Chapters 1–5. Chapter 11 on Green's functions for the heat and wave equation is an exception since it requires Chapters 9 and 10.

New to this edition is Chapter 14 which primarily discusses dispersive waves. Chapter 14 is more advanced, and, unlike the rest of the text, has been written very concisely. Nonetheless, it is mostly self-contained and hence accessible to strong undergraduates. Topics discussed include group velocity and envelope equations for linear dispersive waves and solitary waves and solitons for nonlinear dispersive waves. In addition, instability and bifurcation phenomena for partial differential equations are discussed as well as perturbation methods (multiple scale and boundary layer problems). In Chapter 14, the author has attempted to show the vitality of the contemporary study of partial differential equations in the context of physical problems.

The author has made an effort to preserve the second edition so that previous users will find little disruption. The chapter on numerical methods is now Chapter 6 and all subsequent chapters have been moved back one. Nearly all exercises from the previous edition have been retained with no change in the order to facilitate a transition for previous users. There are over 200 figures to illustrate various concepts, which were prepared for this edition by the author using MATLAB. Modern technology is especially important in its graphical abilities, and I have tried to indicate throughout the text places where three-dimensional visualization is helpful.

Overall my object has been to explain clearly many fundamental aspects of partial differential equations as an introduction to this vast and important field. After achieving a certain degree of competence and understanding, the student can use this text as a reference, but for additional information should be prepared to refer to other books such as the ones cited in the bibliography.

Finally, it is hoped that this text enables the reader to find enjoyment in the study of the relationships between mathematics and the physical sciences.

The author gratefully acknowledges the contributions of the following reviewers of the manuscript: Charles R. MacCluer, Michigan State University; Catherine A. Roberts, Northern Arizona University; William W. Roberts, Jr., University of Virginia; Surendra Singh, University of Tulsa; Alexander P. Stone, University of New Mexico; Ron Lipsman, University of Maryland at College Park; Robert W. Kolkka, Michigan Technological University; Dahlard L. Lukes, University of Virginia; Jonathan Dimock, SUNY at Buffalo; Donald Schwendeman, Rensselaer Polytechnic Institute; William Ames, University of Waterloo; Robert Seeley, University of Massachusettes at Boston; and Arnold L. Villone, San Diego State University.

Richard Haberman
rhaberma@post.smu.edu

Chapter 1

Heat Equation

1.1 Introduction

We wish to discuss the solution of elementary problems involving partial differential equations, the kinds of problems that arise in various fields of science and engineering. A **partial differential equation** (PDE) is a mathematical equation containing partial derivatives, for example,

$$\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0. \tag{1.1.1}$$

We could begin our study by determining what functions $u(x,t)$ satisfy (1.1.1). However, we prefer to start by investigating a physical problem. We do this for two reasons. First, our mathematical techniques probably will be of greater interest to you when it becomes clear that these methods analyze physical problems. Second, we will actually find that physical considerations will motivate many of our mathematical developments.

Many diverse subject areas in engineering and the physical sciences are dominated by the study of partial differential equations. No list could be all-inclusive. However, the following examples should give you a feeling for the type of areas that are highly dependent on the study of partial differential equations: acoustics, aerodynamics, elasticity, electrodynamics, fluid dynamics, geophysics (seismic wave propagation), heat transfer, meteorology, oceanography, optics, petroleum engineering, plasma physics (ionized liquids and gases), quantum mechanics.

We will follow a certain philosophy of applied mathematics in which the analysis of a problem will have three stages:

1. Formulation

2. Solution

3. Interpretation

We begin by formulating the equations of heat flow describing the transfer of thermal energy. Heat energy is caused by the agitation of molecular matter. Two

basic processes take place in order for thermal energy to move: conduction and convection. **Conduction** results from the collisions of neighboring molecules in which the kinetic energy of vibration of one molecule is transferred to its nearest neighbor. Thermal energy is thus spread by conduction even if the molecules themselves do not move their location appreciably. In addition, if a vibrating molecule moves from one region to another, it takes its thermal energy with it. This type of movement of thermal energy is called **convection**. In order to begin our study with relatively simple problems, we will study heat flow only in cases in which the conduction of heat energy is much more significant than its convection. We will thus think of heat flow primarily in the case of solids, although heat transfer in fluids (liquids and gases) is also primarily by conduction if the fluid velocity is sufficiently small.

1.2 Derivation of the Conduction of Heat in a One-Dimensional Rod

Thermal energy density. We begin by considering a rod of constant cross-sectional area A, oriented in the x-direction (from $x = 0$ to $x = L$) as illustrated in Fig. 1.2.1. We temporarily introduce the amount of thermal energy per unit volume as an unknown variable and call it the **thermal energy density**:

$$e(x,t) \equiv \text{ thermal energy density.}$$

We assume that all thermal quantities are constant across a section; the rod is one-dimensional. The simplest way this may be accomplished is to insulate perfectly the lateral surface area of the rod. Then no thermal energy can pass through the lateral surface. The dependence on x and t corresponds to a situation in which the rod is not uniformly heated; the thermal energy density varies from one cross section to another.
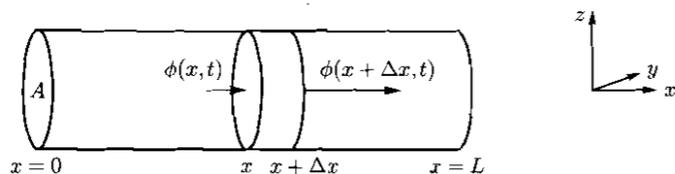

Figure 1.2.1: one-dimensional rod with heat energy flowing into and out of a thin slice.

Heat energy. We consider a thin slice of the rod contained between x and $x + \Delta x$ as illustrated in Fig. 1.2.1. If the thermal energy density is constant throughout the volume, then the total energy in the slice is the product of the thermal energy density and the volume. In general, the energy density is not constant. However, if

Δx is exceedingly small, then $e(x,t)$ may be approximated as a constant throughout the volume so that

$$\text{heat energy } = e(x,t)A \, \Delta x,$$

since the volume of a slice is $A \, \Delta x$.

Conservation of heat energy. The heat energy between x and $x + \Delta x$ changes in time due only to heat energy flowing across the edges (x and $x + \Delta x$) and heat energy generated inside (due to positive or negative sources of heat energy). No heat energy changes are due to flow across the lateral surface, since we have assumed that the lateral surface is insulated. The fundamental heat flow process is described by the word equation

rate of change of heat energy in time	=	heat energy flowing across boundaries per unit time	+	heat energy generated inside per unit time.

This is called **conservation of heat energy**. For the small slice, the rate of change of heat energy is

$$\frac{\partial}{\partial t} \left[e(x,t)A \, \Delta x \right],$$

where the partial derivative $\frac{\partial}{\partial t}$ is used because x is being held fixed.

Heat flux. Thermal energy flows to the right or left in a one-dimensional rod. We introduce the **heat flux**

$$\phi(x,t) \quad = \quad \begin{array}{l} \text{heat flux (the amount of thermal energy } per \ unit \\ time \text{ flowing to the right } per \ unit \ surface \ area). \end{array}$$

If $\phi(x,t) < 0$, it means that heat energy is flowing to the left. Heat energy flowing per unit time across the boundaries of the slice is $\phi(x,t)A - \phi(x + \Delta x, t)A$, since the heat flux is the flow per unit surface area and it must be multiplied by the surface area. If $\phi(x,t) > 0$ and $\phi(x + \Delta x, t) > 0$, as illustrated in Fig. 1.2.1, then the heat energy flowing per unit time at x contributes to an increase of the heat energy in the slice, whereas the heat flow at $x + \Delta x$ decreases the heat energy.

Heat sources. We also allow for internal sources of thermal energy:

$$Q(x,t) = \text{heat energy per unit volume generated per unit time,}$$

perhaps due to chemical reactions or electrical heating. $Q(x,t)$ is approximately constant in space for a thin slice, and thus the total thermal energy generated per unit time in the thin slice is approximately $Q(x,t)A \, \Delta x$.

Conservation of heat energy (thin slice). The rate of change of heat energy is due to thermal energy flowing across the boundaries and internal sources:

$$\frac{\partial}{\partial t}[e(x,t)A \, \Delta x] \approx \phi(x,t)A - \phi(x+\Delta x, t)A + Q(x,t)A \, \Delta x. \qquad (1.2.1)$$

Equation (1.2.1) is not precise because various quantities were assumed approximately constant for the small cross-sectional slice. We claim that (1.2.1) becomes increasingly accurate as $\Delta x \to 0$. Before giving a careful (and mathematically rigorous) derivation, we will just attempt to explain the basic ideas of the limit process, $\Delta x \to 0$. In the limit as $\Delta x \to 0$, (1.2.1) gives no interesting information, namely, $0 = 0$. However, if we first divide by Δx and then take the limit as $\Delta x \to 0$, we obtain

$$\frac{\partial e}{\partial t} = \lim_{\Delta x \to 0} \frac{\phi(x,t) - \phi(x+\Delta x, t)}{\Delta x} + Q(x,t), \qquad (1.2.2)$$

where the constant cross-sectional area has been canceled. We claim that this result is exact (with no small errors), and hence we replace the $\approx$ in (1.2.1) by $=$ in (1.2.2). In this limiting process, $\Delta x \to 0, t$ is being held fixed. Consequently, from the definition of a partial derivative,

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.} \qquad (1.2.3)$$

Conservation of heat energy (exact). An alternative derivation of conservation of heat energy has the advantage of our not being restricted to small slices. The resulting approximate calculation of the limiting process ($\Delta x \to 0$) is avoided. We consider any *finite* segment (from $x = a$ to $x = b$) of the original one-dimensional rod (see Fig. 1.2.2). We will investigate the conservation of heat energy in this region. The total heat energy is $\int_a^b e(x,t)dx$, the sum of the contributions of the infinitesimal slices. Again it changes only due to heat energy flowing through the side edges ($x = a$ and $x = b$) and heat energy generated inside the region, and thus (after canceling the constant A)

$$\boxed{\frac{d}{dt}\int_a^b e \, dx = \phi(a,t) - \phi(b,t) + \int_a^b Q \, dx.} \qquad (1.2.4)$$

Technically, an ordinary derivative d/dt appears in (1.2.4) since $\int_a^b e \, dx$ depends only on t, not also on x. However,

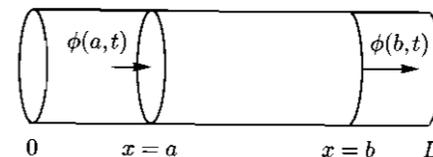$$\frac{d}{dt}\int_a^b e \, dx = \int_a^b \frac{\partial e}{\partial t}dx,$$

Figure 1.2.2: Heat energy flowing into and out of a finite segment of a rod.

if a and b are constants (and if e is continuous). This holds since inside the integral the ordinary derivative now is taken keeping x fixed, and hence it must be replaced by a partial derivative. Every term in (1.2.4) is now an ordinary integral if we notice that

$$\phi(a,t) - \phi(b,t) = -\int_a^b \frac{\partial \phi}{\partial x}dx,$$

(this[1] being valid if ϕ is continuously differentiable). Consequently,

$$\int_a^b \left(\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q\right)dx = 0.$$

This integral must be zero *for arbitrary a and b*; the area under the curve must be zero for arbitrary limits. This is possible only if the integrand itself is identically zero.[2] Thus, we rederive (1.2.3) as

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.} \qquad (1.2.5)$$

Equation (1.2.4), the **integral conservation law**, is more fundamental than the differential form (1.2.5). Equation (1.2.5) is valid in the usual case in which the physical variables are continuous.

A further explanation of the minus sign preceding $\partial \phi / \partial x$ is in order. For example, if $\partial \phi / \partial x > 0$ for $a \leq x \leq b$, then the heat flux ϕ is an increasing function of x. The heat is flowing greater to the right at $x = b$ than at $x = a$ (assuming that $b > a$). Thus (neglecting any effects of sources Q), the heat energy must decrease between $x = a$ and $x = b$, resulting in the minus sign in (1.2.5).

Temperature and specific heat. We usually describe materials by their temperature,

---

[1] This is one of the fundamental theorems of calculus.

[2] Most proofs of this result are inelegant. Suppose that $f(x)$ is continuous and $\int_a^b f(x)dx = 0$ for arbitrary a and b. We wish to prove $f(x) = 0$ for all x. We can prove this by assuming that there exists a point x_0 such that $f(x_0) \neq 0$ and demonstrating a contradiction. If $f(x_0) \neq 0$ and $f(x)$ is continuous, then there exists some region near x_0 in which $f(x)$ is of one sign. Pick a and b to be in this region, and hence $\int_a^b f(x)dx \neq 0$ since $f(x)$ is of one sign throughout. This contradicts the statement that $\int_a^b f(x)dx = 0$, and hence it is impossible for $f(x_0) \neq 0$. Equation (1.2.5) follows.

$$u(x, t) = \text{temperature,}$$

not their thermal energy density. Distinguishing between the concepts of temperature and thermal energy is not necessarily a trivial task. Only in the mid-1700s did the existence of accurate experimental apparatus enable physicists to recognize that it may take different amounts of thermal energy to raise two different materials from one temperature to another larger temperature. This necessitates the introduction of the **specific heat** (or heat capacity):

$$c \quad = \quad \text{specific heat (the heat energy that must be supplied to a unit mass of a substance to raise its temperature one unit).}$$

In general, from experiments (and our definition) the specific heat c of a material depends on the temperature u. For example, the thermal energy necessary to raise a unit mass from $0°C$ to $1°C$ could be different from that needed to raise the mass from $85°C$ to $86°C$ for the same substance. Heat flow problems with the specific heat depending on the temperature are mathematically quite complicated. (Exercise 1.2.1 briefly discusses this situation.) Often for restricted temperature intervals, the specific heat is approximately independent of the temperature. However, experiments suggest that different materials require different amounts of thermal energy to heat up. Since we would like to formulate the correct equation in situations in which the composition of our one-dimensional rod might vary from position to position, the specific heat will depend on x, $c = c(x)$. In many problems the rod is made of one material (a uniform rod), in which case we will let the specific heat c be a constant. In fact, most of the solved problems in this text (as well as other books) correspond to this approximation, c constant.

Thermal energy. The thermal energy in a thin slice is $e(x,t)A \, \Delta x$. However, it is also defined as the energy it takes to raise the temperature from a reference temperature $0°$ to its actual temperature $u(x,t)$. Since the specific heat is independent of temperature, the heat energy per unit mass is just $c(x)u(x,t)$. We thus need to introduce the **mass density** $\rho(x)$:

$$\rho(x) = \text{mass density (mass per unit volume),}$$

allowing it to vary with x, possibly due to the rod being composed of nonuniform material. The total mass of the thin slice is $\rho A \, \Delta x$. The total thermal energy in any thin slice is thus $c(x)u(x,t) \cdot \rho A \, \Delta x$, so that

$$e(x,t)A \, \Delta x = c(x)u(x,t)\rho A \, \Delta x.$$

In this way we have explained the basic relationship between thermal energy and temperature:

$$e(x, t) = c(x)\rho(x)u(x, t). \tag{1.2.6}$$

This states that the thermal energy per unit volume equals the thermal energy per unit mass per unit degree times the temperature times the mass density (mass per unit volume). When the thermal energy density is eliminated using (1.2.6), conservation of thermal energy, (1.2.3) or (1.2.5), becomes

$$c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q. \tag{1.2.7}$$

Fourier's law. Usually, (1.2.7) is regarded as one equation in two unknowns: the temperature $u(x, t)$ and the heat flux (flow per unit surface area per unit time) $\phi(x, t)$. How and why does heat energy flow? In other words, we need an expression for the dependence of the flow of heat energy on the temperature field. First we summarize certain qualitative properties of heat flow with which we are all familiar:

1. If the temperature is constant in a region, no heat energy flows.

2. If there are temperature differences, the heat energy flows from the hotter region to the colder region.

3. The greater the temperature differences (for the same material), the greater is the flow of heat energy.

4. The flow of heat energy will vary for different materials, even with the same temperature differences.

Fourier (1768 – 1830) recognized properties 1 through 4 and summarized them (as well as numerous experiments) by the formula

$$\phi = -K_0 \frac{\partial u}{\partial x}, \tag{1.2.8}$$

known as **Fourier's law of heat conduction.** Here $\partial u/\partial x$ is the derivative of the temperature; it is the slope of the temperature (as a function of x for fixed t); it represents temperature differences (per unit length). Equation (1.2.8) states that the heat flux is proportional to the temperature difference (per unit length). If the temperature u increases as x increases (i.e., the temperature is hotter to the right), $\partial u/\partial x > 0$, then we know (property 2) that heat energy flows to the left. This explains the minus sign in (1.2.8).

We designate the coefficient of proportionality K_0. It measures the ability of the material to conduct heat and is called the **thermal conductivity**. Experiments indicate that different materials conduct heat differently; K_0 depends on the particular material. The larger K_0 is, the greater the flow of heat energy with the same temperature differences. A material with a low value of K_0 would be a poor conductor of heat energy (and ideally suited for home insulation). For a rod composed of different materials, K_0 will be a function of x. Furthermore, experiments show that the ability to conduct heat for most materials is different at different temperatures, $K_0(x, u)$. However, just as with the specific heat c, the dependence on the temperature is often not important in particular problems. Thus, throughout this text we will assume that the thermal conductivity K_0 only depends on x, $K_0(x)$. Usually, in fact, we will discuss uniform rods in which K_0 is a constant.

Heat equation. If Fourier's law, (1.2.8), is substituted into the conservation of heat energy equation, (1.2.7), a partial differential equation results:

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q. \qquad (1.2.9)$$

We usually think of the sources of heat energy Q as being given, and the only unknown being the temperature $u(x, t)$. The thermal coefficients c, ρ, K_0 all depend on the material and hence may be functions of x. In the special case of a uniform rod, in which c, ρ, K_0 are all constants, the partial differential equation (1.2.9) becomes

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q.$$

If, in addition, there are no sources, $Q = 0$, then after dividing by the constant $c\rho$, the partial differential equation becomes

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.} \qquad (1.2.10)$$

where the constant k,

$$k = \frac{K_0}{c\rho}$$

is called the **thermal diffusivity**, the thermal conductivity divided by the product of the specific heat and mass density. Equation (1.2.10) is often called **the heat equation**; it corresponds to no sources and constant thermal properties. If heat energy is initially concentrated in one place, (1.2.10) will describe how the heat energy spreads out, a physical process known as **diffusion**. Other physical quantities besides temperature smooth out in much the same manner, satisfying the same partial differential equation (1.2.10). For this reason (1.2.10) is also known as the **diffusion equation**. For example, the concentration $u(x, t)$ of chemicals

(such as perfumes and pollutants) satisfies the diffusion equation (1.2.8) in certain one-dimensional situations.

Initial conditions. The partial differential equations describing the flow of heat energy, (1.2.9) or (1.2.10), have one time derivative. When an ordinary differential equation has one derivative, the initial value problem consists of solving the differential equation with one initial condition. Newton's law of motion for the position x of a particle yields a second-order ordinary differential equation, $md^2x/dt^2 = $ forces. It involves second derivatives. The initial value problem consists of solving the differential equation with two initial conditions, the initial position x and the initial velocity dx/dt. From these pieces of information (including the knowledge of the forces), by solving the differential equation with the initial conditions, we can predict the future motion of a particle in the x-direction. We wish to do the same process for our partial differential equation, that is, predict the future temperature. Since the heat equations have one time derivative, we must be given one **initial condition** (IC) (usually at $t = 0$), the initial temperature. It is possible that the initial temperature is not constant, but depends on x. Thus, we must be given the initial temperature distribution,

$$u(x, 0) = f(x).$$

Is this enough information to predict the future temperature? We know the initial temperature distribution and that the temperature changes according to the partial differential equation (1.2.9) or (1.2.10). However, we need to know that happens at the two boundaries, $x = 0$ and $x = L$. Without knowing this information, we cannot predict the future. Two conditions are needed corresponding to the second spatial derivatives present in (1.2.9) or (1.2.10), usually one condition at each end. We discuss these boundary conditions in the next section.

EXERCISES 1.2

1.2.1. Suppose that the specific heat is a function of position and temperature, $c(x, u)$.

   (a) Show that the heat energy per unit mass necessary to raise the temperature of a thin slice of thickness Δx from $0°$ to $u(x, t)$ is not $c(x)u(x, t)$, but instead $\int_0^u c(x, \bar{u})d\bar{u}$.

   (b) Rederive the heat equation in this case. Show that (1.2.3) remains unchanged.

1.2.2. Consider conservation of thermal energy (1.2.4) for any segment of a one-dimensional rod $a \leq x \leq b$. By using the fundamental theorem of calculus

$$\frac{\partial}{\partial b}\int_a^b f(x)dx = f(b),$$

derive the heat equation (1.2.9).

✓ *1.2.3. If $u(x,t)$ is known, give an expression for the total thermal energy contained in a rod $(0 < x < L)$.

1.2.4. Consider a thin one-dimensional rod without sources of thermal energy whose lateral surface area is not insulated.

(a) Assume that the heat energy flowing out of the lateral sides per unit surface area per unit time is $w(x,t)$. Derive the partial differential equation for the temperature $u(x,t)$.

(b) Assume that $w(x,t)$ is proportional to the temperature difference between the rod $u(x,t)$ and a known outside temperature $\gamma(x,t)$. Derive that

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}[u(x,t) - \gamma(x,t)]h(x), \qquad (1.2.11)$$

where $h(x)$ is a positive x-dependent proportionality, P is the lateral perimeter, and A is the cross-sectional area.

(c) Compare (1.2.11) to the equation for a one-dimensional rod whose lateral surfaces are insulated, but without heat sources.

(d) Specialize (1.2.11) to a rod of circular cross section with constant thermal properties and $0°$ outside temperature.

*(e) Consider the assumptions in part (d). Suppose that the temperature in the rod is uniform [i.e., $u(x,t) = u(t)$]. Determine $u(t)$ if initially $u(0) = u_0$.

1.3   Boundary Conditions

In solving the heat equation, either (1.2.9) or (1.2.10), one **boundary condition** (BC) is needed at each end of the rod. The appropriate condition depends on the physical mechanism in effect at each end. Often the condition at the boundary depends on both the material inside and outside the rod. To avoid a more difficult mathematical problem, we will assume that the outside environment is known, not significantly altered by the rod.

Prescribed temperature. In certain situations, the temperature of the end of the rod, for example $x = 0$, may be approximated by a **prescribed temperature**,

$$u(0,t) = u_B(t), \qquad (1.3.1)$$

where $u_B(t)$ is the temperature of a fluid bath (or reservoir) with which the rod is in contact.

Insulated boundary. In other situations it is possible to prescribe the heat flow rather than the temperature,

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = \phi(t), \qquad (1.3.2)$$

where $\phi(t)$ is given. This is equivalent to giving one condition for the first derivative, $\partial u/\partial x$, at $x = 0$. The slope is given at $x = 0$. Equation (1.3.2) *cannot* be integrated in x because the slope is known only at one value of x. The simplest example of the prescribed heat flow boundary condition is when an end is **perfectly insulated** (sometimes we omit the "perfectly"). In this case there is no heat flow at the boundary. If $x = 0$ is insulated, then

$$\frac{\partial u}{\partial x}(0,t) = 0. \qquad (1.3.3)$$

Newton's law of cooling. When a one-dimensional rod is in contact at the boundary with a moving fluid (e.g., air), then neither the prescribed temperature nor the prescribed heat flow may be appropriate. For example, let us imagine a very warm rod in contact with cooler moving air. Heat will leave the rod, heating up the air. The air will then carry the heat away. This process of heat transfer is called **convection**. However, the air will be hotter near the rod. Again, this is a complicated problem; the air temperature will actually vary with distance from the rod (ranging between the bath and rod temperatures). Experiments show that, as a good approximation, the heat flow leaving the rod is proportional to the temperature difference between the bar and the prescribed external temperature. This boundary condition is called **Newton's law of cooling**. If it is valid at $x = 0$, then

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = -H[u(0,t) - u_B(t)]. \qquad (1.3.4)$$

where the proportionality constant H is called the **heat transfer coefficient** (or the convection coefficient). This boundary condition[3] involves a linear combination of u and $\partial u/\partial x$. We must be careful with the sign of proportionality. If the rod is hotter than the bath $[u(0,t) > u_B(t)]$, then usually heat flows out of the rod at $x = 0$. Thus, heat is flowing to the left, and in this case the heat flow would be negative. That is why we introduced a minus sign in (1.3.4) (with $H > 0$). The same conclusion would have been reached had we assumed that $u(0,t) < u_B(t)$. Another way to understand the signs in (1.3.4) is to again assume that $u(0,t) > u_B(t)$. The temperature is hotter to the right at $x = 0$ and we should expect the temperature to continue to increase to the right. Thus, $\partial u/\partial x$ should be positive at $x = 0$. Equation (1.3.4) is consistent with this argument. In Exercise 1.3.1 you are asked to derive, in the same manner, that the equation for Newton's law of cooling at a right end point $x = L$ is

$$-K_0(L)\frac{\partial u}{\partial x}(L,t) = H[u(L,t) - u_B(t)], \qquad (1.3.5)$$

where $u_B(t)$ is the external temperature at $x = L$. We immediately note the significant sign difference between the left boundary (1.3.4) and the right boundary (1.3.5).

The coefficient H in Newton's law of cooling is experimentally determined. It depends on properties of the rod as well as fluid properties (including the fluid

[3]For another situation in which (1.3.4) is valid, see Berg and McGregor [1966].

velocity). If the coefficient is very small, then very little heat energy flows across the boundary. In the limit as $H \rightarrow 0$, Newton's law of cooling approaches the insulated boundary condition. We can think of Newton's law of cooling for $H \neq 0$ as representing an imperfectly insulated boundary. If $H \rightarrow \infty$, the boundary condition approaches the one for prescribed temperature, $u(0,t) = u_B(t)$. This is most easily seen by dividing (1.3.4), for example, by H:

$$-\frac{K_0(0)}{H}\frac{\partial u}{\partial x}(0,t) = -[u(0,t) - u_B(t)].$$

Thus, $H \rightarrow \infty$ corresponds to no insulation at all.

Summary. We have described three different kinds of boundary conditions. For example, at $x = 0$:

$$
\begin{array}{llll}
u(0,t) & = & u_B(t) & \text{prescribed temperature} \\
-K_0(0)\frac{\partial u}{\partial x}(0,t) & = & \phi(t) & \text{prescribed heat flux} \\
-K_0(0)\frac{\partial u}{\partial x}(0,t) & = & -H[u(0,t) - u_B(t)] & \text{Newton's law of cooling}
\end{array}
$$

These same conditions could hold at $x = L$, noting that the change of sign ($-H$ becoming H) is necessary for Newton's law of cooling. One boundary condition occurs at each boundary. It is not necessary that both boundaries satisfy the same kind of boundary condition. For example, it is possible for $x = 0$ to have a prescribed oscillating temperature

$$u(0,t) = 100 - 25\cos t,$$

and for the right end, $x = L$, to be insulated,

$$\frac{\partial u}{\partial x}(L,t) = 0.$$

EXERCISES 1.3

1.3.1. Consider a one-dimensional rod, $0 \leq x \leq L$. Assume that the heat energy flowing out of the rod at $x = L$ is proportional to the temperature difference between the end temperature of the bar and the known external temperature. Derive (1.3.5) (briefly, physically explain why $H > 0$).

*1.3.2. Two one-dimensional rods of different materials joined at $x = x_0$ are said to be in **perfect thermal contact** if the temperature is continuous at $x = x_0$:

$$u(x_0-,t) = u(x_0+,t)$$

and no heat energy is lost at $x = x_0$ (i.e., the heat energy flowing out of one flows into the other). What mathematical equation represents the latter condition at $x = x_0$? Under what special condition is $\partial u/\partial x$ continuous at $x = x_0$?

*1.3.3. Consider a bath containing a fluid of specific heat c_f and mass density ρ_f which surrounds the end $x = L$ of a one-dimensional rod. Suppose that the bath is rapidly stirred in a manner such that the bath temperature is approximately uniform throughout, equaling the temperature at $x = L, u(L,t)$. Assume that the bath is thermally insulated except at its perfect thermal contact with the rod, where the bath may be heated or cooled by the rod. Determine an equation for the temperature in the bath. (This will be a boundary condition at the end $x = L$.) (*Hint:* See Exercise 1.3.2.)

1.4 Equilibrium Temperature Distribution

1.4.1 Prescribed Temperature

Let us now formulate a simple, but typical, problem of heat flow. If the thermal coefficients are constant and there are no sources of thermal energy, then the temperature $u(x,t)$ in a one-dimensional rod $0 \leq x \leq L$ satisfies

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}. \tag{1.4.1}$$

The solution of this partial differential equation must satisfy the initial condition

$$u(x,0) = f(x) \tag{1.4.2}$$

and one boundary condition at each end. For example, each end might be in contact with different large baths, such that the temperature at each end is prescribed

$$
\begin{array}{lll}
u(0,t) & = & T_1(t) \\
U(L,t) & = & T_2(t).
\end{array}
\tag{1.4.3}
$$

One aim of this text is to enable the reader to solve the problem specified by (1.4.1 – 1.4.3).

Equilibrium temperature distribution. Before we begin to attack such an initial and boundary value problem for partial differential equations, we discuss a physically related question for ordinary differential equations. Suppose that the boundary conditions at $x = 0$ and $x = L$ were **steady** (i.e., independent of time),

$$u(0,t) = T_1 \qquad \text{and} \qquad u(L,t) = T_2,$$

where T_1 and T_2 are given constants. We define an **equilibrium** or **steady-state** solution to be a temperature distribution that does not depend on time, that is, $u(x,t) = u(x)$. Since $\partial/\partial t\, u(x) = 0$, the partial differential equation becomes $k(\partial^2 u/\partial x^2) = 0$, but partial derivatives are not necessary, and thus

$$\boxed{\frac{d^2u}{dx^2} = 0.} \tag{1.4.4}$$

The boundary conditions are

$$\boxed{\begin{aligned} u(0) &= T_1 \\ u(L) &= T_2. \end{aligned}} \tag{1.4.5}$$

In doing steady-state calculations, the initial conditions are usually ignored. Equation (1.4.4) is a rather trivial second-order ordinary differential equation. Its general solution may be obtained by integrating twice. Integrating (1.4.4) yields $du/dx = C_1$, and integrating a second time shows that

$$u(x) = C_1 x + C_2. \tag{1.4.6}$$

We recognize (1.4.6) as the general equation of a straight line. Thus, from the boundary conditions (1.4.5) the equilibrium temperature distribution is the straight line that equals T_1 at $x = 0$ and T_2 at $x = L$, as sketched in Fig. 1.4.1. Geometrically there is a unique equilibrium solution for this problem. Algebraically, we can determine the two arbitrary constants, C_1 and C_2, by applying the boundary conditions, $u(0) = T_1$ and $u(L) = T_2$:

$$\begin{aligned} u(0) &= T_1 \quad \text{implies} \quad T_1 = C_2 \\ u(L) &= T_2 \quad \text{implies} \quad T_2 = C_1 L + C_2. \end{aligned} \tag{1.4.7}$$

It is easy to solve (1.4.7) for the constants $C_2 = T_1$ and $C_1 = (T_2 - T_1)/L$. Thus, the unique equilibrium solution for the steady-state heat equation with these fixed boundary conditions is

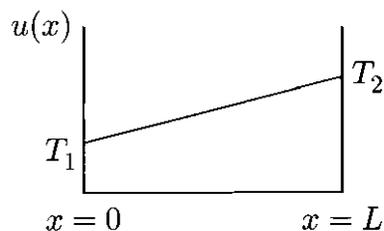$$\boxed{u(x) = T_1 + \frac{T_2 - T_1}{L} x.} \tag{1.4.8}$$

Figure 1.4.1: Equilibrium temperature distribution.

Approach to equilibrium. For the time-dependent problem, (1.4.1) and (1.4.2), with steady boundary conditions (1.4.5), we expect the temperature distribution $u(x, t)$ to change in time; it will not remain equal to its initial distribution $f(x)$. If we wait a very, very long time, we would imagine that the influence of the two ends should dominate. The initial conditions are usually forgotten. Eventually, the temperature is physically expected to approach the equilibrium temperature

distribution, since the boundary conditions are independent of time:

$$\lim_{t \to \infty} u(x, t) = u(x) = T_1 + \frac{T_2 - T_1}{L} x. \tag{1.4.9}$$

In Sec. 7.2 we will solve the time-dependent problem and show that (1.4.9) is satisfied. However, if a steady state is approached, it is more easily obtained by directly solving the equilibrium problem.

1.4.2 Insulated Boundaries

As a second example of a steady-state calculation, we consider a one-dimensional rod again with no sources and with constant thermal properties, but this time with insulated boundaries at $x = 0$ and $x = L$. The formulation of the time-dependent problem is

$$\text{PDE:} \qquad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \tag{1.4.10}$$

$$\text{IC:} \qquad u(x, 0) = f(x) \tag{1.4.11}$$

$$\text{BC1:} \qquad \frac{\partial u}{\partial x}(0, t) = 0 \tag{1.4.12}$$

$$\text{BC2:} \qquad \frac{\partial u}{\partial x}(L, t) = 0. \tag{1.4.13}$$

The equilibrium problem is derived by setting $\partial/\partial t = 0$. The equilibrium temperature distribution satisfies

$$\text{ODE:} \quad \boxed{\frac{d^2 u}{dx^2} = 0} \tag{1.4.14}$$

$$\text{BC1:} \quad \boxed{\frac{du}{dx}(0) = 0} \tag{1.4.15}$$

$$\text{BC2:} \quad \boxed{\frac{du}{dx}(L) = 0,} \tag{1.4.16}$$

where the initial condition is neglected (for the moment). The general solution of $d^2 u/dx^2 = 0$ is again an arbitrary straight line,

$$u = C_1 x + C_2. \tag{1.4.17}$$

The boundary conditions imply that the slope must be zero at both ends. Geometrically, any straight line that is flat (zero slope) will satisfy (1.4.15, 1.4.16) as
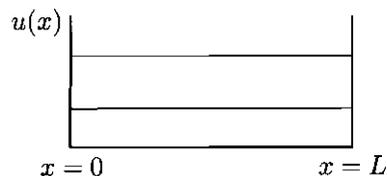
$u(x)$

Figure 1.4.2: Various constant equilibrium temperature distributions (with insulated ends).

$x = 0$ $x = L$

illustrated in Fig. 1.4.2. The solution is any constant temperature. Algebraically, from (1.4.17), $du/dx = C_1$ and both boundary conditions imply $C_1 = 0$. Thus,

$$u(x) = C_2 \tag{1.4.18}$$

for any constant C_2. Unlike the first example (with fixed temperatures at both ends), here there is not a unique equilibrium temperature. Any constant temperature is an equilibrium temperature distribution for insulated boundary conditions. Thus, for the time-dependent initial value problem, we expect

$$\lim_{t \to \infty} u(x, t) = C_2;$$

if we wait long enough a rod with insulated ends should approach a constant temperature. This seems physically quite reasonable. However, it does not make sense that the solution should approach an arbitrary constant; we ought to know what constant it approaches. In this case, the lack of uniqueness was caused by the complete neglect of the initial condition. In general, the equilibrium solution will not satisfy the initial condition. However, the particular constant equilibrium solution is determined by considering the initial condition for the time-dependent problem (1.4.11). Since both ends are insulated, the total thermal energy is constant. This follows from the integral conservation of thermal energy of the entire rod [see (1.2.4)]:

$$\frac{d}{dt} \int_0^L c\rho u \, dx = -K_0 \frac{\partial u}{\partial x}(0, t) + K_0 \frac{\partial u}{\partial x}(L, t). \tag{1.4.19}$$

Since both ends are insulated,

$$\int_0^L c\rho u \, dx = \text{ constant.} \tag{1.4.20}$$

One implication of (1.4.20) is that the initial thermal energy must equal the final ($\lim_{t \to \infty}$) thermal energy. The initial thermal energy is $c\rho \int_0^L f(x)dx$ since $u(x, 0) = f(x)$, while the equilibrium thermal energy is $c\rho \int_0^L C_2 dx = c\rho C_2 L$ since the equilibrium temperature distribution is a constant $u(x, t) = C_2$. The constant C_2 is determined by equating these two expressions for the constant total thermal energy, $c\rho \int_0^L f(x)dx = c\rho C_2 L$. Solving for C_2 shows that the desired unique steady-state solution should be

$$\boxed{u(x) = C_2 = \frac{1}{L} \int_0^L f(x) \, dx,} \tag{1.4.21}$$

the **average** of the initial temperature distribution. It is as though the initial condition is not entirely forgotten. Later we will find a $u(x, t)$ that satisfies (1.4.10 – 1.4.13) and show that $\lim_{t \to \infty} u(x, t)$ is given by (1.4.21).

EXERCISES 1.4

1.4.1. Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:

*(a)	$Q = 0$,	$u(0) = 0$,	$u(L) = T$
(b)	$Q = 0$,	$u(0) = T$,	$u(L) = 0$
(c)	$Q = 0$,	$\frac{\partial u}{\partial x}(0) = 0$,	$u(L) = T$
*(d)	$Q = 0$,	$u(0) = T$,	$\frac{\partial u}{\partial x}(L) = \alpha$
(e)	$\frac{Q}{K_0} = 1$,	$u(0) = T_1$,	$u(L) = T_2$
*(f)	$\frac{Q}{K_0} = x^2$,	$u(0) = T$,	$\frac{\partial u}{\partial x}(L) = 0$
(g)	$Q = 0$,	$u(0) = T$,	$\frac{\partial u}{\partial x}(L) + u(L) = 0$
*(h)	$Q = 0$,	$\frac{\partial u}{\partial x}(0) - [u(0) - T] = 0$,	$\frac{\partial u}{\partial x}(L) = \alpha$

In these you may assume that $u(x, 0) = f(x)$.

1.4.2. Consider the equilibrium temperature distribution for a uniform one-dimensional rod with sources $Q/K_0 = x$ of thermal energy, subject to the boundary conditions $u(0) = 0$ and $u(L) = 0$.

*(a) Determine the heat energy generated per unit time inside the entire rod.

(b) Determine the heat energy flowing out of the rod per unit time at $x = 0$ and at $x = L$.

(c) What relationships should exist between the answers in parts (a) and (b)?

1.4.3. Determine the equilibrium temperature distribution for a one-dimensional rod composed of two different materials in perfect thermal contact at $x = 1$. For $0 < x < 1$, there is one material ($c\rho = 1, K_0 = 1$) with a constant source ($Q = 1$), whereas for the other $1 < x < 2$ there are no sources ($Q = 0, c\rho = 2, K_0 = 2$) (see Exercise 1.3.2) with $u(0) = 0$ and $u(2) = 0$.

1.4.4. If both ends of a rod are insulated, derive *from the partial differential equation* that the total thermal energy in the rod is constant.

✓ **1.4.5.** Consider a one-dimensional rod $0 \le x \le L$ of known length and known constant thermal properties without sources. Suppose that the temperature is an *unknown* constant T at $x = L$. Determine T if we know (in the steady state) both the temperature and the heat flow at $x = 0$.

1.4.6. The two ends of a uniform rod of length L are insulated. There is a constant source of thermal energy $Q_0 \ne 0$ and the temperature is initially $u(x, 0) = f(x)$.

    (a) Show mathematically that there does not exist any equilibrium temperature distribution. Briefly explain physically.

    (b) Calculate the total thermal energy in the entire rod.

✓ **1.4.7.** For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

    * (a) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 1, \qquad u(x, 0) = f(x), \qquad \dfrac{\partial u}{\partial x}(0, t) = 1, \qquad \dfrac{\partial u}{\partial x}(L, t) = \beta$

    (b) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \qquad u(x, 0) = f(x), \qquad \dfrac{\partial u}{\partial x}(0, t) = 1, \qquad \dfrac{\partial u}{\partial x}(L, t) = \beta$

    (c) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + x - \beta, \; u(x, 0) = f(x), \quad \dfrac{\partial u}{\partial x}(0, t) = 0, \quad \dfrac{\partial u}{\partial x}(L, t) = 0$

1.5   Derivation of the Heat Equation in Two or Three Dimensions

Introduction. In Sec. 1.2 we showed that for the conduction of heat in a one-dimensional rod the temperature $u(x, t)$ satisfies

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q.$$

In cases in which there are no sources $(Q = 0)$ and the thermal properties are constant, the partial differential equation becomes

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

where $k = K_0/c\rho$. Before we solve problems involving these partial differential equations, we will formulate partial differential equations corresponding to heat flow problems in two or three spatial dimensions. We will find the derivation to be similar to the one used for one-dimensional problems, although important differences will emerge. We propose to derive new and more complex equations (before solving the

simpler ones) so that when we do discuss *techniques* for the solutions of PDEs, we will have more than one example to work with.

Heat energy. We begin our derivation by considering any *arbitrary subregion* R as illustrated in Fig. 1.5.1. As in the one-dimensional case, conservation of heat energy is summarized by the following word equation:

rate of change of heat energy	=	heat energy flowing across the boundaries per unit time	+	heat energy generated inside per unit time,

where the heat energy within an arbitrary subregion R is

$$\text{heat energy} = \iiint_R c\rho u \, dV,$$

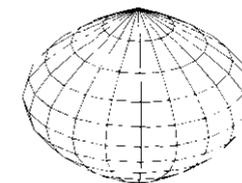instead of the one-dimensional integral used in Sec. 1.2.

Figure 1.5.1: three-dimensional subregion R.

Heat flux vector and normal vectors. We need an expression for the flow of heat energy. In a one-dimensional problem the heat flux ϕ is defined to the right ($\phi < 0$ means flowing to the left). In a three-dimensional problem the heat flows in some direction, and hence the **heat flux is a vector** $\boldsymbol{\phi}$. The magnitude of $\boldsymbol{\phi}$ is the amount of heat energy flowing per unit time per unit surface area. However, in considering conservation of heat energy it is only the heat flowing *across the boundaries* per unit time that is important. If, as at point A in Fig. 1.5.2, the heat flow is parallel to the boundary, then there is no heat energy *crossing* the boundary at that point. In fact, it is only the normal component of the heat flow that contributes (as illustrated by point B in Fig. 1.5.2). At any point there are two normal vectors, an inward and an outward normal $\boldsymbol{n}$. We will use the convention of only utilizing the **unit outward normal vector** $\hat{\boldsymbol{n}}$ (where the ^ stands for a unit vector).

Conservation of heat energy. At each point the amount of heat energy flowing *out* of the region R per unit time per unit surface area is the outward normal component of the heat flux vector. From Fig. 1.5.2 at point B, the outward normal component of the heat flux vector is $|\boldsymbol{\phi}| \cos\theta = \boldsymbol{\phi} \cdot \boldsymbol{n}/|\boldsymbol{n}| = \boldsymbol{\phi} \cdot \hat{\boldsymbol{n}}$. If the heat flux vector $\boldsymbol{\phi}$ is directed inward, then $\boldsymbol{\phi} \cdot \hat{\boldsymbol{n}} < 0$ and the outward flow of heat energy is negative. To calculate the total heat energy flowing out of R per unit time, we
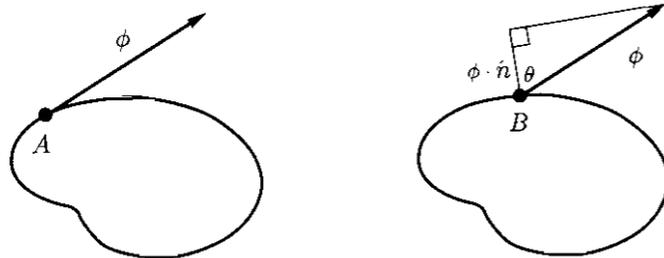
Figure 1.5.2: Outward normal component of heat flux vector.

must multiply $\phi \cdot \hat{n}$ by the differential surface area dS and "sum" over the entire surface that encloses the region R. This[4] is indicated by the closed surface integral $\oint \phi \cdot \hat{n} \, dS$. This is the amount of heat energy (per unit time) leaving the region R and (if positive) results in a decreasing of the total heat energy within R. If Q is the rate of heat energy generated per unit volume, then the total heat energy generated per unit time is $\iiint_R Q \, dV$. Consequently, conservation of heat energy for an arbitrary three-dimensional region R becomes

$$\frac{d}{dt} \iiint_R c\rho u \, dV = -\oiint \phi \cdot \hat{n} \, dS + \iiint_R Q \, dV. \qquad (1.5.1)$$

Divergence theorem. In one dimension, a way in which we derived a partial differential relationship from the integral conservation law was to notice (via the fundamental theorem of calculus) that

$$\phi(a) - \phi(b) = -\int_a^b \frac{\partial \phi}{\partial x} \, dx;$$

that is, the flow through the boundaries can be expressed as an integral over the entire region for one-dimensional problems. We claim that the divergence theorem is an analogous procedure for functions of three variables. The divergence theorem deals with a vector A (with components A_x, A_y and A_z; i.e., $A = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$) and its divergence defined as follows:

$$\nabla \cdot A \equiv \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z. \qquad (1.5.2)$$

Note that the divergence of a vector is a scalar. **The divergence theorem states that the volume integral of the divergence of any continuously differentiable vector A is the closed surface integral of the outward normal component of A :**

---

[4]Sometimes the notation ϕ_n is used instead of $\phi \cdot \hat{n}$, meaning the outward normal component of ϕ.

$$\iiint_R \nabla \cdot A \, dV = \oiint A \cdot \hat{n} \, dS. \qquad (1.5.3)$$

This is also known as Gauss's theorem. It can be used to relate certain surface integrals to volume integrals, and vice versa. It is very important and very useful (both immediately and later in this text). We omit a derivation, which may be based on repeating the one-dimensional fundamental theorem in all three dimensions.

Application of the divergence theorem to heat flow. In particular, the closed surface integral that arises in the conservation of heat energy (1.5.1), corresponding to the heat energy flowing across the boundary per unit time, can be written as a volume integral according to the divergence theorem, (1.5.3). Thus, (1.5.1) becomes

$$\frac{d}{dt} \iiint_R c\rho u \, dV = -\iiint_R \nabla \cdot \phi \, dV + \iiint_R Q \, dV. \qquad (1.5.4)$$

We note that the time derivative in (1.5.4) can be put inside the integral (since R is fixed in space) if the time derivative is changed to a partial derivative. Thus, all the expressions in (1.5.4) are volume integrals over the same volume, and they can be combined into one integral:

$$\iiint_R \left[c\rho \frac{\partial u}{\partial t} + \nabla \cdot \phi - Q \right] dV = 0. \qquad (1.5.5)$$

Since this integral is zero for all regions R, it follows (as it did for one-dimensional integrals) that the integrand itself must be zero:

$$c\rho \frac{\partial u}{\partial t} + \nabla \cdot \phi - Q = 0$$

or, equivalently,

$$c\rho \frac{\partial u}{\partial t} = -\nabla \cdot \phi + Q. \qquad (1.5.6)$$

Equation (1.5.6) reduces to (1.2.3) in the one-dimensional case.

Fourier's law of heat conduction. In one-dimensional problems, from experiments according to Fourier's law, the heat flux ϕ is proportional to the derivative of the temperature, $\phi = -K_0 \partial u / \partial x$. The minus sign is related to the fact that thermal energy flows from hot to cold. $\partial u / \partial x$ is the change in temperature per unit length. These same ideas are valid in three dimensions. In an appendix, we derive that the heat flux vector ϕ is proportional to the temperature gradient $\left(\nabla u \equiv \frac{\partial u}{\partial x} \hat{i} + \frac{\partial u}{\partial y} \hat{j} + \frac{\partial u}{\partial z} \hat{k} \right)$:

$$\boxed{\phi = -K_0 \nabla u,} \tag{1.5.7}$$

known as **Fourier's law of heat conduction**, where again K_0 is called the thermal conductivity. Thus, in three dimensions the gradient ∇u replaces $\partial u/\partial x$.

Heat equation. When the heat flux vector, (1.5.7), is substituted into the conservation of heat energy equation, (1.5.6), a partial differential equation for the temperature results:

$$\boxed{c\rho \frac{\partial u}{\partial t} = \nabla\cdot(K_0 \nabla u) + Q.} \tag{1.5.8}$$

In the cases in which there are no sources of heat energy ($Q = 0$) and the thermal coefficients are constant, (1.5.8) becomes

$$\frac{\partial u}{\partial t} = k\nabla\cdot(\nabla u), \tag{1.5.9}$$

where $k = K_0/c\rho$ is again called the thermal diffusivity. From their definitions, we calculate the divergence of the gradient of u:

$$\nabla\cdot(\nabla u) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \equiv \nabla^2 u.} \tag{1.5.10}$$

This expression $\nabla^2 u$ is defined to be the **Laplacian** of u. Thus, in this case

$$\boxed{\frac{\partial u}{\partial t} = k\nabla^2 u.} \tag{1.5.11}$$

Equation (1.5.11) is often known as the **heat** or **diffusion equation** in three spatial dimensions. The notation $\nabla^2 u$ is often used to emphasize the role of the del operator ∇:

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

Note that ∇u is ∇ operating on u, while $\nabla\cdot A$ is the vector dot product of del with A. Furthermore, ∇^2 is the dot product of the del operator with itself or

$$\nabla\cdot\nabla = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\right)$$

operating on u, hence the notation del squared, ∇^2.

Initial boundary value problem. In addition to (1.5.8) or (1.5.11), the temperature satisfies a given initial distribution.

$$u(x, y, z, 0) = f(x, y, z).$$

The temperature also satisfies a boundary condition at every point on the surface that encloses the region of interest. The boundary condition can be of various types (as in the one-dimensional problem). The temperature could be prescribed,

$$u(x, y, z, t) = T(x, y, z, t),$$

everywhere *on the boundary* where T is a known function of t at each point of the boundary. It is also possible that the flow across the boundary is prescribed. Frequently, we might have the boundary (or part of the boundary) **insulated**. This means that there is no heat flow across that portion of the boundary. Since the heat flux vector is $-K_0\nabla u$, the heat flowing out will be the unit outward normal component of the heat flow vector, $-K_0\nabla u\cdot\hat{n}$, where $\hat{n}$ is a unit outward normal to the boundary surface. Thus, at an insulated surface,

$$\nabla u\cdot\hat{n} = 0.$$

Recall that $\nabla u\cdot\hat{n}$ is the directional derivative of u in the outward normal direction; it is also called the normal derivative.[5]

Often Newton's law of cooling is a more realistic condition at the boundary. It states that the heat energy flowing out per unit time per unit surface area is proportional to the difference between the temperature at the surface u and the temperature outside the surface u_b. Thus, if Newton's law of cooling is valid, then at the boundary

$$-K_0\nabla u\cdot\hat{n} = H(u - u_b). \tag{1.5.12}$$

Note that usually the proportionality constant $H > 0$, since if $u > u_b$, then we expect that heat energy will flow out and $-K_0\nabla u\cdot\hat{n}$ will be greater than zero. Equation (1.5.12) verifies the two forms of Newton's law of cooling for one-dimensional problems. In particular, at $x = 0, \hat{n} = -\hat{i}$ and the left-hand side (l.h.s.) of (1.5.12) becomes $K_0\partial u/\partial x$, while at $x = L$, $\hat{n} = \hat{i}$ and the l.h.s. of (1.5.12) becomes $-K_0\partial u/\partial x$ [see (1.3.4) and (1.3.5)].

Steady state. If the boundary conditions and any sources of thermal energy are independent of time, it is possible that there exist steady-state solutions to the heat equation satisfying the given steady boundary condition:

$$0 = \nabla\cdot(K_0\nabla u) + Q.$$

Note that an equilibrium temperature distribution $u(x, y, z)$ satisfies a partial differential equation when more than one spatial dimension is involved. In the case with

---

[5]Sometimes (in other books and references) the notation $\partial u/\partial n$ is used. However, to calculate $\partial u/\partial n$ one usually calculates the dot product of the two vectors, ∇u and $\hat{n}$, $\nabla u\cdot\hat{n}$, so we will not use the notation $\partial u/\partial n$ in this text.

constant thermal properties, the equilibrium temperature distribution will satisfy

$$\nabla^2 u = -\frac{Q}{K_0}, \tag{1.5.13}$$

known as **Poisson's equation**.

If, in addition, there are no sources ($Q = 0$), then

$$\boxed{\nabla^2 u = 0;} \tag{1.5.14}$$

the Laplacian of the temperature distribution is zero. Equation (1.5.14) is known as **Laplace's equation**. It is also known as the **potential equation**, since the gravitational and electrostatic potentials satisfy (1.5.14) if there are no sources. We will solve a number of problems involving Laplace's equation in later sections.

Two - dimensional problems. All the previous remarks about three-dimensional problems are valid if the geometry is such that the temperature only depends on x, y and t. For example, Laplace's equation in two dimensions, x and y, corresponding to equilibrium heat flow with no sources (and constant thermal properties) is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

since $\partial^2 u/\partial z^2 = 0$. two - dimensional results can be derived directly (without taking a limit of three - dimensional problems), by using fundamental principles in two dimensions. We will not repeat the derivation. However, we can easily outline the results. Every time a volume integral ($\iiint_R \cdots dV$) appears, it must be replaced by a surface integral over the entire two - dimensional plane region ($\iint_R \cdots dS$). Similarly, the boundary contribution for three - dimensional problems, which is the closed surface integral $\oiint \cdots dS$, must be replaced by the closed line integral $\oint \cdots d\tau$, an integration over the boundary of the two - dimensional plane surface. These results are not difficult to derive since the divergence theorem in three dimensions,

$$\iiint_R \nabla \cdot \boldsymbol{A} \, dV = \oiint \boldsymbol{A} \cdot \hat{n} \, dS. \tag{1.5.15}$$

is valid in two dimensions, taking the form

$$\iint_R \nabla \cdot \boldsymbol{A} \, dS = \oint \boldsymbol{A} \cdot \hat{n} \, d\tau. \tag{1.5.16}$$

Sometimes (1.5.16) is called *Green's theorem*, but we prefer to refer to it as the two-dimensional divergence theorem. In this way only one equation need be familiar to the reader, namely (1.5.15); *the conversion to two - dimensional form involves only changing the number of integral signs.*

Polar and cylindrical coordinates. The Laplacian,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \tag{1.5.17}$$

is important for the heat equation (1.5.11) and its steady-state version (1.5.14), as well as for other significant problems in science and engineering. Equation (1.5.17) written as above in Cartesian coordinates is most useful when the geometrical region under investigation is a rectangle or a rectangular box. Other coordinate systems are frequently useful. In practical applications, one may need the formula that expresses the Laplacian in the appropriate coordinate system. In circular cylindrical coordinates, with r the radial distance from the z-axis and θ the angle

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z, \end{aligned}} \tag{1.5.18}$$

the Laplacian can be shown to equal the following formula:

$$\boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.} \tag{1.5.19}$$

There may be no need to memorize this formula, as it can often be looked up in a reference book. As an aid in minimizing errors, it should be noted that every term in the Laplacian has the dimension of u divided by two spatial dimensions [just as in Cartesian coordinates, (1.5.17)]. Since θ is measured in radians, which have no dimensions, this remark aids in remembering to divide $\partial^2 u/\partial \theta^2$ by r^2. In polar coordinates (by which we mean a two - dimensional coordinate system with z fixed, usually $z = 0$), the Laplacian is the same as (1.5.19) with $\partial^2 u/\partial z^2 = 0$ since there is no dependence on z. Equation (1.5.19) can be derived (see the exercises) using the chain rule for partial derivatives, applicable for changes of variables.

In some physical situations it is known that the temperature does *not* depend on the polar angle θ; it is said to be **circularly** or **axially symmetric**. In that case

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}. \tag{1.5.20}$$

Spherical coordinates. Geophysical problems as well as electrical problems with spherical conductors are best solved using spherical coordinates (ρ, θ, ϕ). The radial distance is ρ, the angle from the pole (z-axis) is ϕ, and the cylindrical (or azimuthal) angle is θ. Note that if ρ is constant and the angle ϕ is a constant a

circle is generated with radius $\rho \sin \phi$ (as shown in Fig. 1.5.3) so that

$$
\begin{aligned}
x &= \rho \sin \phi \cos \theta \\
y &= \rho \sin \phi \sin \theta \\
z &= \rho \cos \phi.
\end{aligned}
\tag{1.5.21}
$$

The angle from the pole ranges from 0 to π (while the usual cylindrical angle ranges from 0 to 2π). It can be shown that the Laplacian satisfies

$$
\nabla^2 u = \frac{1}{\rho^2} \frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \phi}\left(\sin \phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2}.
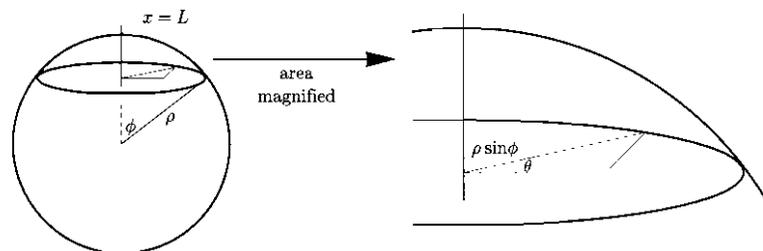\tag{1.5.22}
$$

Figure 1.5.3: Spherical coordinates.

EXERCISES 1.5

1.5.1. Let $c(x, y, z, t)$ denote the concentration of a pollutant (the amount per unit volume).

  (a) What is an expression for the total amount of pollutant in the region R?

  (b) Suppose that the flow $\boldsymbol{J}$ of the pollutant is proportional to the gradient of the concentration. (Is this reasonable?) Express conservation of the pollutant.

  (c) Derive the partial differential equation governing the diffusion of the pollutant.

*1.5.2. For conduction of thermal energy, the heat flux vector is $\boldsymbol{\phi} = -K_0 \nabla u$. If in addition the molecules move at an average velocity $\boldsymbol{V}$ a process called **convection**, then briefly explain why $\boldsymbol{\phi} = -K_0 \nabla u + c\rho u \boldsymbol{V}$. Derive the corresponding equation for heat flow, including both conduction and convection of thermal energy (assuming constant thermal properties with no sources).

1.5.3. Consider the polar coordinates

$$x = r \cos \theta$$

$$y = r \sin \theta.$$

  (a) Since $r^2 = x^2 + y^2$, show that $\frac{\partial r}{\partial x} = \cos \theta$, $\frac{\partial r}{\partial y} = \sin \theta$, $\frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}$, and $\frac{\partial \theta}{\partial x} = \frac{-\sin \theta}{r}$.

  (b) Show that $\hat{\boldsymbol{r}} = \cos \theta \hat{\boldsymbol{i}} + \sin \theta \hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{\theta}} = -\sin \theta \hat{\boldsymbol{i}} + \cos \theta \hat{\boldsymbol{j}}$.

  (c) Using the chain rule, show that $\nabla u = \frac{\partial u}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\boldsymbol{\theta}}$.

  (d) If $\boldsymbol{A} = A_r\hat{\boldsymbol{r}} + A_\theta\hat{\boldsymbol{\theta}}$, show that $\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta)$, since $\partial \hat{\boldsymbol{r}}/\partial \theta = \hat{\boldsymbol{\theta}}$ and $\partial \hat{\boldsymbol{\theta}}/\partial \theta = -\hat{\boldsymbol{r}}$ follows from part (b).

  (e) Show that $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

1.5.4. Using Exercise 1.5.3(a) and the chain rule for partial derivatives, derive the special case of Exercise 1.5.3(e) if $u(r)$ only.

1.5.5. Assume that the temperature is circularly symmetric. $u = u(r, t)$, where $r^2 = x^2 + y^2$. We will derive the heat equation for this problem. Consider any circular annulus $a \le r \le b$.

  (a) Show that the total heat energy is $2\pi \int_a^b c\rho u r\, dr$.

  (b) Show that the flow of heat energy per unit time out of the annulus at $r = b$ is $-2\pi b K_0 \partial u/\partial r \mid_{r=b}$. A similar result holds at $r = a$.

  (c) Use parts (a) and (b) to derive the circularly symmetric heat equation without sources:

$$
\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).
$$

1.5.6. Modify Exercise 1.5.5 if the thermal properties depend on r.

1.5.7. Derive the heat equation in two dimensions by using Green's theorem, (1.5.16), the two-dimensional form of the divergence theorem.

1.5.8. If Laplace's equation is satisfied in three dimensions, show that

$$
\oiint \nabla u \cdot \hat{\boldsymbol{n}}\, dS = 0
$$

for any closed surface. (*Hint*: Use the divergence theorem.) Give a physical interpretation of this result (in the context of heat flow).

1.5.9. Determine the equilibrium temperature distribution inside a circular annulus ($r_1 \le r \le r_2$):

  *(a) if the outer radius is at temperature T_2 and the inner at T_1.

  (b) if the outer radius is insulated and the inner radius is at temperature T_1.

1.5.10. Determine the equilibrium temperature distribution inside a circle $(r \leq r_0)$ if the boundary is fixed at temperature T_0.

*1.5.11. Consider

$$\frac{\partial u}{\partial t} = \frac{k}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \qquad a < r < b$$

subject to

$$u(r,0) = f(r), \ \frac{\partial u}{\partial r}(a,t) = \beta, \ \text{and} \ \frac{\partial u}{\partial r}(b,t) = 1.$$

Using physical reasoning, for what value(s) of β does an equilibrium temperature distribution exist?

1.5.12. Assume that the temperature is spherically symmetric, $u = u(r,t)$, where r is the distance from a fixed point $(r^2 = x^2 + y^2 + z^2)$. Consider the heat flow (without sources) between any two concentric spheres of radii a and b.

(a) Show that the total heat energy is $4\pi \int_a^b c\rho u r^2 \ dr$.

(b) Show that the flow of heat energy per unit time out of the spherical shell at $r = b$ is $-4\pi b^2 K_0 \partial u / \partial r \mid_{r=b}$. A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the spherically symmetric heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right).$$

*1.5.13. Determine the *steady-state* temperature distribution between two concentric *spheres* with radii 1 and 4, respectively, if the temperature of the outer sphere is maintained at $80°$ and the inner sphere at $0°$ (see Exercise 1.5.12).

1.5.14. Isobars are lines of constant temperature. Show that isobars are perpendicular to any part of the boundary that is insulated.

Orthogonal Curvilinear Coordinates. A coordinate system (u, v, w) may be introduced and defined by $x = x(u, v, w), y = y(u, v, w)$ and $z = z(u, v, w)$. The radial vector $r \equiv x\hat{i} + y\hat{j} + z\hat{k}$. Partial derivatives of r with respect to a coordinate are in the direction of the coordinate. Thus, for example, a vector in the u-direction $\partial r / \partial u$ can be made a unit vector $\hat{e}_u$ in the u-direction by dividing by its length $h_u = |\partial r / \partial u|$ called the **scale factor** : $\hat{e}_u = \frac{1}{h_u} \partial r / \partial u$.

1.5.15. Determine the scale factors for cylindrical coordinates.

1.5.16. Determine the scale factors for spherical coordinates.

1.5.17. The gradient of a scalar can be expressed in terms of the new coordinate system $\nabla g = a\partial r / \partial u + b\partial r / \partial v + c\partial r / \partial w$, where you will determine the scalars a, b, c. Using $dg = \nabla g \cdot dr$, derive that the **gradient** in an orthogonal curvilinear coordinate system is given by

$$\nabla g = \frac{1}{h_u} \frac{\partial g}{\partial u} \hat{e}_u + \frac{1}{h_v} \frac{\partial g}{\partial v} \hat{e}_v + \frac{1}{h_w} \frac{\partial g}{\partial w} \hat{e}_w. \qquad (1.5.23)$$

An expression for the divergence is more difficult to derive, and we will just state that if a vector p is expressed in terms of this new coordinate system $p = p_u \hat{e}_u + p_v \hat{e}_v + p_w \hat{e}_w$, then the **divergence** satisfies

$$\nabla \cdot p = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} (h_v h_w p_u) + \frac{\partial}{\partial v} (h_u h_w p_v) + \frac{\partial}{\partial w} (h_u h_v p_w) \right] \quad (1.5.24)$$

1.5.18. Using (1.5.23) and, (1.5.24) derive the **Laplacian** in an orthogonal curvilinear coordinate system:

$$\nabla^2 T = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} \left(\frac{h_v h_w}{h_u} \frac{\partial T}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_u h_w}{h_v} \frac{\partial T}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_u h_v}{h_w} \frac{\partial T}{\partial w} \right) \right].$$
$$(1.5.25)$$

1.5.19. Using (1.5.25), derive the Laplacian for cylindrical coordinates.

1.5.20. Using (1.5.25), derive the Laplacian for spherical coordinates.

Appendix to 1.5: Review of Gradient and a Derivation of Fourier's Law of Heat Conduction

Experimentally, for isotropic[6] materials (i.e., without preferential directions) **heat flows from hot to cold in the direction in which temperature differences are greatest**. The heat flow is proportional (with proportionality constant K_0, the thermal conductivity) to the rate of change of temperature in this direction.

The change in the temperature Δu is

$$\Delta u = u(x + \Delta x, t) - u(x, t) \approx \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \frac{\partial u}{\partial z} \Delta z.$$

In the direction $\hat{\alpha} = \alpha_1 \hat{i} + \alpha_2 \hat{j} + \alpha_3 \hat{k}$, $\Delta x = \Delta s \hat{\alpha}$, where Δs is the distance between x and $x + \Delta x$. Thus, the rate of change of the temperature in the direction $\hat{\alpha}$ is the **directional derivative**:

$$\lim_{\Delta s \to 0} \frac{\Delta u}{\Delta s} = \alpha_1 \frac{\partial u}{\partial x} + \alpha_2 \frac{\partial u}{\partial y} + \alpha_3 \frac{\partial u}{\partial z} = \alpha \cdot \nabla u,$$

where it has been convenient to define the following *vector*:

$$\boxed{\nabla u \equiv \frac{\partial u}{\partial x} \hat{i} + \frac{\partial u}{\partial y} \hat{j} + \frac{\partial u}{\partial z} \hat{k},} \qquad (1.5.26)$$

called the **gradient** of the temperature. From the property of dot products, if θ is the angle between $\hat{\alpha}$ and ∇u, then the directional derivative is $|\nabla u| \cos \theta$ since

---

[6]Examples of nonisotropic materials are certain crystal and grainy woods.

$|\hat{\alpha}| = 1$. The largest rate of change of u (the largest directional derivative) is $|\nabla u| > 0$, and it occurs if $\theta = 0$ (i.e., in the direction of the gradient). Since this $|\nabla u|$ is positive, the temperature increase is greatest in the direction of the derivative. Since heat energy flows in the direction of decreasing temperatures, **the heat flow vector is in the opposite direction to the heat gradient**. It follows that

$$\boxed{\phi = -K_0 \nabla u,} \tag{1.5.27}$$

since $|\nabla u|$ equals the magnitude of the rate of change of u (in the direction of the gradient). This again is called Fourier's law of heat conduction. Thus, in three dimensions, the gradient ∇u replaces $\partial u/\partial x$.
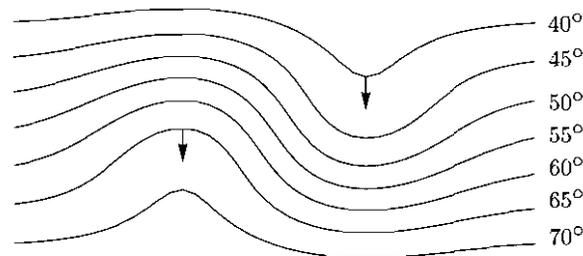

Figure 1.5.4: The gradient is perpendicular to level surfaces of the temperature.

Another fundamental property of the gradient is that it is normal (perpendicular) to the level surfaces. It is easier to illustrate this in a two-dimensional problem (see Fig. 1.5.3) in which the temperature is constant along level curves (rather than level surfaces). To show that the gradient is perpendicular, consider the surface on which the temperature is the constant T_0, $u(x, y, z, t) = T_0$. We calculate the differential of both sides (at a fixed time) along the surface. Since T_0 is constant, $dT_0 = 0$. Therefore, using the chain rule of partial derivatives,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0. \tag{1.5.28}$$

Equation (1.5.28) can be written as

$$\left(\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}\right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) = 0$$

or

$$\nabla u \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) = 0. \tag{1.5.29}$$

$dx\hat{i} + dy\hat{j} + dz\hat{k}$ represents any vector in the tangent plane of the level surface. From (1.5.29), its dot product with ∇u is zero; that is, ∇u is perpendicular to the tangent plane. Thus, ∇u is perpendicular to the surface $u = $ constant.

We have thus learned two properties of the gradient, ∇u:

1. Direction: ∇u is perpendicular to the surface $u = $ constant. ∇u is also in the direction of the largest directional derivative. u increases in the direction of the gradient.

2. Magnitude: $|\nabla u|$ is the largest value of the directional derivative.