Axioms for the Set $\mathbb{R}$ of Real Numbers

Axiom 1 (AE: Addition exists)**.** If $a, b \in \mathbb{R}$, then the sum of a and b, denoted by $a+b$, is a uniquely defined number in $\mathbb{R}$.

Axiom 2 (AA: Addition is associative)**.** For all $a, b, c \in \mathbb{R}$ we have $a + (b + c) = (a + b) + c$.

Axiom 3 (AC: Addition is commutative)**.** For all $a, b \in \mathbb{R}$ we have $a + b = b + a$.

Axiom 4 (AZ: 0 is a neutral element for addition)**.** There is an element 0 in $\mathbb{R}$ such that $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$.

Axiom 5 (AO: Opposites exist)**.** If $a \in \mathbb{R}$, then the equation $a + x = 0$ has a solution $-a \in \mathbb{R}$. The number $-a$ is called the *opposite* of a.

Axiom 6 (ME: Multiplication exists)**.** If $a, b \in \mathbb{R}$, then the product of a and b, denoted by ab, is a uniquely defined number in $\mathbb{R}$.

Axiom 7 (MA: Multiplication is associative)**.** For all $a, b, c \in \mathbb{R}$ we have $a(bc) = (ab)c$.

Axiom 8 (MC: Multiplication is commutative)**.** For all $a, b \in \mathbb{R}$ we have $ab = ba$.

Axiom 9 (MO: 1 is a neutral element for multiplication)**.** There is an element $1 \neq 0$ in $\mathbb{R}$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$.

Axiom 10 (MR: Reciprocals exist)**.** If $a \in \mathbb{R}$ is such that $a \neq 0$, then the equation $a \cdot x = 1$ has a solution $a^{-1} = \dfrac{1}{a}$ in $\mathbb{R}$. The number $a^{-1} = \dfrac{1}{a}$ is called the *reciprocal* of a.

Axiom 11 (DL: Distributive law, the connection between addition and multiplication)**.** For all $a, b, c \in \mathbb{R}$ we have $a(b + c) = ab + ac$.

Axiom 12 (OE: Order exists)**.** Given any $a, b \in \mathbb{R}$, exactly one of these statements is true: $a < b$, $a = b$, or $b < a$.

Axiom 13 (OT: Order is transitive)**.** Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

Axiom 14 (OA: Order and addition)**.** Given any $a, b, c \in \mathbb{R}$, if $a < b$ then $a + c < b + c$.

Axiom 15 (OM: Order and multiplication)**.** Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $0 < c$, then $ac < bc$.

Axiom 16 (CA: Completeness Axiom)**.** If A and B are nonempty subsets of $\mathbb{R}$ such that for every $a \in A$ and for every $b \in B$ we have $a \leq b$, then there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and all $b \in B$.

All statements about real numbers that are studied in beginning mathematical analysis courses can be deduced from these sixteen axioms.

The formulation of the Completeness Axiom given as Axiom 16 above is not standard. This version I found in the book *Mathematical analysis* by Vladimir Zorich, published by Springer in 2004. The standard formulation of the Completeness Axiom is in the notes. We will prove that the standard formulation is equivalent to Zorich's Completeness Axiom.