The table below was constructed to help us understand the proof of **Proposition 3.2.22** in the notes:

Proposition. *Let S be a nonempty set. The function $\Phi : \mathcal{P}(S) \to \{0,1\}^S$ defined by*

$$\forall A \in \mathcal{P}(S) \qquad \Phi(A) = \chi_A$$

is a bijection.

In the table below, the set S consists of four symbols:

$$S = \Big\{\triangle,\ \square,\ \bigcirc, \bigstar\Big\}.$$

The top part of the table consists of sixteen indicator functions on the power set of of S, and the bottom part of the table consists of the sixteens subsets of S, presented vertically for better space management.

To strengthen your understanding of the proof of **Proposition 3.2.22**, recognize the function $\Phi : \mathcal{P}(S) \to \{0,1\}^S$ and its inverse $\Psi : \{0,1\}^S \to \mathcal{P}(S)$ in the table below.

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
$\triangle$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
$\square$	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
$\bigcirc$	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
$\bigstar$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
subset	$\emptyset$	$\{\bigstar\}$	$\{\bigcirc\}$	$\{\bigcirc, \bigstar\}$	$\{\square\}$	$\{\square, \bigstar\}$	$\{\square, \bigcirc\}$	$\{\square, \bigcirc, \bigstar\}$	$\{\triangle\}$	$\{\triangle, \bigstar\}$	$\{\triangle, \bigcirc\}$	$\{\triangle, \bigcirc, \bigstar\}$	$\{\triangle, \square\}$	$\{\triangle, \square, \bigstar\}$	$\{\triangle, \square, \bigcirc\}$	$\{\triangle, \square, \bigcirc, \bigstar\}$

Next, we want to use this specific example to illustrate the proof of Cantor's Theorem, **Theorem 3.2.25** in the notes:

Theorem (Cantor's Theorem)**.** *Let S be a nonempty set. Then there is no surjection with domain S and codomain $\mathcal{P}(S)$.*

We have to prove

$$\forall \Theta : S \to \mathcal{P}(S) \quad \exists A \in \mathcal{P}(S) \ \text{ such that } \ \forall x \in S \ \text{ we have } \ A \neq \Theta(x).$$

In the table below, I give random values to the function $\Theta : S \to \mathcal{P}(S)$ with S as in the example above:

$$\Theta(\triangle) = \Big\{\square,\ \bigcirc, \bigstar\Big\}, \quad \Theta(\square) = \Big\{\square,\ \bigcirc\Big\}, \quad \Theta(\bigcirc) = \Big\{\triangle,\ \square,\ \bigcirc, \bigstar\Big\}, \quad \Theta(\bigstar) = \emptyset.$$

Then, in the next four columns, the values of Θ, which are subsets of S, are encoded by their indicator functions. Then, in the column "to find A" I present the algorithm that we give us the indicator function for A in the orange boxes. Finally, I fold the indicator function for A in one column and write the elements of A. This algorithm works for every nonempty set S and every $\Theta : S \to \mathcal{P}(S)$.

		indicator function of $\Theta()$					indicator function of A					
	the value of $\Theta()$	$\triangle$	$\square$	$\bigcirc$	$\bigstar$	to find A	$\triangle$	$\square$	$\bigcirc$	$\bigstar$	fold	A
$\triangle$	$\{\square, \bigcirc, \bigstar\}$	0	1	1	1	$1-$	1				1	$\triangle$
$\square$	$\{\square, \bigcirc\}$	0	1	0	1	$1-$		0			0	
$\bigcirc$	$\{\triangle, \square, \bigcirc, \bigstar\}$	1	1	1	1	$1-$			0		0	
$\bigstar$	$\emptyset$	0	0	0	0	$1-$				1	1	$\bigstar$

$$\text{Thus, } A = \Big\{\triangle,\ \bigstar\Big\}.$$

Verify:

$$\Big\{\triangle,\ \bigstar\Big\} \neq \Big\{\square,\ \bigcirc, \bigstar\Big\}$$

$$\Big\{\triangle,\ \bigstar\Big\} \neq \Big\{\square,\ \bigcirc\Big\}$$

$$\Big\{\triangle,\ \bigstar\Big\} \neq \Big\{\triangle,\ \square,\ \bigcirc, \bigstar\Big\}$$

$$\Big\{\triangle,\ \bigstar\Big\} \neq \emptyset$$

Explore the proof of Cantor's Theorem. Here $S = \{\triangle, \square, \bigcirc, \star\}$. Invent your own $\theta : S \to \mathcal{P}(S)$ and use the algorithm from the proof of Cantor's Theorem to construct A which is NOT in the range of Θ.

		indicator function of $\Theta(\,)$					indicator function of A					
	the value of $\Theta(\,)$	$\triangle$	$\square$	$\bigcirc$	$\star$	to find A	$\triangle$	$\square$	$\bigcirc$	$\star$	fold	A
$\triangle$	$\{\quad\}$					$1-$						
$\square$	$\{\quad\}$					$1-$						
$\bigcirc$	$\{\quad\}$					$1-$						
$\star$	$\{\quad\}$					$1-$						

Thus, $A = \{\qquad\}$. Verify $A \notin \operatorname{ran}\Theta$:

$$\{\qquad\} \neq \{\qquad\}$$

$$\{\qquad\} \neq \{\qquad\}$$

$$\{\qquad\} \neq \{\qquad\}$$

$$\{\qquad\} \neq \{\qquad\}$$