CHAPTER 3

Continuous functions

In this chapter $I$ will always denote a non-empty subset of $\mathbb{R}$. This includes more general sets, but the most common examples of $I$ are intervals.

3.1. The $\epsilon$-$\delta$ definition of a continuous function

Definition 3.1.1. A function $f : I \to \mathbb{R}$ is continuous at a point $x_0 \in I$ if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0) > 0$ such that

$$x \in (x_0 - \delta, x_0 + \delta) \cap I \implies |f(x) - f(x_0)| < \epsilon.$$  

The function $f$ is continuous on $I$ if it is continuous at each point of $I$.

Note that the implication in (3.1.1) can be restated as

$$x \in I \text{ and } |x - x_0| < \delta(\epsilon, x_0) \implies |f(x) - f(x_0)| < \epsilon.$$  

Next we restate Definition 3.1.1 using the terminology introduced in Section 2.14.

For a function $f : I \to \mathbb{R}$ and a subset $A \subseteq I$ we will use the notation $f(A)$ to denote the set $\{y \in \mathbb{R} : \exists x \in A \text{ s.t. } f(x) = y\} = \{f(x) : x \in A\}$.

A function $f : I \to \mathbb{R}$ is continuous at a point $x_0 \in I$ if for each neighborhood $V$ of $f(x_0)$ there exists a neighborhood $U$ of $x_0$ such that

$$f(I \cap U) \subseteq V.$$  

3.2. Finding $\delta(\epsilon)$ for a given function at a given point

In this and the next section we will prove that some familiar functions are continuous. This should be a review of what was done in Math 226.

A general strategy for proving that a given function $f$ is continuous at a given point $x_0$ is as follows:

Step 1. Simplify the expression $|f(x) - f(x_0)|$ and try to establish a simple connection with the expression $|x - x_0|$. The simplest connection is to discover positive constants $\delta_0$ and $K$ such that

$$x \in I \text{ and } x_0 - \delta_0 < x < x_0 + \delta_0 \implies |f(x) - f(x_0)| \leq K|x - x_0|.$$  

Constants $\delta_0$ and $K$ might depend on $x_0$. Formulate your discovery as a lemma.

Step 2. Let $\epsilon > 0$ be given. Use the result in Step 1 to define your $\delta(\epsilon, x_0)$. For example, if (3.2.1) holds, then $\delta(\epsilon, x_0) = \min\{\epsilon/K, \delta_0\}$.

Step 3. Use the definition of $\delta(\epsilon, x_0)$ from Step 2 and the lemma from Step 1 to prove the implication (3.1.1).
Example 3.2.1. We will show that the function $f(x) = x^2$ is continuous at $x_0 = 3$. Here $I = \mathbb{R}$ and we do not need to worry about the domain of $f$.

Step 1. First simplify

$(3.2.2) \quad |f(x) - f(x_0)| = |x^2 - 3^2| = |(x + 3)(x - 3)| = |x + 3||x - 3|.$

Now we notice that if $2 < x < 4$ we have $|x + 3| = x + 3 \leq 7$. Thus $(3.2.1)$ holds with $\delta_0 = 1$ and $K = 7$. We formulate this result as a lemma.

Lemma. Let $f(x) = x^2$ and $x_0 = 3$. Then

$(3.2.3) \quad |x - 3| < 1 \quad \Rightarrow \quad |x^2 - 3^2| < 7|x - 3|.$

Proof. Let $|x - 3| < 1$. Then $2 < x < 4$. Therefore $x + 3 > 0$ and $|x + 3| = x + 3 < 7$. By $(3.2.2)$ we now have $|x^2 - 3^2| < 7|x - 3|$. \hfill \Box

Step 2. Now we define $\delta(x) = \min\{\epsilon/7, 1\}$.

Step 3. It remains to prove $(3.1.1)$. To this end, assume $|x - 3| < \min\{\epsilon/7, 1\}$. Then $|x - 3| < 1$. Therefore, by Lemma we have $|x^2 - 3^2| < 7|x - 3|$. Since by the assumption $|x - 3| < \epsilon/7$, we have $7|x - 3| < 7\epsilon/7\epsilon$. Now the inequalities $|x^2 - 3^2| < 7|x - 3|$ and $7|x - 3| < \epsilon$ imply that $|x^2 - 3^2| < \epsilon$. This proves $(3.1.1)$ and completes the proof that the function $f(x) = x^2$ is continuous at $x_0 = 3$.

Exercise 3.2.2. Prove that the reciprocal function $x \mapsto \frac{1}{x}$, $x \neq 0$, is continuous at $x_0 = 1/2$.

Exercise 3.2.3. State carefully what it means for a function $f$ not to be continuous at a point $x_0$ in its domain. (Express this as a formal mathematical statement.)

Exercise 3.2.4. Consider the function $f(x) = \text{sgn} x$. Find a point $x_0$ at which the function $f$ is not continuous. Provide a formal proof.

Exercise 3.2.5. Show that the function $f(x) = x^2$ is continuous on $\mathbb{R}$.

Exercise 3.2.6. Prove that $q(x) = 3x^2 + 5$ is continuous on $\mathbb{R}$.

3.3. Familiar continuous functions

Exercise 3.3.1. Let $m, k \in \mathbb{R}$ and $m \neq 0$. Prove that the linear function $\ell(x) = mx + k$ is continuous on $\mathbb{R}$.

Exercise 3.3.2. Let $a, b, c \in \mathbb{R}$ and $a \neq 0$. Prove that the quadratic function $q(x) = ax^2 + bx + c$ is continuous on $\mathbb{R}$.

Exercise 3.3.3. Let $n \in \mathbb{N}$ and let $x, x_0 \in \mathbb{R}$ be such that $x_0 - 1 \leq x \leq x_0 + 1$. Prove the following inequality

$$|x^n - x_0^n| \leq n(|x_0| + 1)^{n-1}|x - x_0|.$$  

Hint: First notice that the assumption $x_0 - 1 \leq x \leq x_0 + 1$ implies that $|x| < |x_0| + 1$. Then use the Mathematical Induction and the identity

$$|x^{n+1} - x_0^{n+1}| = |x^{n+1} - x x_0^n + x_0^n - x_0^{n+1}|.$$

Exercise 3.3.4. Let $n \in \mathbb{N}$. Prove that the power function $x \mapsto x^n$, $x \in \mathbb{R}$, is continuous on $\mathbb{R}$. 
Exercise 3.3.5. Let \( n \in \mathbb{N} \) and let \( a_0, a_1, \ldots, a_n \in \mathbb{R} \) with \( a_n \neq 0 \). Prove that the \( n \)-th order polynomial
\[
p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n
\]
is a continuous function on \( \mathbb{R} \).

Exercise 3.3.6. Prove that the reciprocal function \( x \mapsto \frac{1}{x}, x \neq 0 \), is continuous on its domain.

Exercise 3.3.7. Prove that the square root function \( x \mapsto \sqrt{x}, x \geq 0 \), is continuous on its domain.

Exercise 3.3.8. Let \( n \in \mathbb{N} \) and let \( x \) and \( a \) be positive real numbers. Prove that
\[
|\sqrt[n]{x} - \sqrt[n]{a}| \leq \frac{\sqrt[n]{a}}{a} |x - a|.
\]
HINT: Notice that the given inequality is equivalent to
\[
b^{n-1} |y - b| \leq |y^n - b^n|, \quad y, b > 0.
\]
This inequality can be proved using Exercise 2.7.7 (with \( a = 1 \) and \( x = y/b \)).

Exercise 3.3.9. Let \( n \in \mathbb{N} \). Prove that the \( n \)-th root function \( x \mapsto \sqrt[n]{x}, x \geq 0 \), is continuous on its domain.

3.4. Various properties of continuous functions

Exercise 3.4.1. Let \( f : I \to \mathbb{R} \) be continuous at \( x_0 \in I \) and let \( y \) be a real number such that \( f(x_0) < y \). Then there exists \( \alpha > 0 \) such that
\[
x \in I \cap (x_0 - \alpha, x_0 + \alpha) \quad \Rightarrow \quad f(x) < y.
\]
Illustrate with a diagram.

Exercise 3.4.2. Let \( f : I \to \mathbb{R} \) be a continuous function on \( I \). Let \( S \) be a non-empty bounded above subset of \( I \) such that \( u = \sup S \) belongs to \( I \). Let \( y \in \mathbb{R} \). Prove: If \( f(x) \leq y \) for each \( x \in S \), then \( f(u) \leq y \).

3.5. Algebra of continuous functions

All exercises in this section have the same structure. With the exception of Exercise 3.5.3 there are three functions in each exercise: \( f \), \( g \) and \( h \). The function \( h \) is always related in a simple (green) way to the functions \( f \) and \( g \). Based on the given (green) information about \( f \) and \( g \) you are asked to prove a claim (red) about the function \( h \).

Exercise 3.5.1. Let \( f : I \to \mathbb{R} \) and \( g : I \to \mathbb{R} \) be given functions with a common domain. Define the function \( h : I \to \mathbb{R} \) by
\[
h(x) = f(x) + g(x), \quad x \in I.
\]
(a) If \( f \) and \( g \) are continuous at \( x_0 \in I \), then \( h \) is continuous at \( x_0 \).
(b) If \( f \) and \( g \) are continuous on \( I \), then \( h \) is continuous on \( I \).

Exercise 3.5.2. Let \( f : I \to \mathbb{R} \) and \( g : I \to \mathbb{R} \) be given functions with a common domain. Define the function \( h : I \to \mathbb{R} \) by
\[
h(x) = f(x)g(x), \quad x \in I.
\]
(a) If $f$ and $g$ are continuous at $x_0 \in I$, then $h$ is continuous at $x_0$.
(b) If $f$ and $g$ are continuous on $I$, then $h$ is continuous on $I$.

**Exercise 3.5.3.** Let $g : I \to \mathbb{R}$ be a given functions such that $g(x) \neq 0$ for all $x \in I$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = \frac{1}{g(x)}, \quad x \in I.$$

(a) If $g$ is continuous at $x_0 \in I$, then $h$ is continuous at $x_0$.
(b) If $g$ is continuous on $I$, then $h$ is continuous on $I$.

**Exercise 3.5.4.** Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Assume that $g(x) \neq 0$ for all $x \in I$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = \frac{f(x)}{g(x)}, \quad x \in I.$$

(a) If $f$ and $g$ are continuous at $x_0 \in I$, then $h$ is continuous at $x_0$.
(b) If $f$ and $g$ are continuous on $I$, then $h$ is continuous on $I$.

**Exercise 3.5.5.** Let $I$ and $J$ be non-empty subsets of $\mathbb{R}$. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be given functions. Assume that the range of $f$ is contained in $J$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = g(f(x)), \quad x \in I.$$

(a) If $f$ is continuous at $x_0 \in I$ and $g$ is continuous at $f(x_0) \in J$, then $h$ is continuous at $x_0$.
(b) If $f$ is continuous on $I$ and $g$ is continuous on $J$, then $h$ is continuous on $I$.

### 3.6. Continuous functions on a closed bounded interval $[a, b]$

In this section we assume that $a, b \in \mathbb{R}$ and $a < b$.

**Exercise 3.6.1.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a) < 0$ and $f(b) > 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.

**HINT:** Consider the set

$$W = \{ w \in [a, b] : \forall x \in [a, w] \ f(x) < 0 \}.$$

Prove the following properties of $W$:

(i) $W$ does not have a maximum.
(ii) $W$ has a supremum. Set $w = \sup W$.
(iii) Review Exercise 3.4.2.
(iv) Connect the dots.

**Exercise 3.6.2.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a, b]$ such that $f(x) \leq f(c)$ for all $x \in [a, b]$.

**HINT:** Consider the set

$$W = \{ v \in [a, b] : \exists z \in [a, b] \text{ such that } \forall x \in [a, v] \ f(x) < f(z) \}.$$

Here $[a, a]$ denotes the set $\{ a \}$. Prove the following properties of the set $W$:

(i) If $a < u$ and $[a, u] \subseteq W$ and there exists $t \in [a, b]$ such that $f(t) > f(u)$, then $u \in W$.
(ii) $W$ does not have a maximum.
(iii) $W$ has a supremum. Set $w = \sup W$ and prove $[a, w] \subseteq W$. 
(iv) The items \( \text{(iii)} \) and \( \text{(iv)} \) yield information about \( w \).

**Exercise 3.6.3.** Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Then there exists \( d \in [a, b] \) such that \( f(d) \leq f(x) \) for all \( x \in [a, b] \).

**Hint:** Use Exercise 3.6.2

**Exercise 3.6.4.** Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Then the range of \( f \) is a closed bounded interval.

**Hint:** Use Exercises 3.6.2, 3.6.3, and 3.6.1

**Exercise 3.6.5.** Consider the function \( f(x) = x^2, x \in \mathbb{R} \).

(a) Prove that 2 is in the range of \( f \).

(b) Prove that the range of \( f \) equals \([0, +\infty)\).

**Definition 3.6.6.** A function \( f \) is **increasing** on an interval \( I \) if \( x, y \in I \) and \( x < y \) imply \( f(x) < f(y) \). A function \( f \) is **decreasing** if \( x, y \in I \) and \( x < y \) imply \( f(x) > f(y) \). A function which is increasing or decreasing is said to be **strictly monotonic**.

**Exercise 3.6.7.** If \( f \) is continuous and increasing on \([a, b]\) or continuous and decreasing on \([a, b]\), then for each \( y \) between \( f(a) \) and \( f(b) \) there is exactly one \( x \in [a, b] \) such that \( f(x) = y \).

**Exercise 3.6.8.** Let \( f(x) = x^3 + x, x \in \mathbb{R} \). Prove that \( f \) has an inverse. That is, prove that for each \( y \in \mathbb{R} \) there exists unique \( x \in \mathbb{R} \) such that \( f(x) = y \).