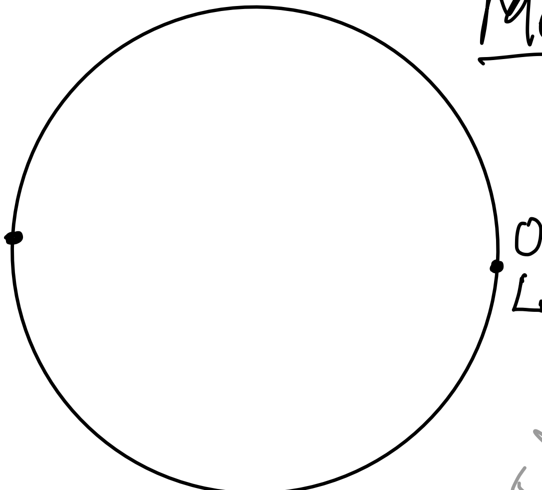
The Heat Eq in a Ring



I did this on Tuesday

Me: The total length is L

$$\frac{\partial u}{\partial t}(x,t) = \kappa \frac{\partial^2 u}{\partial x^2}(x,t)$$

$$0 \le x \le L, \quad t \ge 0$$

BC. $u(0,t) = u(L,t) \quad \forall t \ge 0$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) \quad \forall t \ge 0$$

IC: $u(x,0) = f(x) \quad 0 \le x \le L$

BOOK: The total length is $2L$; the ring starts from $-L$ and ends at L.

Change my coordinates and set the length to be $2L$, but start from 0 and end at $2L$.

We look for a sequence of product solutions which satisfy BCs.

$$u(x,t) = A(x)B(t)$$

the PDE turns into

$$\frac{B'(t)}{\varkappa B(t)} = \frac{A''(x)}{A(x)} = -\lambda$$

$$\boxed{B(t) = e^{-\lambda \varkappa t}}$$

$$\boxed{\begin{aligned} A(0) &= A(2L) \\ A'(0) &= A'(2L) \end{aligned}}$$

Now we are looking for the eigenvalues and the corresponding eigenfunctions for the problem

$$\left(-\frac{d^2}{dx^2}\right)A = \lambda A, \quad A(0) = A(2L), \; A'(0) = A'(2L)$$

the eq is $A''(x)+\lambda A(x)=0$

Case 1 $\lambda>0$ set $\lambda=\mu^2$, with $\boxed{\mu>0}$

$A''+\mu^2 A=0$ The general sol. is

$$A(x)=C_1\cos(\mu x)+C_2\sin(\mu x)$$

Now choose which of these functions satisfy BCs

$$A'(x)=-C_1\mu s(\mu x)+C_2\mu c(\mu x)$$

Now impose the BC: $A(0)=A(2L)$, $A'(0)=A'(2L)$

$$\to C_1=C_1 c(\mu 2L)+C_2 s(\mu 2L)$$

$$\to C_2\cancel{\mu}=-C_1\cancel{\mu}s(\mu 2L)+C_2\cancel{\mu}c(\mu 2L) \qquad \mu>0$$

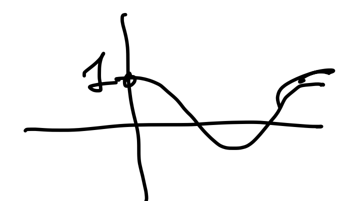
Write the system in matrix form 204

$$\begin{bmatrix} -1+c(\mu 2L) & s(\mu 2L) \\ -s(\mu 2L) & -1+c(\mu 2L) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A nontrivial sol. exists iff $\det(\;) = 0$.

$$\det(\;) = 1 - 2c(\mu 2L) + \underbrace{(c(\mu 2L))^2 + (s(\mu 2L))^2}_{1}$$

$$= 2\left(1 - c(\mu 2L)\right)$$



$$\det(\;) = 0 \iff \cos(\mu 2L) = 1$$

$\iff \mu 2L = 2m\pi$, $m \in \mathbb{N}$ pos. integer

$\iff \mu = \frac{m\pi}{L}$ $m \in \mathbb{N}$.

Now, find c_1 and c_2. Solve

$$m \in \mathbb{N} \quad \begin{bmatrix} -1 + c(2m\pi) & s(2m\pi) \\ -s(2m\pi) & -1 + c(2m\pi) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, we conclude that the eigenvalues are

$$\boxed{\lambda_m = \left(\frac{m\pi}{L}\right)^2 \text{ with } \underline{\underline{\text{two}}} \text{ eigenfunctions } \cos\left(\frac{m\pi}{L}x\right) \text{ and } \sin\left(\frac{m\pi}{L}x\right), \quad m \in \mathbb{N}}$$

Case 2 $\lambda = 0$. The general sol. is

$$A(x) = c_1 + c_2 x$$

BCs:

$$c_1 = c_1 + c_2 \cdot 2L$$

$$c_2 = c_2$$

$$c_1 = 1, \quad c_2 = 0$$

Conclusion: $\lambda = 0$ is an eigenvalue and the corresp. eigenfun. is 1

Case 3 $\lambda < 0$, set $\lambda = -\mu^2$ with $\boxed{\mu > 0}$

The eq. now is $A'' - \mu^2 A = 0$. The fund. set of sols is $\{ch(\mu x), sh(\mu x)\}$

The gen. sol. is

$$A(x) = c_1 ch(\mu x) + c_2 sh(\mu x)$$

$$A'(x) = c_1 \mu sh(\mu x) + c_2 \mu ch(\mu x)$$

BCs:

$$c_1 = c_1 ch(\mu 2L) + c_2 sh(\mu 2L)$$

$$c_2 \cancel{\mu} = c_1 \cancel{\mu} sh(\mu 2L) + c_2 \cancel{\mu} ch(\mu 2L) \quad \mu > 0$$

In matrix form:

Always Nonsingular

$$\begin{bmatrix} -1+ch(\mu 2L) & sh(\mu 2L) \\ sh(\mu 2L) & -1+ch(\mu 2L) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This system has a nontrivial sol. $\Longleftrightarrow$ $\det() = 0$

$$1 - 2ch(\mu 2L) + \underbrace{(ch(\mu 2L))^2 - (sh(\mu 2L))^2}_{1} = 0$$

$$\Updownarrow$$

$$ch(\mu 2L) = 1$$

$$\Updownarrow$$

$$\mu 2L = 0$$

$\mu > 0$

$L > 0$

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = $$

$$= \frac{1}{4}\left(\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - (\cancel{e^{2x}} - 2 + \cancel{e^{-2x}})\right)$$

$$= 1$$

$$A'' - \mu^2 A = 0$$

From two

$A(x) = \boxed{e^{\mu x}}$ $\quad A'' = \mu^2 e^{\mu x}$

$A(x) = e^{-\mu x}$ $\quad A'' = \mu^2 e^{-\mu x}$

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$$ay'' + by' + cy = 0$$

lin. hom. eq.

$y_1(x)$ $\quad$ $y_2(x)$, if these are lin. ind. sols

then $\boxed{C_1 y_1(x) + C_2 y_2(x)}$ is the gen. sol. of

ALL SOLS!

ALL

$$y'' - \mu^2 y = 0$$

ALL SOL $\Rightarrow$ Are $\Rightarrow$ $\boxed{c_1 e^{-\mu x} + c_2 e^{\mu x}}$

der $\Rightarrow$ $\boxed{-c_1 \mu e^{-\mu x} + c_2 \mu e^{\mu x}}$

Out of these so many, which are the nicest?

What is nice?

It is nice to take value 0 or 1 at 0. And it is nice for derivative to do the same.

only nice

at 0	nice 1	nice 2
fun	1	0
der	0	1

$\mu = 1$

c_1

$c_1 + c_2 = 1$

$-c_1\mu + c_2\mu = 0$

$c_1 = c_2 = 1/2$

Nice 1 is $\boxed{\frac{1}{2}e^{\mu x} + \frac{1}{2}e^{-\mu x}}$

$c_1 + c_2 = 0$

$-\cancel{\mu} c_1 + \cancel{\mu} c_2 = 1$

$2c_2 = 1$

$c_2 = 1/2$

$c_1 = -1/2$

$\boxed{-\frac{1}{2}e^{-x} + \frac{1}{2}e^{x}}$

$$y'' - y = 0$$

Nice sols are $\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$ and $\frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$

$ch(x)$ $sh(x)$