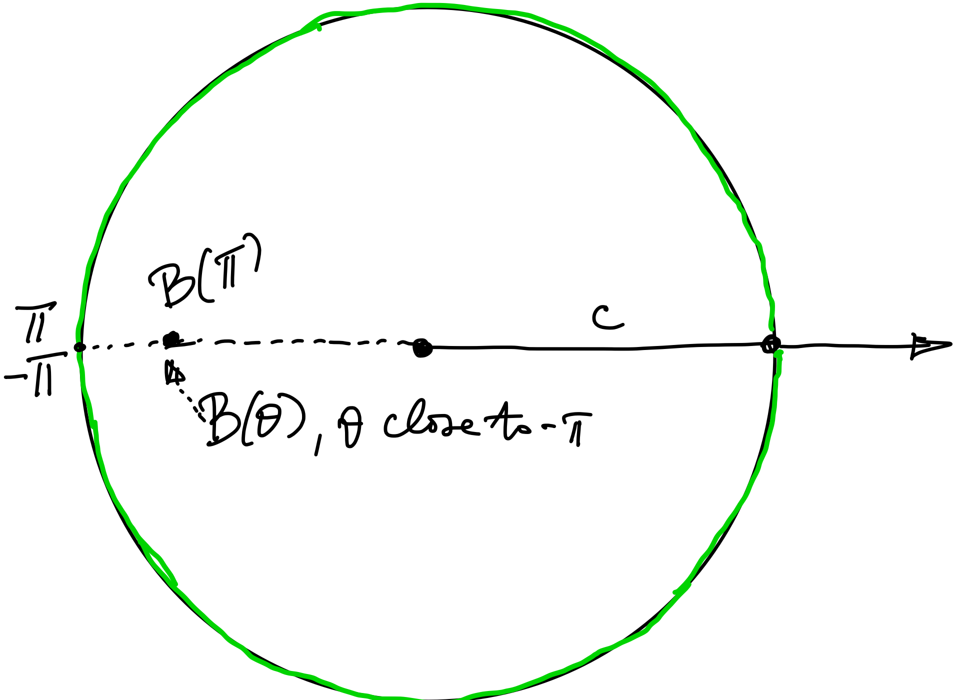
Laplace's Equation in a Disk



Laplacian in polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}=0$$

$$0\le r\le c$$
$$-\pi<\theta\le\pi$$

BC $w(c,\theta)=f(\theta)$

$$-\pi<\theta\le\pi$$

$$w(r,\theta)=A(r)B(\theta)$$

it is differentiable in the disk

$$A(c)B(\theta)=f(\theta)$$

It would be a very exceptional f if this cond. is satisfied.

$A:[0,a]\to\mathbb{R}$ $\quad B:(-\pi,\pi]\to\mathbb{R}$

Differentiability of w translates into the following conditions for $B(\theta)$:

$$\boxed{\begin{aligned} B(-\pi)&=B(\pi)\\ B'(-\pi)&=B'(\pi)\end{aligned}}$$

These are NOW the boundary conditions for B

PERIODIC

Substitute $A(r)B(\theta)$ into Laplace's eq.:

$$\frac{1}{r}\left(rA'(r)\right)'B(\theta)+\frac{1}{r^2}A(r)B''(\theta)=0 \quad / * r^2$$

$$\underbrace{r\left(rA'(r)\right)'}_{r^2A''(r)+rA'(r)}B(\theta)+A(r)B''(\theta)=0 \quad / \div AB$$

$$\frac{r^2 A''(r) + rA'(r)}{A(r)} = -\frac{B''(\theta)}{B(\theta)} = \lambda$$ separation constant

$$\boxed{r^2 A''(r) + rA'(r) = \lambda A(r)}$$

need to solve this, after we have λ Postpone!

$$\left(-\frac{d^2}{d\theta^2}\right) B(\theta) = \lambda B(\theta)$$

PBC $\quad B(-\pi) = B(\pi)$

$\quad B'(-\pi) = B'(\pi)$

$\cos(m\theta)$, $\sin(m\theta)$ $\quad m \in \mathbb{N}$

$\lambda = 0$ is an eigenvalue and 1 is the corresp. eigenfunction.

Recall, for the problem on $[-L, L]$

$$\lambda_m = \left(\frac{m\pi}{L}\right)^2$$

with corresp. eigenfunctions

but $L = \pi$

$$\lambda_m = m^2, \quad m \in \mathbb{N}$$

Now go back to $r^2 A''(r) + r A'(r) = m^2 A(r)$

$m = 0$ $\quad r^2 A'' + r A' = 0 \quad rA'' + A' = 0$

$r y' + y = 0$

$r \underbrace{\frac{y'}{y}}_{(\ln y)'} = -1, \quad r(\ln y)' = -1$

$(\ln y)' = -\frac{1}{r}$

$\ln y = -\ln r + C$

$\ln y = \ln \frac{1}{r}$ $\quad y(r) = \frac{1}{r}$

$y'(r) = -\frac{1}{r^2}$

The fund. set of solutions is $\{1, \ln r\}$

The logic of the problem tells me that $A(0)$ must be defined. So, I must reject $\ln r$ as a solution. $m = 0$ my sol. of A is $\underline{1}$

So the product solution is $\underline{1}$

$\lambda_m = m^2, m \in \mathbb{N}$ → Euler's Equation

$$\boxed{r^2 A''(r) + r A'(r) - m^2 A(r) = 0}$$

In the previous solution ($\lambda = 0$) the powers of r played a special role. So I will try $A(r) = r^\alpha$, $\alpha \in \mathbb{R}$ as a solution. $A'(r) = \alpha r^{\alpha-1}$, $A'' = \alpha(\alpha-1) r^{\alpha-2}$

$$r^2 \alpha(\alpha-1) r^{\alpha-2} + r\alpha r^{\alpha-1} - m^2 r^\alpha = 0 \quad \Big/ \div r^\alpha$$

$$\alpha(\alpha-1) + \alpha - m^2 = 0$$

$$\alpha^2 \cancel{-\alpha} \cancel{+\alpha} - m^2 = 0$$

$$\alpha \in \{-m, m\}$$

→ the fundamental set of solutions is $\{r^{-m}, r^m\}$

must reject !

For $\lambda = m^2$ my product solutions are

$$\underbrace{r^m \cos(m\theta)}_{\downarrow}, \quad r^m \sin(m\theta)$$

$$m \in \mathbb{N}$$

$m = 0 \longrightarrow \qquad = 1$

By superposition principle the infinite linear combination is also a solution:

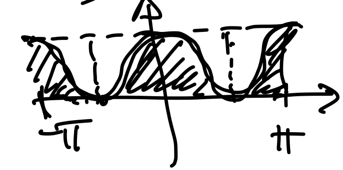
$$w(r,\theta) = \sum_{m=0}^{\infty} \hat{a}_m r^m \cos(m\theta) + \sum_{m=1}^{\infty} \hat{b}_m r^m \sin(m\theta)$$

now go back to BC $w(c,\theta) = f(\theta)$

So $$\sum_{m=0}^{\infty} \hat{a}_m c^m \cos(m\theta) + \sum_{m=1}^{\infty} \hat{b}_m c^m \sin(m\theta) = f(\theta)$$

Using the orthogonality of $\cos(m\theta), \sin(m\theta)$

$\cos(k\theta)$



$$f(\theta)\cos(k\theta) = \Big(\sum$$

$$\int_{-\pi}^{\pi} f(\theta)\cos(k\theta)\,d\theta = a_k c^k \underbrace{\int_{-\pi}^{\pi} (\cos(k\theta))^2\,d\theta}$$

$k=0 \quad 2\pi$

$k>0 \quad \pi$

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta}$$

$k>0$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(k\theta)\,d\theta$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(k\theta)\,d\theta$$

$$\hat{a}_k = \frac{a_k}{c^k} \quad , \quad \hat{b}_k = \frac{b_k}{c^k}$$

Laplace's Eq

The solution of the boundary value problem is

$$w(r,\theta) = \sum_{m=0}^{\infty} a_m \left(\frac{r}{c}\right)^m \cos(m\theta) + \sum_{m=1}^{\infty} b_m \left(\frac{r}{c}\right)^m \sin(m\theta)$$