Topology of $\mathbb{R}$

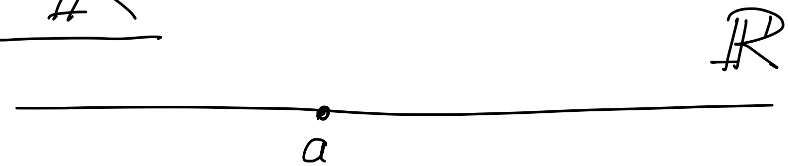
at the end, I proved that

E closed $\Rightarrow E^c$ open

by proving the contrapositive

Topology of $\mathbb{R}$

$a \in \mathbb{R}$



a neighborhood of a is $(a-\varepsilon, a+\varepsilon) = \{x \in \mathbb{R} : |x-a| < \varepsilon\}$

$\mathbb{R}^n$

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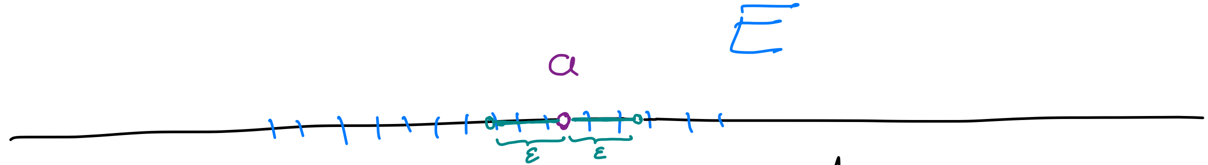
$$N(a, \varepsilon)$$

Punctured neighborhood:

$$N(a, \varepsilon) \setminus \{a\}$$

$E \subseteq \mathbb{R}$ $\quad a \in \mathbb{R}$ is an <u>accumulation</u> point of E

$$\forall \varepsilon > 0 \quad E \cap \big(\underbrace{N(a,\varepsilon) \setminus \{a\}}\big) \neq \emptyset$$



When we are proving that a is an acc. point of a given E, then it is just algebra to prove

$$\forall \varepsilon > 0 \ \exists x \in E \text{ s.t. }$$

$$x \neq a \text{ and } x \in N(a, \varepsilon)$$

punctured

The most important example is

A bdd above

$(\sup A) \notin A$

then $\sup A$ is an acc. pnt. of A !

To some extend, the acc. points

of E "belong" to E

E attracts its $\underbrace{\text{acc. pnts.}}_{E'}$ of E

closure of E $\quad \bar{E} = E \cup E'$

$$Cl(E) = \overline{E} \overset{\text{def.}}{=} E \cup E'$$

E is said to **closed** if $E' \subseteq E$.

$$E = \overline{E}$$

$E = (0,1)$

$0 \notin E$

$1 \notin E$

$\sup E = 1$

$\inf E = 0$

$0, 1 \in E'$

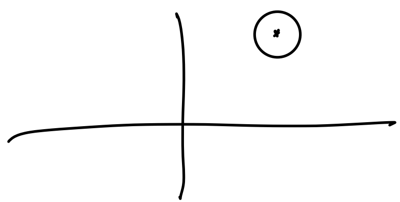
$\overline{E} = [0,1]$

Dual concepts: $E \subseteq \mathbb{R}$

$a \in \mathbb{R}$ is an <u>interior point</u> of E

$\exists\, \varepsilon > 0$ s.t. $N(a, \varepsilon) \subseteq E$

⊛ The set of all interior points of E is denoted by E°

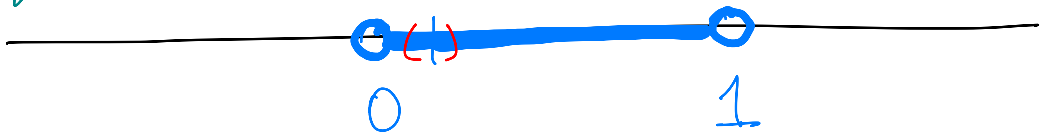


OPEN SET in $\mathbb{R}$

E is $\qquad E^\circ = E$

Example $\boxed{(0,1)}$ is an open set

$\forall a \in (0,1)$ a $0 < a < 1$



$\varepsilon > 0$ such that

$N(a,\varepsilon) = (a-\varepsilon, a+\varepsilon) \subseteq (0,1)$

ALGEBRA Prove it!

$$\boxed{\varepsilon = \frac{1}{2}\min\{1-a, a\}}$$

$E \neq \emptyset$ is an open set

$\forall a \in E \;\; \exists \varepsilon > 0$ such that

$N(a, \varepsilon) \subseteq E$

Definition of an open set.

$E \subseteq \mathbb{R}^2$

E is closed if and only if the complement of E is open.

$E^c = \mathbb{R} \setminus E$

~~Assume~~ $\boxed{E \text{ is closed.}} \Rightarrow \boxed{E^c \text{ is open}}$

Try contrapositive:

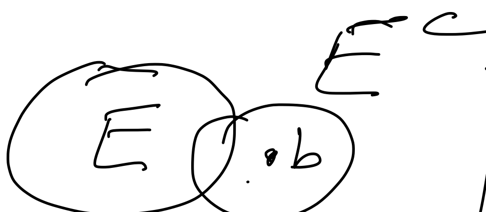
$\boxed{E^c \text{ is NOT OPEN}} \Rightarrow \boxed{E \text{ is not closed.}}$

$\exists\, b \in E^c$ such that $\forall\, \varepsilon > 0$

$$\boxed{N(b,\varepsilon) \subseteq E^c \text{ NOT TRUE}}$$

$$\Updownarrow$$

$$\boxed{N(b,\varepsilon) \cap E \neq \emptyset}$$



Recall $b \in E^c$, so $b \notin E$. Therefore

$(N(b,\varepsilon)\setminus\{b\}) \cap E \neq \emptyset.$

Thus, we proved that

b is an ACC Point of E

and, by assmption $b \in E^c$,

that is $b \notin E$. This

proves that E is NOT

closed.

Now prove

$\boxed{E^c \text{ open}} \Rightarrow \boxed{E \text{ closed}}$

Let us try the contrapositive again.

$\boxed{E \text{ not closed}} \Rightarrow \boxed{E^c \text{ not open}}$

$\Downarrow$

$E' \not\subseteq E$ means that there exists an accumulation point a of E such that $a \notin E$

$\exists\, a \notin E$ such that $\forall \varepsilon > 0$

$$(N(a,\varepsilon)\setminus\{a\}) \cap E \neq \emptyset$$

Since $a \notin E$, this means

$$\boxed{N(a,\varepsilon) \cap E \neq \emptyset.} \Longleftrightarrow \boxed{N(a,\varepsilon) \not\subseteq E^c}$$

Thus $\exists\, a \in E^c$ such that $\forall \varepsilon > 0$ $N(a,\varepsilon) \not\subseteq E^c$.

that is $a \in E^c$ is NOT an interior point of E^c, that is E^c is NOT OPEN.