Boundary value problems for the Heat Equation

2nd order Linear ODE with constant coefficients:

$$\boxed{y'' + \alpha y' + \beta y = 0} \quad \text{homogeneous}$$

$\boxed{r^2 + \alpha r + \beta = 0}$ char. eq.

$y = y(x)$

$\boxed{\text{A review of 331}}$

How to solve?

(1) $y'' = 0$

$y(x) = c_1 x + c_2 \cdot 1$ c_1, c_2 const.

general solution

What is the spirit here? We have two special solutions x and 1 and then ALL other solutions are obtained as linear combinations of of x & 1. Form the fundamental set of solutions: $\{x, 1\}$

(2) $\alpha \neq 0$ $y'' + \alpha y' = 0$

$(y')' = -\alpha (y')$

$y(x) = \frac{c_1}{-\alpha} e^{-\alpha x} + c_2 \Longleftarrow y'(x) = c_1 e^{-\alpha x}$

Fund. set of sols $\{e^{-\alpha x}, 1\}$ $\quad -\alpha e^{-\alpha x}, \alpha^2 e^{-\alpha x}$

The gen. sol. is $y(x) = C_1 e^{-\alpha x} + C_2 1$

$$\ast \quad y'' + \beta y = 0$$

try $y(x) = e^{rx}$ $r \in \mathbb{C}$

$y'(x) = r e^{rx}$, $y'' = r^2 e^{rx}$

$$\boxed{r^2 \cancel{e^{rx}} + \beta \cancel{e^{rx}} = 0}$$

Charact. equation

$$\boxed{r^2 = -\beta}$$

Case 1 $\quad \beta < 0 \quad r = \sqrt{-\beta}, r = -\sqrt{-\beta}$

Case 2 $\quad \beta > 0 \quad r = i\sqrt{\beta}, r = -i\sqrt{\beta}$

Case 1 Fund set of sols: $\left\{e^{\sqrt{-\beta}x}, e^{-\sqrt{-\beta}x}\right\}$ ($-\beta > 0$)

Case 2 Fund set of sol. $\left\{e^{i\sqrt{\beta}x}, e^{-i\sqrt{\beta}x}\right\}$

It is convenient to set:

<u>Case 1</u>. $\beta = -\mu^2$ ($\mu > 0$) the gen. sol. $c_1 e^{\mu x} + c_2 e^{-\mu x}$

<u>Case 2</u> $\beta = \mu^2$ the fund set of sols is $\{ e^{i\mu x}, e^{-i\mu x} \}$

complex functions not useful.

Euler's formula: $\boxed{e^{i\mu x} = \cos\mu x + i \sin\mu x}$

$\boxed{e^{-i\mu x} = \cos\mu x - i \sin\mu x}$

$\frac{1}{2}(e^{i\mu x} + e^{-i\mu x}) = \cos\mu x$ a solution as well

$-\frac{i}{2}(e^{i\mu x} - e^{-i\mu x}) = \sin\mu x$ a solution as well

Thus: $\{ \cos(\mu x), \sin(\mu x) \}$ Fund. set of S.

In Case 2 the gen. sol. is

$$\boxed{C_1 \cos(\mu x) + C_2 \sin(\mu x)}$$

!!

Case 1

An alternative gen. sol. is

$$C_1 \cosh(\mu x) + C_2 \sinh(\mu x)$$

The big prop. of the equation $y'' + \alpha y' + \beta y = 0$ behind the above construction is

THE LINEARITY:

If y_1 and y_2 are solutions, then $c_1 y_1 + c_2 y_2$ is also a sol.

$L(y) = y'' + \alpha y' + \beta y$

$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$

L is a Linear Transformation

The nullspace of L is a vector space

(2-dim. vector space)

Just verify!!

$$y'' - \mu^2 y = 0$$

$\sin(\mu x)$ is a sol.

$$(\sin(\mu x))' = \mu \cos(\mu x)$$

$$(\sin \mu x)'' = -\mu^2 \sin(\mu x)$$

The same with $\cos(\mu x)$

$$\underbrace{y'' + \alpha y' + \beta y = 0}_{\text{ODE (333 Math)}} \quad \text{set } y(x) = e^{rx} \quad \underbrace{(r^2 + \alpha r + \beta)}_{\text{High school. } = 0} \underbrace{e^{rx}}_{\neq 0} = 0$$

the charact. equation

Back to heat equation (the simplest)

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

BCs $u(0,t) = 0$ $u(L,t) = 0$ $\forall t \geq 0$

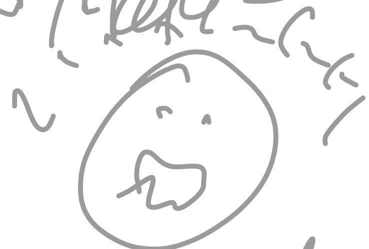
IC $u(x,0) = f(x)$ Solve !!

Ingeneous idea Fourier's IDEA

LOOK for a solution in the form

$$u(x,t) = A(x)B(t)$$

Math 224



not highschool

two single var. functions

high-school

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(A(x)B(t)\right) = A(x)B'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x}\left(A(x)B(t)\right) = A''(x)B(t)$$