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Exercises

12. The initial value problem to be solved is

$$0.1x''(t) + 3.6x(t) = 0, \quad x(0) = 0, \quad x'(0) = -0.4.$$

The solution of this initial value problem is $x(t) = -\frac{1}{15}\sin(6t)$. From the solution we see that the amplitude is $1/15$, the natural frequency is 6. To get the phase we rewrite the solution in the form

$$A\cos(6t - \phi)$$

To get ϕ we write in the polar form the complex number $0 - \frac{1}{15}i$. Clearly $0 - \frac{1}{15}i = \frac{1}{15}e^{-i\frac{\pi}{2}}$. Therefore, the phase is $-\frac{\pi}{2}$.

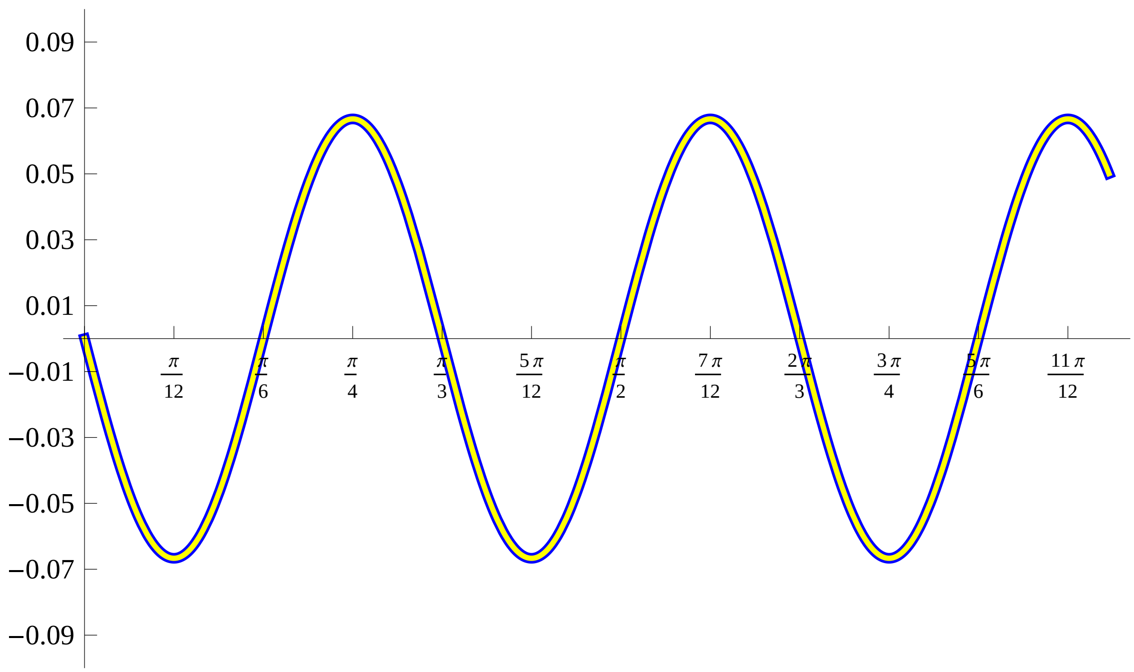

Figure 1: Problem 12

15. We are given an LC circuit with the $C = 2 \times 10^{-6}$F capacitor and $L = 6 \times 10^{-6}$H inductor. It is given that initially the current is 0. That is $Q'(0) = I(0) = 0$. The equation governing this circuit is

$$L\,Q''(t) + \frac{1}{C}\,Q(t) = 0, \qquad Q(0) = ?, \quad Q'(0) = 0.$$

How do we determine $Q(0)$? It is given that the capacitor is charged to 20V and then connected to the circuit. This implies that the voltage drop at the capacitor at $t = 0$ is 20V. The capacitance law establishes the connection between the voltage drop at the capacitor and the charge:

$$E_C = \frac{1}{C} Q.$$

Therefore

$$Q(0) = C\, E_C(0) = 2 \times 10^{-6} \times 20 = 4 \times 10^{-5}.$$

This is the initial charge at the capacitor: $Q(0) = 4 \times 10^{-5}$C (coulombs).
The solution of the initial value problem

$$L\, Q''(t) + \frac{1}{C}\, Q(t) = 0, \qquad Q(0) = Q_0, \quad Q'(0) = 0,$$

is

$$Q(t) = Q_0 \, \cos\left(\frac{1}{\sqrt{LC}}\, t\right).$$

This is the equation for the charge. The solution for the current is

$$I(t) = Q'(t) = -\frac{Q_0}{\sqrt{LC}} \, \sin\left(\frac{1}{\sqrt{LC}} t\right) = \frac{Q_0}{\sqrt{LC}} \, \cos\left(\frac{1}{\sqrt{LC}}\, t + \frac{\pi}{2}\right).$$

Thus the amplitude for the current is

$$\frac{Q_0}{\sqrt{LC}} = \frac{4 \times 10^{-5}}{\sqrt{6 \times 10^{-6} \times 2 \times 10^{-6}}} = \frac{4 \times 10}{\sqrt{12}} = \frac{20}{\sqrt{3}} \approx 11.547.$$

The natural frequency for the current is the same as natural frequency for the charge:

$$\frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{6 \times 10^{-6} \times 2 \times 10^{-6}}} = \frac{10^6}{\sqrt{12}} \approx 2.88675 \times 10^5.$$

The phase is $-\pi/2$. In this problem book's solution seems to be wrong. The question in the problem is about the current. The answer is about the charge. Even the graph in the answer indicated the charge. Clearly inconsistent with the question.

16. The spring constant k is determined from the equation $1 \cdot 9.8 = 4.8 \cdot k$. Hence $k = 2$. The initial value problem to be solved is

$$x''(t) + 3x'(t) + 2x(t) = 0, \quad x(0) = -1, \quad x'(0) = -1.$$

The solution is $x(t) = -3e^{-t} + 2e^{-2t}$. Hence this is an overdamped case.
An additional interesting question here is to find the maximal displacements of the mass. To answer this question find the derivative of the solution and find where derivative is

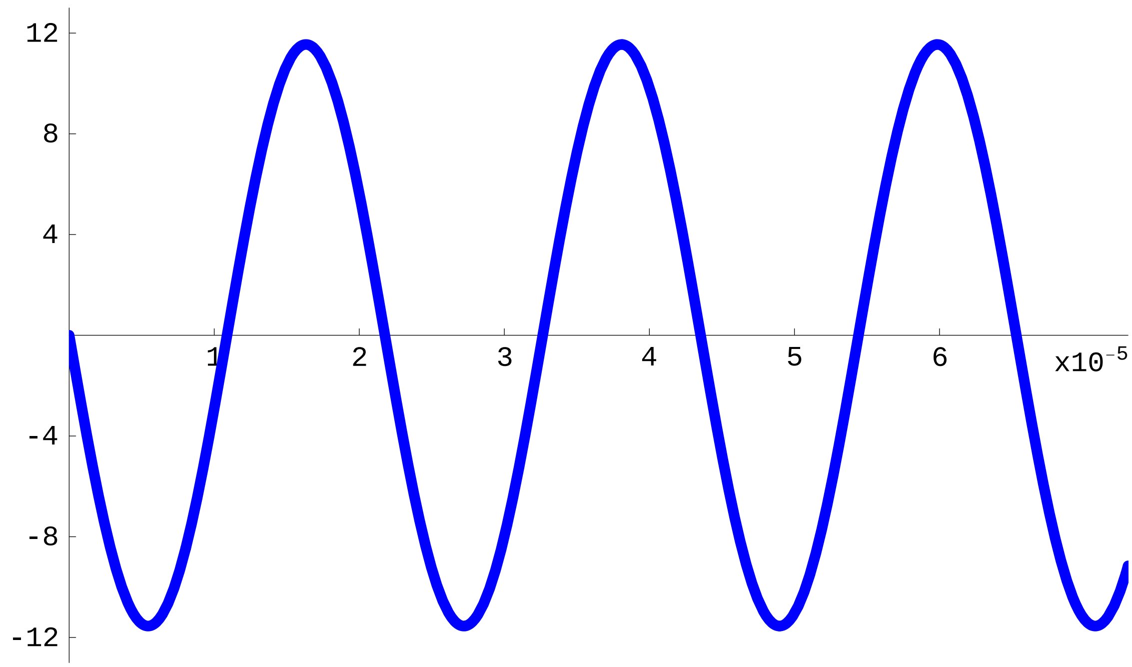

Figure 2: Problem 15

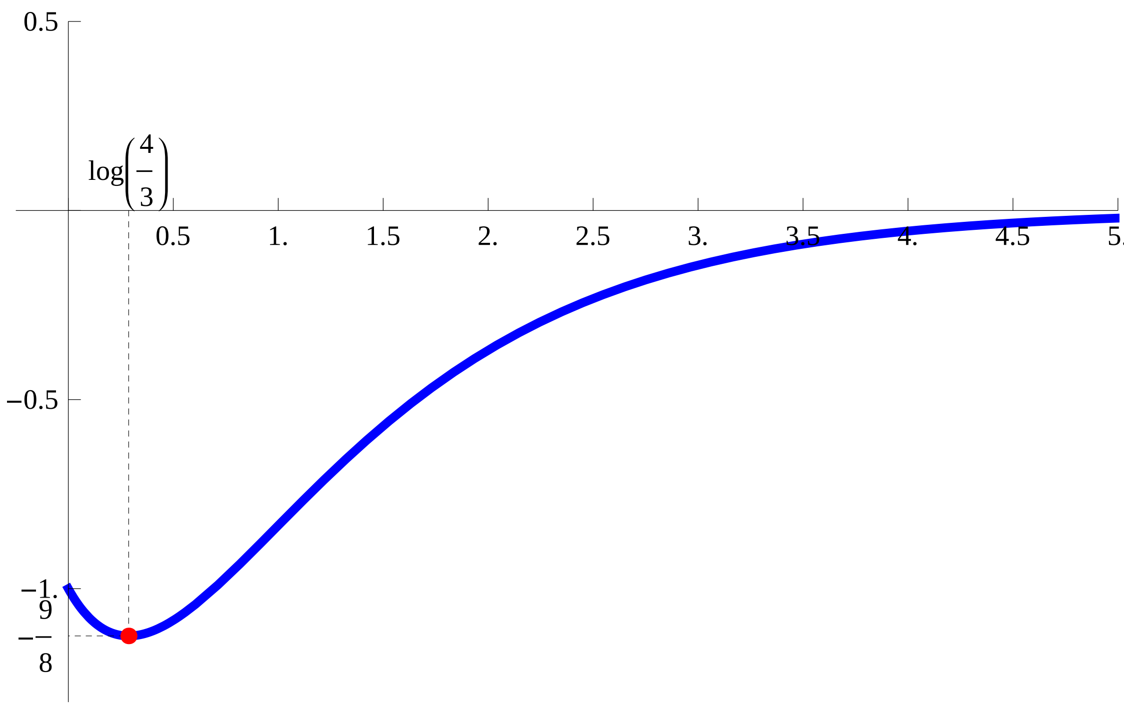

Figure 3: Problem 16

0. Then use this value in the solution. The zero of the derivative is $t = \ln(4/3)$. The corresponding value of the function is $-9/8$. Hence the maximum displacement of the mass from the equilibrium is 9/8m.

18. The spring constant k is determined from the equation $0.05 \cdot 9.8 = 0.2 \cdot k$. Hence $k = 49/20$. To find μ so that the system is critically damped we solve the equation $\mu^2 - 4 \cdot 0.05 \cdot \frac{49}{20} = 0$. We get $\mu = 7/10$. The initial value problem to be solved is

$$\frac{1}{20}x''(t) + \frac{7}{10}x'(t) + \frac{49}{20}x(t) = 0, \quad x(0) = \frac{15}{100}, \quad x'(0) = 0.$$

The solution is $x(t) = \dfrac{3}{20}\, e^{-7t}\big(1 + 7t\big)$.

22. The coefficient μ satisfies the equation $0.3 = 0.2\mu$. Hence $\mu = 3/2$. The initial value problem to be solved is

$$\frac{1}{10}x''(t) + \frac{3}{2}x'(t) + \frac{98}{10}x(t) = 0, \quad x(0) = \frac{1}{10}, \quad x'(0) = 0.$$

The symbolic solution of this equation is

$$x(t) = e^{-15t/2}\left(\frac{1}{10}\cos(\omega t) + \frac{3}{2\sqrt{167}}\sin(\omega t)\right) \quad \text{where} \quad \omega = \frac{\sqrt{167}}{2} \approx 6.4614$$

To write this solution in amplitude-phase form we calculate

$$\frac{1}{10} + \frac{3}{2\sqrt{167}}\,i = \frac{7}{5}\sqrt{\frac{2}{167}}\,e^{i\phi}, \quad \text{where} \quad \phi = \arctan\left(\frac{15}{\sqrt{167}}\right) \approx 0.85965$$

Now we can write the solution as

$$x(t) = \frac{7}{5}\sqrt{\frac{2}{167}}\,e^{-15t/2}\cos(\omega t - \phi)$$

From this formula we can read that amplitude decays exponentially, the quasi-period of the motion is $T = 2\pi/\omega \approx 0.972415$. The first passage through the equilibrium occurs at

$$\frac{\phi + \pi/2}{\omega} \approx 0.37615$$

The subsequent passages occur for $k = 1, 2, 3, \ldots$ at

$$\frac{\phi + \pi/2}{\omega} + k\frac{T}{2} \approx 0.37615 + k \cdot 0.48621$$

Since the derivative of the solution $x'(0) = 0$, all maximal displacements in positive direction occur at integer multiples of T, that is $T, 2T, 3T, \ldots$, and the maximum displacements in negative direction occur at odd multiples of $T/2$, that is, $T/2, T/2 + T, T/2 + 2T, \ldots$.

23. We will formulate an initial value problem for the charge $Q(t)$. It is given that the initial voltage drop on the capacitor is 50V. Since the capacitance of the capacitor is $C = 0.008$F and the charge is given by $Q(0) = CE(0)$ we have $Q(0) = 2/5$ coulombs. The initial value problem to be solved is

$$4\,Q''(t) + 20\,Q'(t) + 125\,Q(t) = 0, \quad Q(0) = \frac{2}{5}, \quad Q'(0) = 0.$$

The solution is

$$Q(t) = \frac{1}{5}\,e^{-5t/2}\big(2\cos(5t) + \sin(5t)\big)$$

The problem asks for the current:

$$I(t) = Q'(t) = -\frac{5}{2}\,e^{-5t/2}\sin(5t)$$

To get the expression in amplitude-phase form we calculate

$$0 - \frac{5}{2}\,i = \frac{5}{2}\,e^{-\frac{\pi}{2}i}$$

Then

$$I(t) = \frac{5}{2}\,e^{-5t/2}\cos\left(5t + \frac{\pi}{2}\right)$$

An additional interesting question here is to calculate the local maximums and the local minimums of the current, see Figure 4. We first find the derivative of the current

$$I'(t) = \frac{25}{4}\,e^{-5t/2}\big(-2\cos(5t) + \sin(5t)\big)$$

Then we write the derivative of the current in the amplitude-phase form. The first step is the calculation of the polar form (notice that since the real part is negative and the imaginary part is positive the argument is between $\pi/2$ and π)

$$-2 + i = \sqrt{5}\,e^{i\phi} \quad \text{where} \quad \phi = \arccos\left(-\frac{2}{\sqrt{5}}\right) \approx 2.67795.$$

So the amplitude-phase of the derivative is

$$I'(t) = \frac{25\sqrt{5}}{4}\,e^{-5t/2}\cos(5t - \phi)$$

We need zeros of this function. The smallest positive zero is the solution of

$$5t - \phi = -\frac{\pi}{2}, \qquad t_1 = -\frac{\pi}{10} + \frac{\phi}{5} \approx 0.22143$$

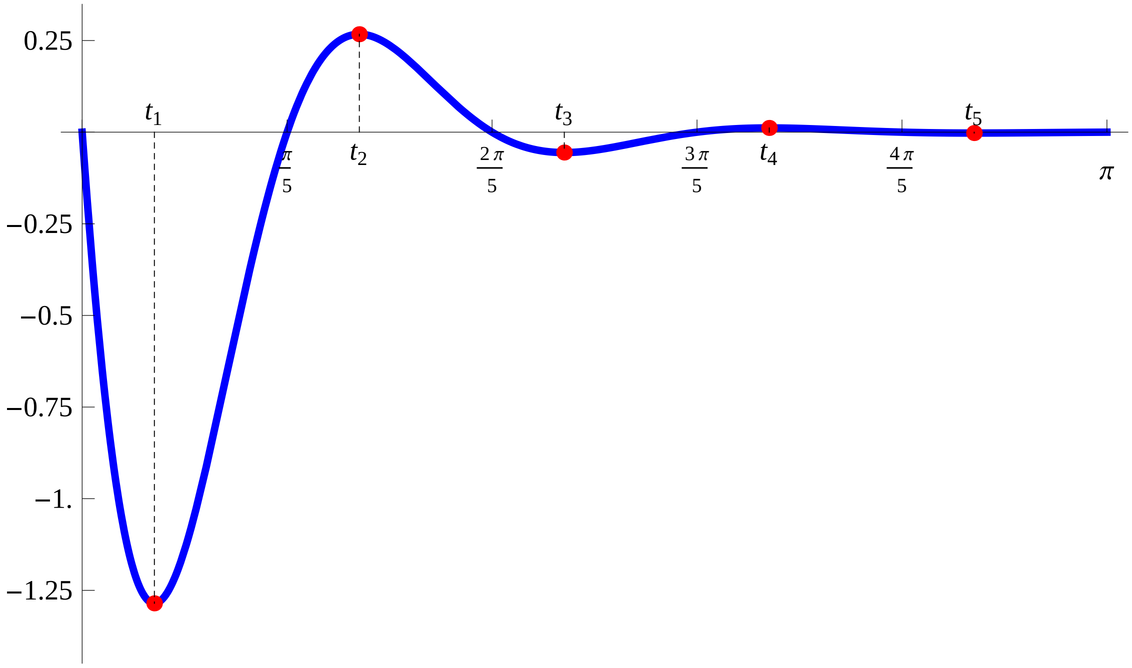

Figure 4: Problem 23

Since the period of this function is $2\pi/5$ all other zeros appear at distances $\pi/5$, that is $t_2 = t_1 + \pi/5,\ t_3 = t_1 + 2\pi/5, \ldots$.
Thus the first few maximums of the current are

$$I(t_1) = -\sqrt{5}\, e^{\frac{\pi}{4}-\frac{\phi}{2}} \approx -1.2855,$$
$$I(t_2) = \sqrt{5}\, e^{-\frac{\pi}{4}-\frac{\phi}{2}} \approx 0.26723$$
$$I(t_3) = -\sqrt{5}\, e^{-3\frac{\pi}{4}-\frac{\phi}{2}} \approx -0.055551$$

One can see that the next extreme current is obtained from the previous one just by multiplication by

$$-e^{-\frac{\pi}{2}} \approx 0.20788$$

24. First we calculate the spring constant k. We use the fact that the weight $10 \cdot 9.8$ kg m/s^2 stretches the spring 1m. Therefore, $10 \cdot 9.8 = 1 \cdot k$; that is $k = 98$. Now the initial value problem to be solved is

$$10\, y''(t) + 20y'(t) + 98y(t) = 0, \qquad y(0) = 0, \quad y'(0) = -1.2.$$

The characteristic equation is $10\lambda^2 + 20\lambda + 98 = 0$. The solution of this quadratic equation are

$$\lambda_1 = -1 + i\sqrt{8.8}, \qquad \lambda_2 = -1 + i\sqrt{8.8}.$$

The general solution is

$$y(t) = C_1 e^{-t} \cos\big(\sqrt{8.8}\,t\big) + C_2 e^{-t} \sin\big(\sqrt{8.8}\,t\big).$$

We use the initial conditions to determine C_1 and C_2, but first calculate the derivative of $y(t)$:

$$y(t) = e^{-t}\Big(C_1 \cos\big(\sqrt{8.8}\,t\big) + C_2 \sin\big(\sqrt{8.8}\,t\big)\Big);$$

$$y'(t) = -e^{-t}\Big(C_1 \cos\big(\sqrt{8.8}\,t\big) + C_2 \sin\big(\sqrt{8.8}\,t\big)\Big) + e^{-t}\Big(-C_1\sqrt{8.8}\sin\big(\sqrt{8.8}\,t\big) + C_2\sqrt{8.8}\cos\big(\sqrt{8.8}\,t\big)\Big);$$

Hence

$$0 = C_1 \qquad \text{and} \qquad -1.2 = -C_1 + \sqrt{8.8}C_2,$$

and consequently

$$C_1 = 0, \qquad C_2 = -\frac{1.2}{\sqrt{8.8}}.$$

Finally, the solution is

$$y(t) = -\frac{1.2}{\sqrt{8.8}}\, e^{-t} \sin\big(\sqrt{8.8}\,t\big).$$

The next step is to write the solution in the form $Ae^{-t}\cos(\omega t - \phi)$. The amplitude and the natural frequency are clear (respectively)

$$\frac{1.2}{\sqrt{8.8}}\, e^{-t} \qquad \text{and} \qquad \sqrt{8.8}.$$

To find the phase we need to express $-\sin\big(\sqrt{8.8}\,t\big)$ as $\cos\big(\sqrt{8.8}\,t - \phi\big)$:

$$-\sin\big(\sqrt{8.8}\,t\big) = \mathrm{Re}\Big(ie^{i\sqrt{8.8}\,t}\Big) = \mathrm{Re}\Big(e^{i\sqrt{8.8}\,t + i\pi/2}\Big) = \cos\big(\sqrt{8.8}\,t + \pi/2\big)$$

Thus the phase is $\phi = -\pi/2$.

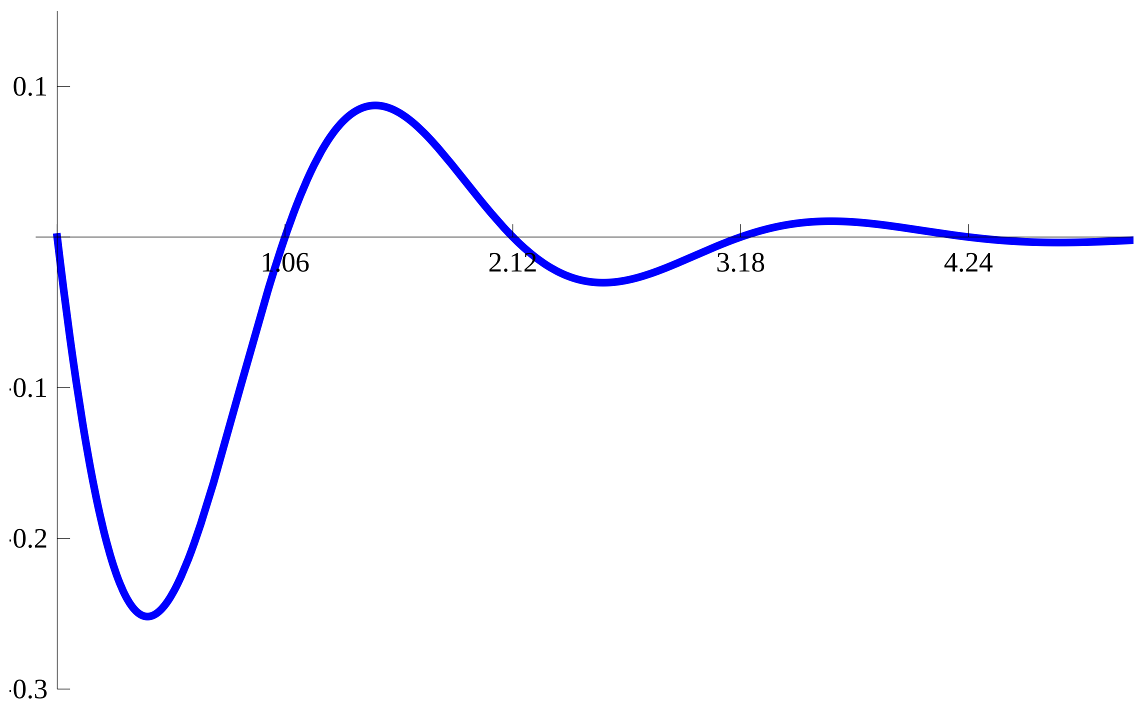

Figure 5: Problem 24