Heat Equation,

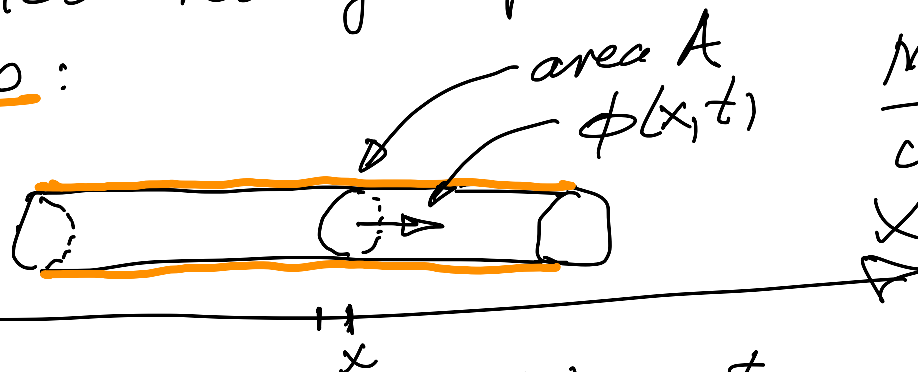
Boundary conditions

a thin heated rod of uniform cross-section A

Math 331

insulated sides:

cooling problems



$T(x,t)$ temp at pos. x at time t

$\rho(x)$ mass density

$\phi(x,t)$ heat flux at pos. x at time t

Heat energy: is a quantity that changes temp.

1 cal h.e. needed to raise temp. of 1g water by 1°C

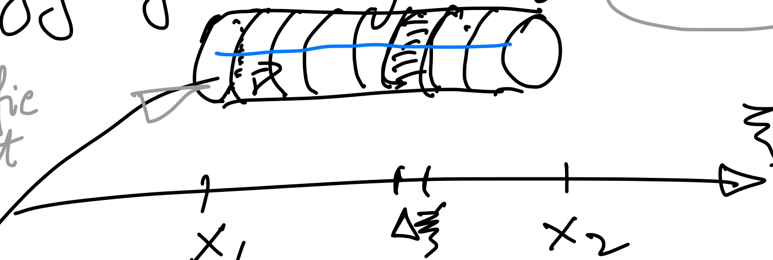
specific heat of a particular substance is amount of h.e. needed to raise temp of 1g of subst. by 1°C

Object with spec. heat c mass m at tem T °C

heat energy of this object: cmT

$Q(x,t)$

heat sources



$T(x,t)$

$\rho(x)$, $c(x)$ specific heat

total heat energy here

$$\underbrace{T(\xi_i,t)\,c(\xi_i)\,\rho(\xi_i)\underbrace{A\cdot(\Delta\xi)}_{\text{volume}}}_{\text{heat energy in tiny slice}}$$

$$\sum_{i=1}^{n} T(\xi_i,t)\,c(\xi_i)\,\rho(\xi_i)\,A\cdot\Delta\xi$$

$$A\int_{x_1}^{x_2}\underbrace{c(\xi)\,\rho(\xi)\,T(\xi,t)}_{\substack{\text{heat energy density}\\ e(\xi,t)}}\,d\xi$$

at time t

heat energy in the slice between x_1 and x_2

Phys. Laws: Conservation of heat energy

$$A \frac{d}{dt}\int_{x_1}^{x_2} e(\xi,t)\,d\xi = -\big(\phi(x_2,t)-\phi(x_1,t)\big)A + A\int_{x_1}^{x_2} Q(\xi,t)\,d\xi$$

Fourier's Law of heat conduction

$$\phi(x,t) = -\underbrace{K_0(x)}_{>0}\frac{\partial T}{\partial x}(x,t)$$

The heat flows from hotter to colder region; heat flux is proportional to the derivative of the temperature with a negative constant of proportionality.

Now consider the conservation of heat energy law in a tiny slice $x, x+\Delta x$

$$\frac{d}{dt}\left(\cancel{A}\int_x^{x+\Delta x} e(\xi,t)\,d\xi\right) = -\big(\phi(x+\Delta x,t)-\phi(x,t)\big)\cancel{A} + \cancel{A}\int_x^{x+\Delta x} Q(\xi,t)\,d\xi$$

$$\parallel$$

$$\int_x^{x+\Delta x} \frac{\partial e}{\partial t}(\xi,t)\,d\xi$$

$$\underbrace{\frac{1}{\Delta x}\int_x^{x+\Delta x}\frac{\partial e}{\partial t}(\xi,t)\,d\xi}_{\text{FTC}} = -\frac{1}{\Delta x}\,\underbrace{\frac{\phi(x+\Delta x,t)-\phi(x,t)}{\Delta x}}_{\text{def. } \frac{\partial \phi}{\partial x}} + \underbrace{\frac{1}{\Delta x}\int_x^{x+\Delta x} Q(\xi,t)\,d\xi}_{\text{FTC}}$$

let $\Delta x \to 0$

$$\frac{\partial e}{\partial t}(x,t) = -\frac{\partial \phi}{\partial x}(x,t) + Q(x,t)$$

$$e(x,t) = c(x)\rho(x)T(x,t)$$

use Fourier's Law

$$\phi(x,t) = -K_0(x)\frac{\partial T}{\partial x}(x,t)$$

$$c(x)\rho(x)\frac{\partial T}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial T}{\partial x}(x,t)\right) + Q(x,t)$$

Write $T(x,t) = u(x,t)$: Heat Equation

$$\boxed{\begin{gathered}\underbrace{c(x)}_{>0}\,\underbrace{\rho(x)}_{>0}\,\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}(x,t)\right) + Q(x,t) \\ u(x,0) = \underline{f(x)}\end{gathered}}$$

nonhomog. term

For most part c, ρ, K_0 are constants and $Q = 0$

$$\boxed{\frac{\partial u}{\partial t}(x,t) = \underbrace{\frac{K_0}{c\rho}}_{>0} \frac{\partial^2 u}{\partial x^2}(x,t)}$$

homogen. equation

k thermal diffusivity

In addition to the I.C. $u(x,0) = f(x)$

it natural to impose Boundary Conditions.

Say, the rod is between $x=a$ and $x=b$, $a<b$

we keep boundry points at CONSTANT temp.

$$u(a,t) = \underset{0}{T_1}, \quad u(b,t) = \underset{0}{T_2} \quad \forall t \geq 0$$

Dirichlet b.c.'s