbb{R}$.

EXERCISE 4.3.3. Let $n \in \mathbb{N}$ and let $x, x_0 \in \mathbb{R}$ be such that $x_0 - 1 \leq x \leq x_0 + 1$. Prove the following inequality

$$\left|x^n - x_0^n\right| \leq n\big(|x_0| + 1\big)^{n-1}\left|x - x_0\right|.$$

HINT: First notice that the assumption $x_0 - 1 \leq x \leq x_0 + 1$ implies that $|x| < |x_0| + 1$. Then use the Mathematical Induction and the identity

$$\left|x^{n+1} - x_0^{n+1}\right| = \left|x^{n+1} - x\, x_0^n + x\, x_0^n - x_0^{n+1}\right|.$$

EXERCISE 4.3.4. Let $n \in \mathbb{N}$. Prove that the power function $x \mapsto x^n$, $x \in \mathbb{R}$, is continuous on $\mathbb{R}$.

EXERCISE 4.3.5. Let $n \in \mathbb{N}$ and let $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$. Prove that the n-th order polynomial

$$p(x) = a_0 + a_1\, x + \cdots + a_{n-1}\, x^{n-1} + a_n\, x^n$$

is a continuous function on $\mathbb{R}$.

EXERCISE 4.3.6. Prove that the reciprocal function $x \mapsto \dfrac{1}{x}$, $x \neq 0$, is continuous on its domain.

EXERCISE 4.3.7. Prove that the square root function $x \mapsto \sqrt{x}$, $x \geq 0$, is continuous on its domain.

EXERCISE 4.3.8. Let $n \in \mathbb{N}$ and let x and a be positive real numbers. Prove that

$$\left| \sqrt[n]{x} - \sqrt[n]{a} \right| \leq \frac{\sqrt[n]{a}}{a}\, \left|x - a\right|.$$

HINT: Notice that the given inequality is equivalent to

$$b^{n-1}\, \left|y - b\right| \leq \left|y^n - b^n\right|, \quad y, b > 0.$$

This inequality can be proved using Exercise 2.7.7 (with $a = 1$ and $x = y/b$).

EXERCISE 4.3.9. Let $n \in \mathbb{N}$. Prove that the n-th root function $x \mapsto \sqrt[n]{x}$, $x \geq 0$, is continuous on its domain.

4.4. Various properties of continuous functions

EXERCISE 4.4.1. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$ and let y be a real number such that $f(x_0) < y$. Then there exists $\alpha > 0$ such that

$$x \in I \cap (x_0 - \alpha, x_0 + \alpha) \quad \Rightarrow \quad f(x) < y.$$

Illustrate with a diagram.

EXERCISE 4.4.2. Let $f : I \to \mathbb{R}$ be a continuous function on I. Let S be a non-empty bounded above subset of I such that $u = \sup S$ belongs to I. Let $y \in \mathbb{R}$. Prove: If $f(x) \leq y$ for each $x \in S$, then $f(u) \leq y$.

The following exercise establishes a connection between continuous functions and convergent sequences.

EXERCISE 4.4.3. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$. Let $\{t_n\}$ be a sequence in I that converges to $x_0 \in I$. Then $f\,(t_n) \to f(x_0)$ as $n \to \infty$.

EXERCISE 4.4.4. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$. Let $\{t_n\}$ be a sequence in I that converges to $x_0 \in I$. Assume that there is a real number y such that $f(t_n) \leq y$ for all $n \in \mathbb{N}$. Then $f(x_0) \leq y$.

EXERCISE 4.4.5. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$. Let $\{x_n\}$ be a sequence in I that converges to $x_0 \in I$. Assume that there is a real number y such that $f(t_n) \geq y$ for all $n \in \mathbb{N}$. Then $f(x_0) \geq y$.

4.5. Algebra of continuous functions

All exercises in this section have the same structure. With the exception of Exercise 4.5.3, there are three functions in each exercise: f, g and h. The function h is always related in a simple (green) way to the functions f and g. Based on the given (green) information about f and g you are asked to prove a claim (red) about the function h.

EXERCISE 4.5.1. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = f(x) + g(x), \qquad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If f and g are continuous on I, then h is continuous on I.

EXERCISE 4.5.2. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = f(x)g(x), \qquad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If f and g are continuous on I, then h is continuous on I.

EXERCISE 4.5.3. Let $g : I \to \mathbb{R}$ be a given functions such that $g(x) \neq 0$ for all $x \in I$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = \frac{1}{g(x)}, \qquad x \in I.$$

(a) If g is continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If g is continuous on I, then h is continuous on I.

EXERCISE 4.5.4. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Assume that $g(x) \neq 0$ for all $x \in I$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = \frac{f(x)}{g(x)}, \qquad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If f and g are continuous on I, then h is continuous on I.

EXERCISE 4.5.5. Let I and J be non-empty subsets of $\mathbb{R}$. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be given functions. Assume that the range of f is contained in J. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = g(f(x)), \qquad x \in I.$$

(a) If f is continuous at $x_0 \in I$ and g is continuous at $f(x_0) \in J$, then h is continuous at x_0.
(b) If f is continuous on I and g is continuous on J, then h is continuous on I.

4.6. Continuous functions on a closed bounded interval $[a, b]$

In this section we assume that $a, b \in \mathbb{R}$ and $a < b$.

EXERCISE 4.6.1. Let $\alpha, \beta, \gamma \in \mathbb{R}$. If $\alpha\beta \leq 0$, then $\alpha\gamma \leq 0$ or $\beta\gamma \leq 0$.

EXERCISE 4.6.2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a)f(b) \leq 0$, then there exists $z \in [a, b]$ such that $f(z) = 0$.

HINT 1: Use Cantor's intersection theorem. Define a sequence of closed intervals $[a_n, b_n]$, $n \in \mathbb{N}$, such that

$$[a_n, b_n] \subseteq [a, b], \qquad [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n], \qquad b_n - a_n = (b - a)/2^{n-1},$$

and, most importantly, $f\bigl(a_n\bigr)f\bigl(b_n\bigr) \leq 0$ for all $n \in \mathbb{N}$.

HINT 2: Assume that $f(a) < 0$ and $f(b) > 0$ and consider the set

$$W = \bigl\{w \in [a, b) : f(x) < 0 \ \forall x \in [a, w]\bigr\}.$$

EXERCISE 4.6.3. Let $f : D \to \mathbb{R}$ be a function defined on a nonempty set D. If $D = A \cup B$, then one of the following two statements hold:

(a) For each $x \in D$ there exists $y \in A$ such that $f(x) \leq f(y)$.
(b) For each $x \in D$ there exists $y \in B$ such that $f(x) \leq f(y)$.

EXERCISE 4.6.4. Let $f : [a, b] \to \mathbb{R}$ be a function defined on $[a, b]$. Then for each $\eta > 0$ there exists $c, d \in [a, b]$ such that $0 < d - c < \eta$ and for each $x \in [a, b]$ there exists $y \in [c, d]$ such that $f(x) \leq f(y)$.

HINT: Use a part of the hint for Exercise 4.6.2 and Exercise 4.6.3.

EXERCISE 4.6.5. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $w \in [a, b]$ such that $f(x) \leq f(w)$ for all $x \in [a, b]$.

HINT 1: Use Cantor's intersection theorem, a part of the hint for Exercise 4.6.2 and Exercise 4.6.3.

HINT 2: Consider the set

$$W = \Bigl\{w \in [a, b) : \exists\, z \in (w, b] \ \text{ such that } \ f(x) < f(z) \ \ \forall\, x \in [a, w] \Bigr\}.$$

Here $[a, a]$ denotes the set $\{a\}$. Prove that the set W has the following property: If $[a, v) \subset W$, with $a < v$, and if there exists $t \in [a, b]$ such that $f(t) > f(v)$, then $v \in W$.

EXERCISE 4.6.6. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $v \in [a, b]$ such that $f(v) \leq f(x)$ for all $x \in [a, b]$.

HINT: Use Exercise 4.6.5.

EXERCISE 4.6.7. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the range of f is a closed bounded interval.

HINT: Use Exercises 4.6.5, 4.6.6, and 4.6.2.

EXERCISE 4.6.8. Consider the function $f(x) = x^5 - x$, $x \in \mathbb{R}$.

(a) Prove that 1 is in the range of f.
(b) Prove that the range of f equals $\mathbb{R}$.

DEFINITION 4.6.9. A function f is *increasing* on an interval I if $x, y \in I$ with $x < y$ imply $f(x) < f(y)$. A function f is *decreasing* if $x, y \in I$ with $x < y$ imply $f(x) > f(y)$. A function which is increasing or decreasing is said to be *strictly monotonic*.

EXERCISE 4.6.10. If f is continuous and increasing on $[a, b]$ or continuous and decreasing on $[a, b]$, then for each y between $f(a)$ and $f(b)$ there is exactly one $x \in [a, b]$ such that $f(x) = y$.

EXERCISE 4.6.11. Let $f(x) = x^3 + x$, $x \in \mathbb{R}$. Prove that f has an inverse. That is, prove that for each $y \in \mathbb{R}$ there exists unique $x \in \mathbb{R}$ such that $f(x) = y$.