Solving the Heat Eq with different boundary conditions

$$\frac{\partial u}{\partial t}(x,t) = \varkappa \frac{\partial^2 u}{\partial x^2}(x,t) \qquad 0 \le x \le L$$
$$t \ge 0$$

BC

Dirichlet BCs
$u(0,t) = 0$ $\forall t \ge 0$
$u(L,t) = 0$

Neumann BCs
$\frac{\partial u}{\partial x}(0,t) = 0$ $\forall t \ge 0$
$\frac{\partial u}{\partial x}(L,t) = 0$

Periodic BC.
$u(0,t) = u(L,t)$ $\forall t \ge 0$
$\frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial t}(L,t)$

IC. $u(x,0) = f(x) \qquad 0 \le x \le L$

Separation of Variables method
$$u(x,t) = A(x)B(t)$$

substitute in PDE

$$\frac{B'(t)}{\varkappa B(t)} = \frac{A''(x)}{A(x)} = -\lambda$$

$B'(t) = -\lambda \varkappa B(t)$

$B(t) = b\, e^{-\lambda \varkappa t}$

$$\boxed{\left(-\frac{d^2}{dx^2}\right) A = \lambda A}$$

the BCs transfer to A :

Dirchlet BCs	Neumann BCs	Periodic BCs
$A(0)=0, A(L)=0$	$A'(0)=0, A'(L)=0$	$A(0)=A(L)$ $A'(0)=A'(L)$
$\lambda_m = \left(\frac{m\pi}{L}\right)^2, m \in \mathbb{N}$ corresp. $A_m(x) = \sin\left(\frac{m\pi}{L}x\right)$ the corresponding sols of PDE are $u_m(x,t) = b_m e^{-\left(\frac{m\pi}{L}\right)^2 \varkappa t} \sin\left(\frac{m\pi}{L}x\right)$		
find b_m-s to satisfy IC: $u(x,0)=f(x)$		

$$f(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi}{L}x\right)$$

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx$$

The solution of the PDE + BC + IC

$$u(x,t) = \sum_{m=1}^{\infty} b_m e^{-\left(\frac{m\pi}{L}\right)^2 \varkappa t} \sin\left(\frac{m\pi}{L}x\right)$$

Find eigenvalues and the corresp. eigenfunctions for Neumann BCs. First look at the ODE $-A'' = \lambda A$, that is $\underline{A'' + \lambda A = 0}$. The nature of this equation is that we need to consider 3 cases $\lambda > 0, \lambda = 0, \lambda < 0$.

Case 1 $\lambda > 0$, my style is $\lambda = \mu^2$, $\boxed{\mu > 0}$

$A'' + \mu^2 A = 0$ A Fundamental set of sols is $\{\cos \mu x, \sin \mu x\}$

The general solution is

$$A(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)$$

Now from this two dimensional space of functions select those that satisfy BCs.

Neumann BC

$A'(x) = -\mu C_1 s(\mu x) + \mu C_2 c(\mu x)$

$0 = A'(0) = \mu C_2 \Rightarrow C_2 = 0$

$\mu > 0$ $\quad A(x) = C_1 c(\mu x)$

$0 = A'(L) = -\underset{>0}{\mu} \underset{\neq 0}{C_1} \sin(\mu L)$

$\Rightarrow \mu L = m\pi$, $m \in \mathbb{N}$ $\quad \mu > 0$

Periodic BCs

$A(0) = A(L)$

$A'(0) = A'(L)$

$C_1 = C_1 c(\mu L) + C_2 s(\mu L)$

$\mu C_2 = -\mu C_1 s(\mu L) + \mu C_2 c(\mu L)$ $\quad / \mu$, $\mu > 0$

$\mu_m = \frac{m\pi}{L}$, $\boxed{\lambda_m = \left(\frac{m\pi}{L}\right)^2, m \in \mathbb{N}}$

$$\boxed{A_m = \cos\left(\frac{m\pi}{L}x\right)}$$

Case 2 $\boxed{\lambda = 0}$, $A''(x) = 0$

$$A(x) = C_1 + C_2 x$$

$$A'(x) = C_2 \Rightarrow C_2 = 0$$

$\lambda = 0$ is an eigenvalue, the corresp. eigenfunction is $\boxed{1}$.

Case 3 $\lambda < 0$, my style is $\lambda = -\mu^2$, $\mu > 0$

$$A'' - \mu^2 A = 0$$

A fund. set of solutions is

Math 204

$$\begin{bmatrix} -1+c(\mu L) & s(\mu L) \\ -s(\mu L) & -1+c(\mu L) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Has a nontrivial sol $\Leftrightarrow$ det $= 0$

$$(-1 + c(\mu L))^2 + (s(\mu L))^2 = 0$$

$$1 - 2c(\mu L) + \underbrace{c(\mu L)^2 + (s(\mu L))^2}_{1} = 0$$

$$\boxed{c(\mu L) = 1}$$

$\{ ch(\mu x), \; sh(\mu x) \}$.

The general sol. is:

$$A(x) = c_1 ch(\mu x) + c_2 sh(\mu x)$$

$$A'(x) = \mu c_1 sh(\mu x) + \mu c_2 ch(\mu x)$$

$$0 = A'(0) = \mu c_2 \Rightarrow c_2 = 0$$

$$0 = A'(L) = \mu c_1 sh(\mu L) \qquad \mu L > 0, \; sh(\mu L) > 0$$

$$c_1 = 0$$

$\lambda < 0$ is NOT an eigenvalue ever

$\lambda = 0$

$$u_0(x,t) = a_0$$

$$u_m(x,t) = a_m e^{-\left(\frac{m\pi}{L}\right)^2 \varkappa t} \cos\left(\frac{m\pi}{L} x\right)$$

To satisfy ICs we need to find $a_0, a_1, a_2, \ldots$ such that

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi}{L}x\right)$$

$$0 \le x \le L$$

$$1 = \cos\left(\frac{0\pi}{L}x\right)$$

We have the orthogonality relation

$$\int_0^L \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{k\pi}{L}x\right)dx = 0$$

whenever $m, k \in \mathbb{N} \cup \{0\}$ and $m \neq k$