2.9.5) Prove that the function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$A(s,t) = \frac{(s+t-2)(s+t-1)}{2} + s \qquad s,t \in \mathbb{N}$$

is a bijection. That is, prove that there exists a function $B : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that $B \circ A = id_{\mathbb{N}\times\mathbb{N}}$ and $A \circ B = id_{\mathbb{N}}$.

Proof. ① Let $A(s,t) = \frac{(s+t-2)(s+t-1)}{2} + s, \quad s,t \in \mathbb{N}$.

② By 2.7.5, $A(s,t) = T(s+t-2) + s, \quad s,t \in \mathbb{N}$.

③ Let $B(n) = \left(n - \frac{(R_n - 1)R_n}{2}, \frac{R_n(R_n+1)}{2} - n + 1 \right), \quad n \in \mathbb{N}$.

④ By 2.7.5, $B(n) = (n - T(R_n - 1), T(R_n) - n + 1), \quad n \in \mathbb{N}$.

⑤ Let $s,t \in \mathbb{N}$.

⑥ By ①, ③,
$B(A(s,t)) = (A(s,t) - T(R(A(s,t)) - 1), T(R(A(s,t))) - A(s,t) + 1)$.

⑦ By 2.9.3,
$T(s+t-2) + 1 \leq T(s+t-2) + s \leq T(s+t-2) + s + t + 1$.

⑧ By 2.9.3, $R(T(s+t-2)+s) = s+t-1$.

⑨ By ⑥, ⑧,
$B(A(s,t)) = (A(s,t) - T(s+t-1-1), T(s+t-1) - A(s,t) + 1)$

⑩ $= (A(s,t) - T(s+t-2), T(s+t-1) - A(s,t) + 1)$

⑪ $= (T(s+t-2) + s - T(s+t-2), T(s+t-1) - (T(s+t-2) + s) + 1)$

⑫ $= \left(s, \frac{(s+t-1)(s+t)}{2} - \frac{(s+t-2)(s+t-1)}{2} - \frac{2s+2}{2} \right)$

⑬ $= \left(s, \frac{(s+t-1)(s+t-(s+t-2)) - 2s + 2}{2} \right)$

⑭ $= \left(s, \frac{(s+t-1)(2) - 2s + 2}{2} \right)$

⑮ $= (s, s+t-1-s+1)$

⑯ $= (s,t) = id_{s,t}$.

(17) By (1), (3),

$$A(B(n)) = T\left(n - T(R_n - 1) + T(R_n) - n + 1 - 2\right) + n - T(R_n - 1).$$

(18) $$= T\left(T(R_n) - T(R_n - 1) - 1\right) + n - T(R_n - 1).$$

(19) $$= T\left(\frac{(R_n)(R_n+1)}{2} - \frac{(R_n-1)(R_n)}{2} - 1\right) + n - \frac{(R_n-1)(R_n)}{2}$$

(20) $$= \left(\frac{(R_n)(R_n+1)}{2} - \frac{(R_n-1)(R_n)}{2} - 1\right)\left(\frac{(R_n)(R_n+1)}{2} - \frac{(R_n-1)(R_n)}{2}\right) + n - \frac{(R_n-1)(R_n)}{2}$$

(21) $$= \left(\frac{R_n^2 + R_n - R_n^2 + R_n - 2}{2}\right)\left(\frac{R_n^2 + R_n - R_n^2 + R_n}{2}\right) + n - \frac{(R_n-1)(R_n)}{2}$$

(22) $$= \left(\frac{2R_n - 2}{2}\right)\left(\frac{2R_n}{2}\right) + n - \frac{(R_n-1)(R_n)}{2}$$

(23) $$= \frac{(R_n-1)(R_n)}{2} + n - \frac{(R_n-1)(R_n)}{2}$$

(24) $$= n = id_n.$$

(25) By (16), (24), 1.6.3, $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection.

$\boxed{/\!/\!/}$.