Section 4.4

EXERCISES

12. The initial value problem to be solved is

$$0.1x''(t) + 3.6x(t) = 0, \quad x(0) = 0, \quad x'(0) = -0.4.$$

The solution of this initial value problem is $x(t) = -\frac{1}{15}\sin(6t)$. From the solution we see that the amplitude is $1/15$, the natural frequency is 6. To get the phase we rewrite the solution in the form

$$A\cos(6t - \phi)$$

To get ϕ we write in the polar form the complex number $0 - \frac{1}{15}i$. Clearly $0 - \frac{1}{15}i = \frac{1}{15}e^{-i\frac{\pi}{2}}$. Therefore, the phase is $-\frac{\pi}{2}$.

---

16. The spring constant k is determined from the equation $1 \cdot 9.8 = 4.8 \cdot k$. Hence $k = 2$. The initial value problem to be solved is

$$x''(t) + 3x'(t) + 2x(t) = 0, \quad x(0) = -1, \quad x'(0) = -1.$$

The solution is $x(t) = -3e^{-t} + 2e^{-2t}$. Hence this is an overdamped case.

An additional interesting question here is to find the maximal displacements of the mass. To answer this question find the derivative of the solution and find where derivative is 0. Then use this value in the solution. The zero of the derivative is $t = \ln(4/3)$. The corresponding value of the function is $-9/8$. Hence the maximum displacement of the mass from the equilibrium is $9/8$m.

---

18. The spring constant k is determined from the equation $0.05 \cdot 9.8 = 0.2 \cdot k$. Hence $k = 49/20$. To find μ so that the system is critically damped we solve the equation $\mu^2 - 4 \cdot 0.05 \cdot \frac{49}{20} = 0$. We get $\mu = 7/10$. The initial value problem to be solved is

$$\frac{1}{20}x''(t) + \frac{7}{10}x'(t) + \frac{49}{20}x(t) = 0, \quad x(0) = \frac{15}{100}, \quad x'(0) = 0.$$

The solution is $x(t) = \dfrac{3}{20}\, e^{-7t}\big(1+7t\big)$.

---

22. The coefficient μ satisfies the equation $0.3 = 0.2\mu$. Hence $\mu = 3/2$. The initial value problem to be solved is

$$\frac{1}{10}x''(t) + \frac{3}{2}x'(t) + \frac{98}{10}x(t) = 0, \quad x(0) = \frac{1}{10}, \quad x'(0) = 0.$$

The symbolic solution of this equation is

$$x(t) = e^{-15t/2}\left(\frac{1}{10}\cos(\omega t) + \frac{3}{2\sqrt{167}}\sin(\omega t)\right) \quad \text{where} \quad \omega = \frac{\sqrt{167}}{2} \approx 6.4614$$

To write this solution in amplitude-phase form we calculate

$$\frac{1}{10} + \frac{3}{2\sqrt{167}}\, i = \frac{7}{5}\sqrt{\frac{2}{167}}\, e^{i\phi}, \quad \text{where} \quad \phi = \arctan\left(\frac{15}{\sqrt{167}}\right) \approx 0.85965$$

Now we can write the solution as

$$x(t) = \frac{7}{5}\sqrt{\frac{2}{167}}\, e^{-15t/2}\cos(\omega t - \phi)$$

From this formula we can read that amplitude decays exponentially, the quasi-period of the motion is $T = 2\pi/\omega \approx 0.972415$. The first passage through the equilibrium occurs at

$$\frac{\phi + \pi/2}{\omega} \approx 0.37615$$

The subsequent passages occur for $k = 1, 2, 3, \ldots$ at

$$\frac{\phi + \pi/2}{\omega} + k\frac{T}{2} \approx 0.37615 + k \cdot 0.48621$$

Since the derivative of the solution $x'(0) = 0$, all maximal displacements in positive direction occur at integer multiples of T, that is $T, 2T, 3T, \ldots$, and the maximum displacements in negative direction occur at odd multiples of $T/2$, that is, $T/2, T/2+T, T/2+2T, \ldots$.

---

23. We will formulate an initial value problem for the charge $Q(t)$. It is given that the initial voltage drop on the capacitor is 50V. Since the capacitance of the capacitor is $C = 0.008$F and the charge is given by $Q(0) = CE(0)$ we have $Q(0) = 2/5$ coulombs. The initial value problem to be solved is

$$4\,Q''(t) + 20\,Q'(t) + 125\,Q(t) = 0, \quad Q(0) = \frac{2}{5}, \quad Q'(0) = 0.$$

The solution is

$$Q(t) = \frac{1}{5}\,e^{-5t/2}\big(2\cos(5t) + \sin(5t)\big)$$

The problem asks for the current:

$$I(t) = Q'(t) = -\frac{5}{2}\,e^{-5t/2}\sin(5t)$$

To get the expression in amplitude-phase form we calculate

$$0 - \frac{5}{2}\,i = \frac{5}{2}\,e^{-\frac{\pi}{2}i}$$

Then

$$I(t) = \frac{5}{2}\,e^{-5t/2}\cos\left(5t + \frac{\pi}{2}\right)$$

An additional interesting question here is to calculate the maximums and the minimums of the decreasing current. We first find the derivative of the current

$$I'(t) = \frac{25}{4}\,e^{-5t/2}\big(-2\cos(5t) + \sin(5t)\big)$$

Then we write the current in amplitude-phase form. The first step is the calculation of the polar form (notice that since the real part is negative and the imaginary part is positive the argument is between $\pi/2$ and π)

$$-2 + i = \sqrt{5}\,e^{i\phi} \quad \text{where} \quad \phi = \arccos\left(-\frac{2}{\sqrt{5}}\right) \approx 2.67795.$$

So the amplitude-phase of the derivative is

$$I'(t) = \frac{25\sqrt{5}}{4}\,e^{-5t/2}\cos(5t - \phi)$$

We need zeros of this function. The smallest positive zero is the solution of

$$5t - \phi = -\frac{\pi}{2}, \qquad t_1 = -\frac{\pi}{10} + \frac{\phi}{5} \approx 0.22143$$

Since the period of this function is $2\pi/5$ all other zeros appear at distances $\pi/5$, that is $t_2 = t_1 + \pi/5,\ t_3 = t_1 + 2\pi/5, \ldots$.

Thus the first few maximums of the current are

$$I(t_1) = -\sqrt{5}\, e^{\frac{\pi}{4} - \frac{\phi}{2}} \approx -1.2855,$$

$$I(t_2) = \sqrt{5}\, e^{-\frac{\pi}{4} - \frac{\phi}{2}} \approx 0.26723$$

$$I(t_3) = -\sqrt{5}\, e^{-3\frac{\pi}{4} - \frac{\phi}{2}} \approx -0.055551$$

One can see that the next extreme current is obtained from the previous one just by multiplication by

$$-e^{-\frac{\pi}{2}} \approx 0.20788$$