ON A ZERO OF A CONTINUOUS FUNCTION

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In this note a and b are real numbers and $a < b$.

Definition 1. A function $f : [a, b] \to \mathbb{R}$ is *continuous at a point* $x_0 \in [a, b]$ if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0) > 0$ such that

$$x \in \big(x_0 - \delta, x_0 + \delta\big) \cap [a, b] \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon.$$

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. If $f(a) < 0$ and $f(b) > 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.*

Proof. Assume $f(a) < 0$ and $f(b) > 0$.

Step 1. Set

$$W = \big\{ x \in [a, b] \,:\, f(x) < 0 \big\}.$$

Clearly $a \in W$, $b \notin W$ and $W \subseteq [a, b)$. Therefore, $c = \sup W$ exists by the Completeness Axiom. Since $a \in W$ and b is an upper bound for W we have $c \in [a, b]$.

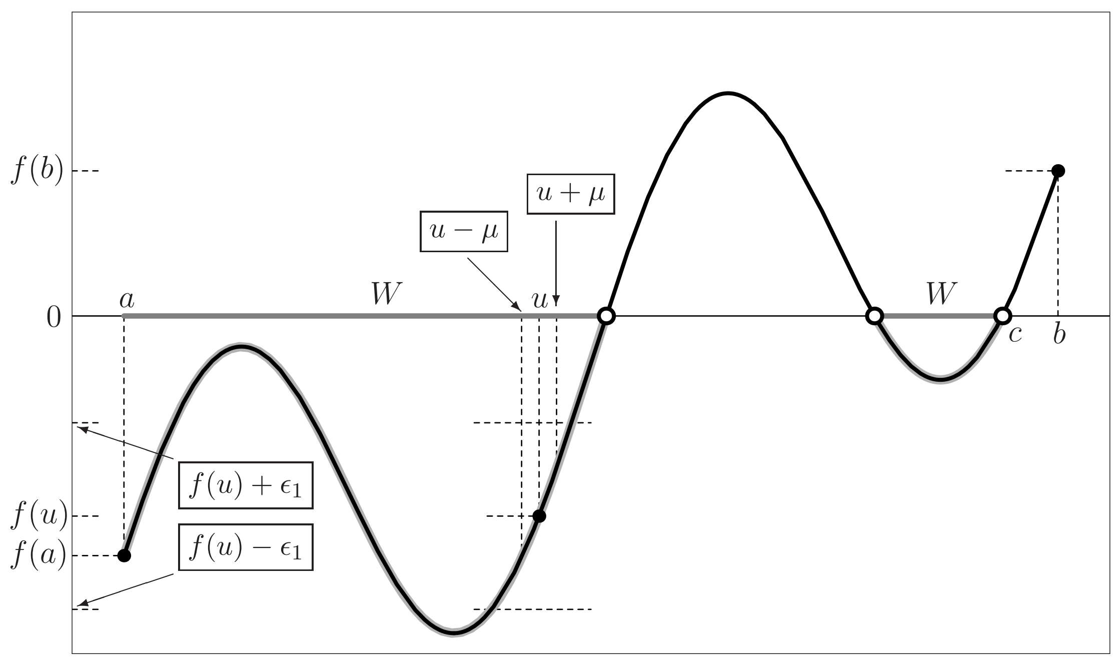

FIGURE 1. An illustration of Step 2 of the proof

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Step 2. Here we show that W does not have a maximum. Let $u \in W$ be arbitrary. Then $u < b$ and $f(u) < 0$. Set $\epsilon_1 = -f(u)/2 > 0$. Since $\epsilon_1 > 0$ and f is continuous at u there exists $\delta_1 = \delta(\epsilon_1, u) > 0$ such that

$$x \in [a,b] \cap \big(u - \delta_1, u + \delta_1\big) \quad \Rightarrow \quad f(u) - \epsilon_1 < f(x) < f(u) + \epsilon_1. \tag{1}$$

Set

$$\mu = \tfrac{1}{2} \min\big\{\delta_1, b - u\big\}.$$

Then $\mu > 0$, $u + \mu < b$ and $u < u + \mu < u + \delta_1$. It follows from (1) that

$$f(u + \mu) < f(u) + \epsilon_1 = f(u) + \left(-\frac{f(u)}{2}\right) = \frac{f(u)}{2} < 0.$$

Thus $u + \mu \in W$. Since $u + \mu > u$, we proved that u is not a maximum of W.

Step 3. As W does not have a maximum, $c \notin W$. Since $c \in [a,b]$ and $c \notin W$ we conclude that $f(c) \geq 0$.

Step 4. Here we show that $f(c) \leq 0$. Let $\epsilon > 0$ be arbitrary. Since f is continuous at c, there exists $\delta = \delta(\epsilon, c) > 0$ such that

$$x \in [a,b] \cap \big(c - \delta, c + \delta\big) \quad \Rightarrow \quad f(c) - \epsilon < f(x) < f(c) + \epsilon. \tag{2}$$

Since $c = \sup W$ and $\delta > 0$ there exists $w \in W$ such that

$$c - \delta < w < c.$$

Now (2) and $f(w) < 0$ yield $f(c) - \epsilon < f(w) < 0$. Since $\epsilon > 0$ was arbitrary, we proved that $f(c) < \epsilon$ for all $\epsilon > 0$. Consequently $f(c) \leq 0$.

Step 5. In Step 3 we proved $f(c) \geq 0$. In Step 4 we proved $f(c) \leq 0$. Thus $f(c) = 0$. This completes the proof. $\square$