Section 2.2 version October 3, 2011 at 14:16

Assigned problems: 1-5, 8, 9, 13-17, 19, 20, 22, 24-27, 34, 35, 36, 37, 40.

Selected solutions:

1. $y(x) = C\,e^{\frac{x^2}{2}}$

2. The general solution is $y(x) = C\,x^2$

Solution. The given equation is $xy' = 2y$. We first separate variables:

$$\frac{1}{y}y' = 2x.$$

Next objective is to write the left-hand side as a derivative. To do that we find the antiderivative of $1/y$. The antiderivative is $\ln\bigl(|y|\bigr)$. Then by the chain rule

$$\frac{d}{dx}\ln\bigl(|y(x)|\bigr) = \frac{1}{y(x)}y'(x).$$

This is exactly the left-hand side of the given equation. This the given equation is equivalent to

$$\frac{d}{dx}\ln\bigl(|y(x)|\bigr) = 2x.$$

The last equation yields

$$\ln\bigl(|y(x)|\bigr) = x^2 + C_1.$$

Solving for $y(x)$ we get

$$|y(x)| = e^{x^2+C_1}, \quad \text{that is} \quad |y(x)| = e^{x^2}\,e^{C_1}, \quad \text{or} \quad y(x) = \pm e^{C_1}\,e^{x^2}.$$

Since $\pm e^{C_1}$ can be an arbitrary nonzero constant and since the constant function $y(x) = 0$ is also a solution, the general solution is

$$y(x) = C\,e^{x^2}$$

3. $y(x) = \ln\bigl(e^x + C\bigr)$

4. $y(x) = \tan\bigl(e^x + C\bigr)$

5. $y(x) = C\, e^{x+\frac{x^2}{2}}$

8. $y(x) = C\, e^x\ (x - 1)$

9. $y(x) = e^{Ce^{\arctan(x)}}$

Solution. The given equation is $x^2y' = y\ln y - y'$. We first write the equation in normal form

$$y' = \big(y\ln y\big)\,\frac{1}{1+x^2}.$$

Then we separate variables:

$$\frac{1}{y\ln y}y' = \frac{1}{1+x^2}.$$

Next objective is to write the left-hand side as a derivative. To do that we find the antiderivative of

$$\frac{1}{y\ln y}.$$

Thus find

$$\int \frac{1}{y\ln y}\,dy = \left|\begin{array}{l} w = \ln y \quad \leftarrow \boxed{\text{This is a “natural” substitution.}} \\ \hline \dfrac{dw}{dy} = \dfrac{1}{y} \\ dw = \dfrac{1}{y}\,dy \quad \leftarrow \boxed{\text{Thus } dw \text{ is already in the integral.}} \end{array}\right|$$

$$\begin{aligned} &= \int \frac{1}{w}\,dw \\ &= \ln|w| + C \\ &= \ln\big|\ln y\big| + C. \end{aligned}$$

We need only one function, so we can use $C = 0$. Since

$$\frac{d}{dy}\ln\big|\ln y\big| = \frac{1}{y\ln y},$$

the chain rule yields

$$\frac{d}{dx}\ln\big|\ln y(x)\big| = \frac{1}{y(x)\;\ln y(x)}\,y'(x).$$

This is exactly the left-hand side of the given equation. Therefore the given equation is equivalent to

$$\frac{d}{dx}\ln\big|\ln y(x)\big| = \frac{1}{1+x^2}.$$

The last equation yields

$$\ln\big|\ln y(x)\big| = \int \frac{1}{1+x^2}\,dx.$$

Since

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C_1,$$

we have

$$\ln\big|\ln y(x)\big| = \arctan x + C_1.$$

The first step in solving this equation for $y(x)$ is

$$\big|\ln y(x)\big| = e^{\arctan x + C_1}.$$

As in Problem 2, the last can be written as

$$\ln y(x) = Ce^{\arctan x}.$$

Finally, the solution for $y(x)$ is

$$y(x) = e^{Ce^{\arctan x}}.$$

13. $y(x) = -2x$ The interval of existence is $\mathbb{R}$.

14. $y(t) = \sqrt{-1 + 2\,e^{-2\,t^2}}$ The interval of existence is $\left(-\sqrt{\frac{\ln(2)}{2}}, \sqrt{\frac{\ln(2)}{2}}\right)$

Solution. The given initial value problem is

$$y' = -2t\,\frac{1+y^2}{y}, \quad y(0) = 1.$$

Then we separate variables:

$$\frac{y}{1+y^2}y' = -2t.$$

As always in separable equations, the next objective is to write the left-hand side as a derivative. To do that we find the antiderivative of

$$\frac{y}{1+y^2}.$$

Thus find

$$\int \frac{y}{1+y^2}\,dy = \left|\begin{array}{l} w = 1+y^2 \quad \leftarrow \boxed{\text{This is a “natural” substitution.}} \\ \hline \dfrac{dw}{dy} = 2y \\ ydy = \dfrac{1}{2}\,dw \quad \leftarrow \boxed{\text{Thus } ydy \text{ will be substituted by } \tfrac{1}{2}dw.} \end{array}\right|$$

$$\begin{aligned} &= \int \frac{1}{w}\,\frac{1}{2}dw \\ &= \frac{1}{2}\ln|w| + C \\ &= \frac{1}{2}\ln\bigl(1+y^2\bigr) + C. \end{aligned}$$

We need only one function, so we can use $C = 0$. Since

$$\frac{d}{dy}\frac{1}{2}\ln\bigl(1+y^2\bigr) = \frac{y}{1+y^2},$$

the chain rule yields

$$\frac{d}{dt}\frac{1}{2}\ln\bigl(1+y(t)^2\bigr) = \frac{y(t)}{1+y(t)^2}\,y'(t).$$

This is exactly the left-hand side of the given equation after the separation of variables. Therefore the given equation is equivalent to

$$\frac{d}{dt}\frac{1}{2}\ln\bigl(1+y(t)^2\bigr) = -2t.$$

The last equation yields

$$\frac{1}{2}\ln\bigl(1+y(t)^2\bigr) = \int -2t\,dt.$$

Since

$$\int -2t\,dt = -t^2 + C,$$

we have

$$\frac{1}{2}\ln\bigl(1 + y(t)^2\bigr) = -t^2 + C.$$

Since we are asked to solve an initial value problem, it is best to calculate C now:

$$\frac{1}{2}\ln\bigl(1 + 1^2\bigr) = 0 + C, \quad \text{that is} \quad C = \frac{1}{2}\ln 2.$$

The next step is to solve

$$\frac{1}{2}\ln\bigl(1 + y(t)^2\bigr) = -t^2 + \frac{1}{2}\ln 2$$

for $y(t)$:

$$\begin{aligned}
\ln\bigl(1 + y(t)^2\bigr) &= -2t^2 + \ln 2\\
1 + y(t)^2 &= e^{-2t^2 + \ln 2}\\
y(t)^2 &= 2e^{-2t^2} - 1\\
y(t) &= \sqrt{2e^{-2t^2} - 1} \quad \leftarrow
\end{aligned}$$

We choose $\sqrt{\ }$ since $y(0) = 1$. If the initial condition was $y(0) = -1$ we would choose $-\sqrt{\ }$.

The interval of existence is the domain of the function above. In the given differential equation y is in the denominator, so it must not be 0. To determine the domain of

$$y(t) = \sqrt{2e^{-2t^2} - 1}$$

we solve the inequality $2e^{-2t^2} - 1 > 0$, or $e^{-2t^2} > 1/2$. We first solve the equation $e^{-2t^2} = 1/2$. Its solutions are

$$-\sqrt{\frac{\ln(2)}{2}}, \quad \sqrt{\frac{\ln(2)}{2}}.$$

Looking at the graph of e^{-2t^2} and knowing these two values, see Figure 1, we find that the domain of $y(t)$ is

$$-\sqrt{\frac{\ln(2)}{2}} < t < \sqrt{\frac{\ln(2)}{2}}.$$

The above interval is the interval of existence of the solution. Finally, in Figure 2 is a graph of the solution together with the direction field. The interval of existence is the green interval.

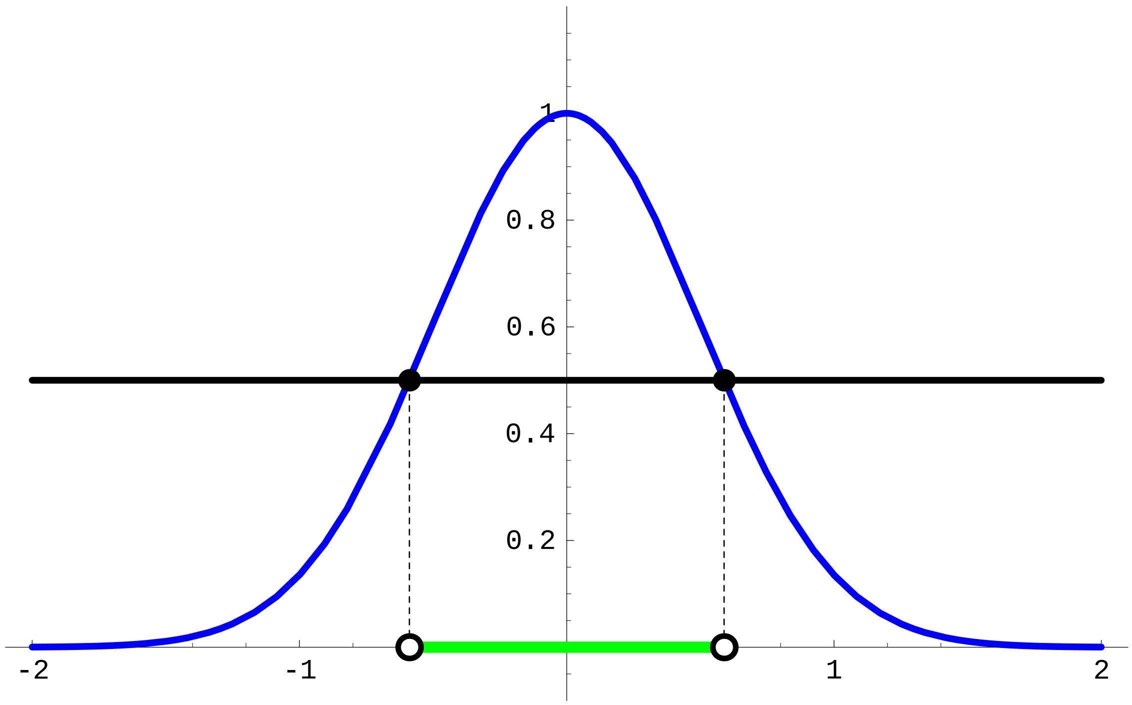

Figure 1: Problem 14

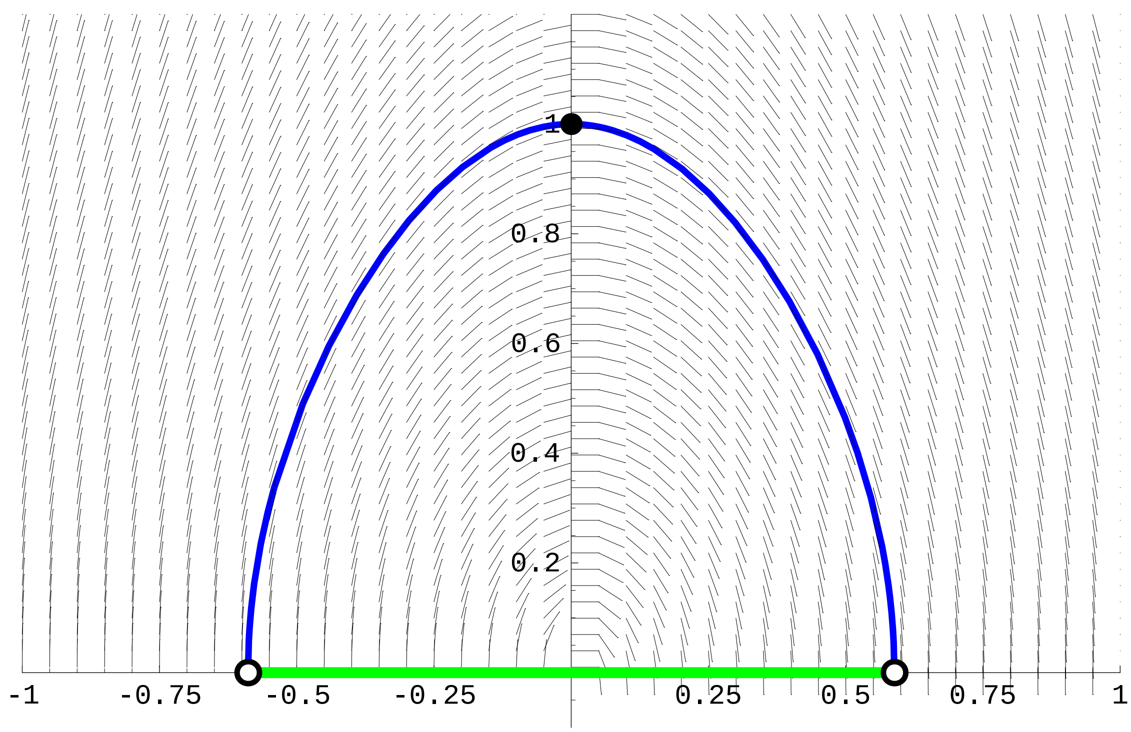

Figure 2: Problem 14

15. $y(x) = \sqrt{1 - 2\,\cos(x)}$. The interval of existence is $\left(\dfrac{\pi}{3}, \dfrac{5\pi}{3}\right)$

16. $y(x) = -\ln(2 - e^x)$ The interval of existence is $(-\infty, \ln 2\,)$
Solution. The given initial value problem is

$$y' = e^{t+y}, \quad y(0) = 0.$$

First we separate variables:

$$e^{-y} y' = e^t.$$

As always in separable equations, the next objective is to write the left-hand side as a derivative. To do that we find the antiderivative of

$$e^{-y}.$$

This is almost a table integral

$$\int e^{-y}\, dy = -e^{-y} + C.$$

We need only one function, so we can choose $C = 0$. Since

$$\frac{d}{dy} - e^{-y} = e^{-y},$$

the chain rule yields

$$\frac{d}{dt} - e^{-y(t)} = e^{-y(t)} y'(t).$$

This is exactly the left-hand side of the given equation after the separation of variables. Therefore the given equation is equivalent to

$$\frac{d}{dt} - e^{-y(t)} = e^t.$$

The last equation yields

$$-e^{-y(t)} = \int e^t\, dt.$$

Since

$$\int e^t\, dt = e^t + C,$$

we have

$$-e^{-y(t)} = e^t + C.$$

Since we are asked to solve an initial value problem, it is best to calculate C now:

$$-e^0 = e^0 + C, \quad \text{hence} \quad C = -2.$$

The next step is to solve

$$-e^{-y(t)} = e^t - 2.$$

for $y(t)$:

$$\begin{aligned} -e^{-y(t)} &= e^t - 2 \\ e^{-y(t)} &= 2 - e^t \\ -y(t) &= \ln\bigl(2 - e^t\bigr) \\ y(t) &= -\ln\bigl(2 - e^t\bigr) \end{aligned}$$

The interval of existence is the domain of the function $y(t)$ above. In the given differential equation y is in the denominator, so it must not be 0. To determine the domain of

$$y(t) = -\ln\bigl(2 - e^t\bigr)$$

we solve the inequality $2 - e^t > 0$, or $2 > e^t$. We first solve the equation $e^t = 2$. Its solution is $t = \ln 2$. Looking at the graph of e^t we find that the domain of $y(t)$ is

$$-\infty < t < \ln 2.$$

The above interval is the interval of existence of the solution. Finally, in Figure 3 is a graph of the solution together with the direction field. The interval of existence is the green interval.

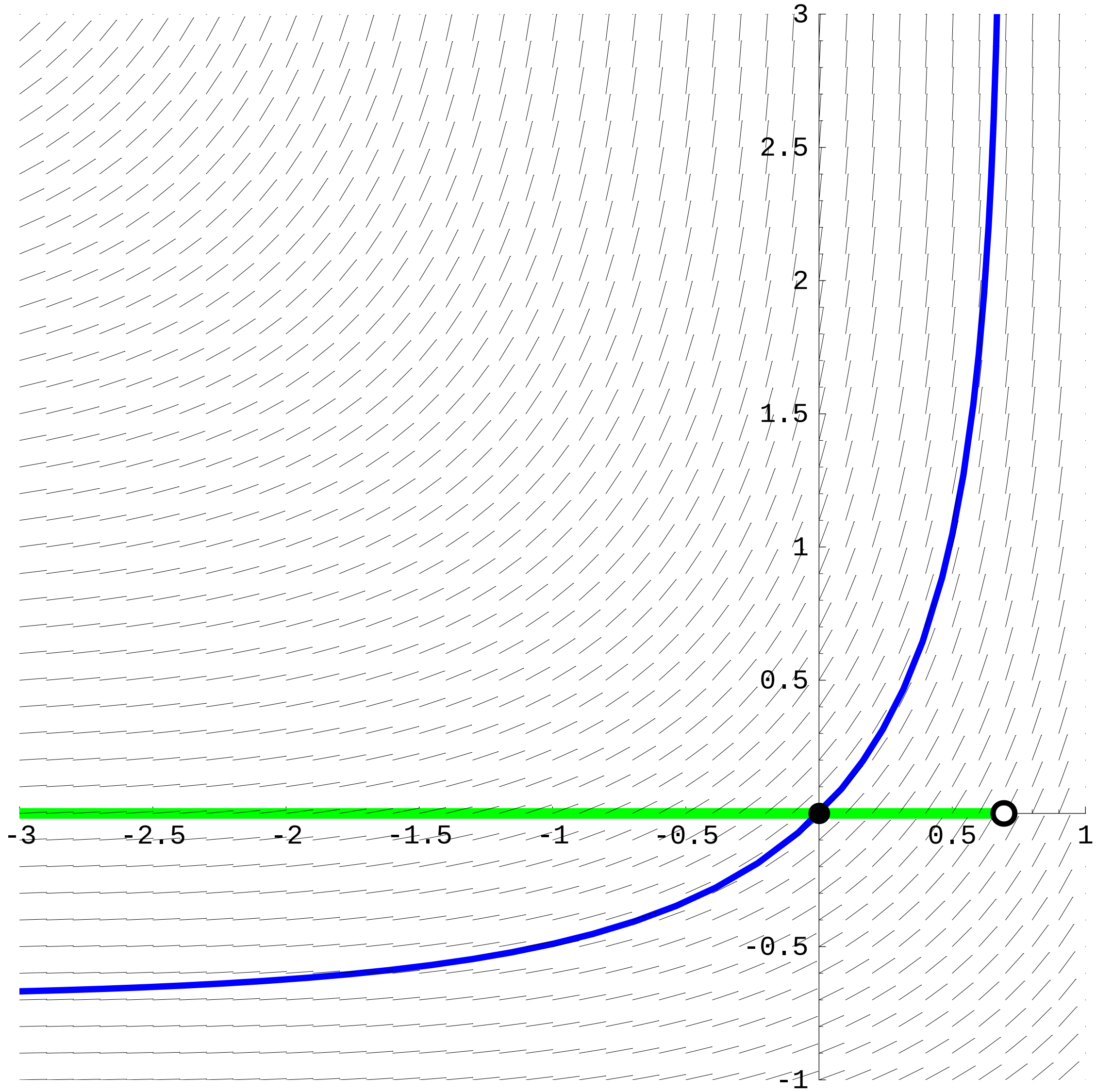

Figure 3: Problem 16

17. $y(x) = \tan\left(\frac{\pi}{4} + x\right)$ The interval of existence is $\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$

19. Solutions $y(x) = \sqrt{1 + x^2}$ and $-\sqrt{1 + x^2}$. Intervals of existence $\mathbb{R}$.

20. Solutions $y(x) = \sqrt{4 - x^2}$ and $-\sqrt{4 - x^2}$. Intervals of existence $(-2, 2)$.
Solution. The given initial value problems is

$$y' = -\frac{t}{y}, \quad y(0) = -2, \quad y(0) = 2.$$

First separate the variables

$$yy' = -t.$$

As always in separable equations, the next objective is to write the left-hand side as a derivative. To do that we find the antiderivative of

$$y$$

This is easy

$$\int y\, dy = \frac{1}{2}\, y^2 + C.$$

We need only one function, so we can choose $C = 0$. Since

$$\frac{d}{dy}\frac{1}{2}\, y^2 = y,$$

the chain rule yields

$$\frac{d}{dt}\frac{1}{2}\, y(t)^2 = y(t) y'(t).$$

This is exactly the left-hand side of the given equation after the separation of variables. Therefore the given equation is equivalent to

$$\frac{d}{dt}\frac{1}{2}\, y(t)^2 = -t.$$

The last equation yields

$$\frac{1}{2}\, y(t)^2 = \int -t\, dt.$$

Since

$$\int -t\, dt = -\frac{1}{2}\, t^2 + C,$$

we have

$$\frac{1}{2}\, y(t)^2 = -\frac{1}{2}\, t^2 + C.$$

Since we are asked to solve an initial value problem, it is best to calculate C now:

$$\frac{1}{2}\,(-2)^2 = 0 + C, \quad \text{hence} \quad C = 2.$$

It is interesting that we get the same value of C for the second initial condition. The next step is to solve

$$\frac{1}{2}\, y(t)^2 = -\frac{1}{2}\, t^2 + 2$$

for $y(t)$:

$$y(t)^2 = -t^2 + 4$$

Solving this equation leads to two solutions:

$$y(t) = -\sqrt{-t^2 + 4}$$
$$y(t) = \sqrt{-t^2 + 4}$$

The first solutions solves the first initial value problem, the second solution solves the second initial value problem. The interval of existence is the same for both solutions. It is the domain of the function $y(t)$ above. In the given differential equation y is in the denominator, so it must not be 0. Thus the interval of existence for both solutions is

$$-2 < t < 2.$$

Finally, in Figure 4 is a graph of the solution together with the direction field. The interval of existence is the green interval. There are blue graphs and red graphs in Figure 4. The blue graphs are plotted using the formula for the solution, while the red graphs are obtained using numerical solver.

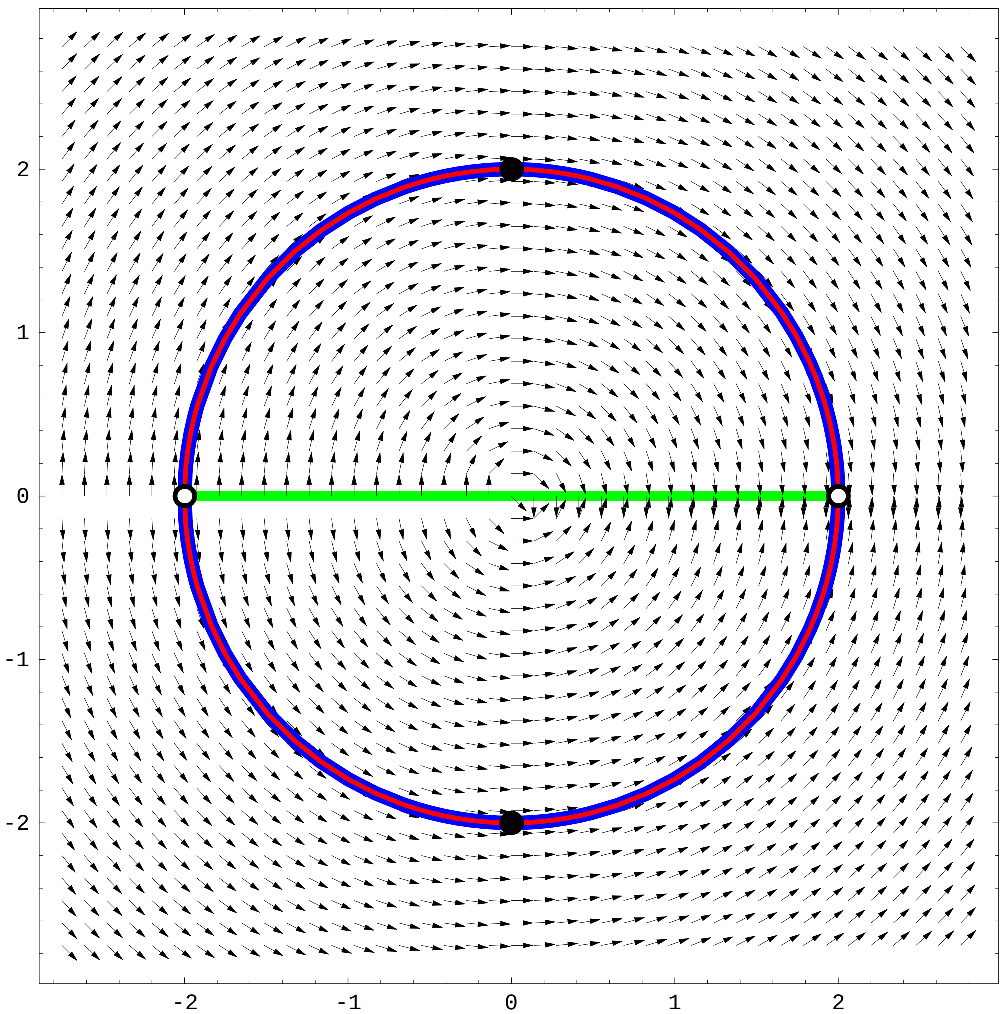

Figure 4: Problem 20

22. $y(x) = \sqrt{-1 + 5\, e^{2\,(x-1)}}$. The interval of existence is $\left(1 - \frac{1}{2}\ln(5)\,, +\infty\right)$.

24. Done in class.

25. Done in class.

26. (a) 1.55066×10^{-8}; (b) 1.4849×10^{8}

27. 14.2584 hours.

30. 89.1537 mg

34. (b) 3878.99 years

35. (a) done in class. (b) 21:53:37.392

Solution. (b) We set the coordinate system for the time to start at midnight. Denote by $B(t)$ the body temperature at time t. Then we have the initial value problem

$$B'(t) = k(21 - B(t)), \quad B(0) = 31.$$

The solution is

$$B(t) = 21 + 10e^{-kt}.$$

But, here k is unknown. We determine k from $B(1) = 29$. Thus, solve for k

$$21 + 10e^{-k} = 29.$$

The solution is $k = \ln(5/4)$. Now we need to find the time when the temperature was 37. Thus solve for t

$$21 + 10e^{-(\ln(5/4))t} = 37.$$

The solution is

$$t = -\frac{\ln(8/5)}{\ln(5/4)} =\approx -2.10628.$$

Thus, victims's time of death occurred 2.10628 hours before midnight; that is at 9.89372 pm. It is more natural to write this in terms of minutes and seconds: 9:53:37pm.

36. 56.1842°F

37. 5 minutes 51 seconds.

40. $y(x) = 2\sqrt{x + C}$.