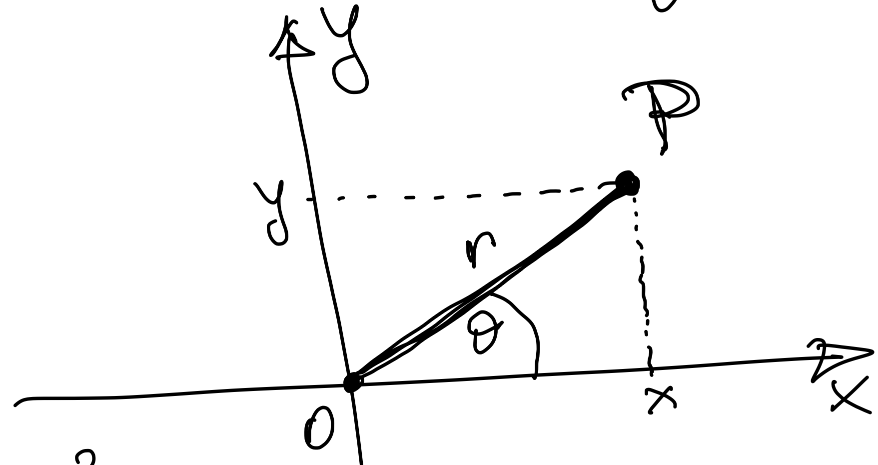
Laplacian in Polar Coordinates, Equilibrium Temp. in a Disk

$$u(x,y)$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$



P Cartesian (x,y)

Polar

P (r,θ)

$$u(x,y) = x^2 + y^2 \qquad r^2$$

$$w(r,\theta) = u(r\,c\theta,\; r\,s\theta)$$

Chain rule

$$\frac{\partial w}{\partial r} = \boxed{c\theta\,\frac{\partial u}{\partial x} + s\theta\,\frac{\partial u}{\partial y}} \quad (*)$$

$$\frac{\partial w}{\partial \theta} = -r s\theta \frac{\partial u}{\partial x} + r c\theta \frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial^2 w}{\partial r^2} = (c\theta)^2 \frac{\partial^2 u}{\partial x^2} + s\theta c\theta \frac{\partial^2 u}{\partial y \partial x} + c\theta s\theta \frac{\partial^2 u}{\partial x \partial y} + (s\theta)^2 \frac{\partial^2 u}{\partial y^2}}$$

$$\frac{\partial^2 w}{\partial \theta^2} = \boxed{-r c\theta \frac{\partial u}{\partial x} - r s\theta \frac{\partial u}{\partial y}} + r^2 (s\theta)^2 \frac{\partial^2 u}{\partial x^2} - r^2 c\theta s\theta \frac{\partial^2 u}{\partial y \partial x} - r^2 s\theta c\theta \frac{\partial^2 u}{\partial x \partial y} + r^2 (c\theta)^2 \frac{\partial^2 u}{\partial y^2}$$

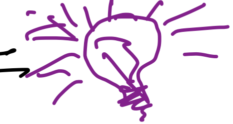
$(*)$

$$\boxed{\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial \theta^2} + r \frac{\partial w}{\partial r}\right) = (s\theta)^2 \frac{\partial^2 u}{\partial x^2} - c\theta s\theta \frac{\partial^2 u}{\partial y \partial x} - s\theta c\theta \frac{\partial^2 u}{\partial x \partial y} + (c\theta)^2 \frac{\partial^2 u}{\partial y^2}}$$

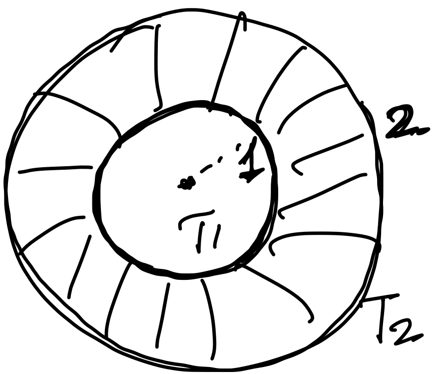
$\square + \square$ DONE

$$\frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} (r c\theta, r s\theta)$$

$$\Delta w = \frac{1}{r^2}\frac{\partial^2 w}{\partial \vartheta^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2}$$



$$= \frac{1}{r^2}\frac{\partial^2 w}{\partial \vartheta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$$



$w(r,\vartheta,t)$

$R_1 = 1$

$R_2 = 2$

Calculate $\boxed{\text{Equilibrium}}$ temp. distribution.

$c = 1,\ \rho = 1,\ K_0 = 1$

$$\underbrace{\frac{\partial w}{\partial t}}_{=0} = \Delta w$$

$w(1,\vartheta,t) = T_1$

$w(2,\vartheta,t) = T_2$

w does not dep. on ϑ

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right) = 0$$

$$\frac{\partial}{\partial r}\left(r\,\frac{\partial w}{\partial r}\right) = 0$$

$$r\,\frac{\partial w}{\partial r} = C_1$$

$$\frac{\partial w}{\partial r} = \frac{C_1}{r}$$

$$\boxed{w(r) = C_1 \ln r + C_2}$$

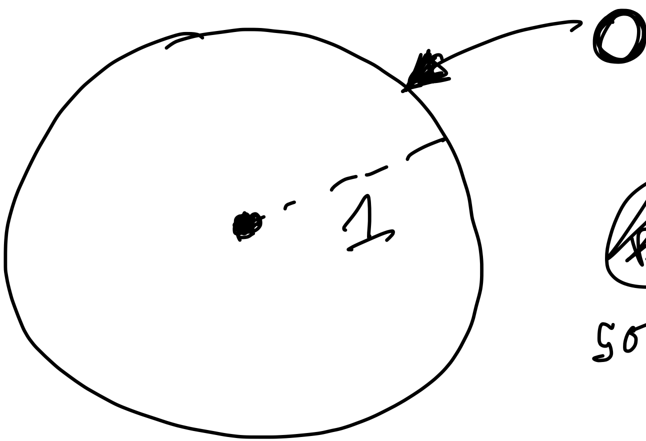
$$C_1 \ln 1 + C_2 = T_1$$

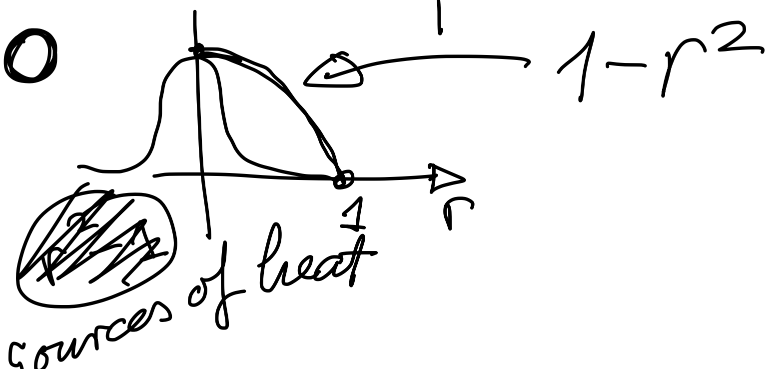
$$C_1 \ln 2 + C_2 = T_2$$

$$C_1 \ln 2 + T_1 = T_2$$

$$C_1 = \frac{T_2 - T_1}{\ln 2}$$

$$\boxed{w(r) = \frac{T_2 - T_1}{\ln 2} \ln r + T_1}$$





$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+1-r^2=0$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)=r^3-r$$

$$r\frac{\partial w}{\partial r}=\frac{1}{4}r^4-\frac{1}{2}r^2+c_1$$

$$\frac{\partial w}{\partial r}=\frac{1}{4}r^3-\frac{1}{2}r+\frac{c_1}{r}$$

Does not make sense

Infinite at the origin

$$\frac{\partial w}{\partial r}=\frac{1}{4}r^3-\frac{1}{2}r$$

$$w(r)=\frac{1}{12}r^4-\frac{1}{4}r^2+\cancel{c_2}\ 1/6$$

BC $\underline{w(1)=0}$ $\quad \frac{1}{12}-\frac{1}{4}+\frac{1}{6}=0$