Fourier Series and Even and Odd Functions

$$f : [-L, L] \to \mathbb{R}, \quad L > 0$$

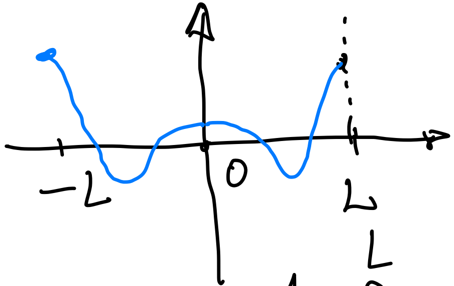
f piecewise smooth

$$f \sim \overbrace{a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{L} x\right)}^{\text{even function}} + \overbrace{\sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L} x\right)}^{\text{odd function}}$$

$$\boxed{a_0 = \frac{1}{2L} \int_{-L}^{L} f(\xi)\, d\xi}$$

$$\boxed{a_k = \frac{1}{L} \int_{-L}^{L} f(\xi) \cos\left(\frac{k\pi}{L} \xi\right) d\xi} \quad k \in \mathbb{N}$$

$$\boxed{b_k = \frac{1}{L} \int_{-L}^{L} f(\xi) \sin\left(\frac{k\pi}{L} \xi\right) d\xi} \quad k \in \mathbb{N}$$



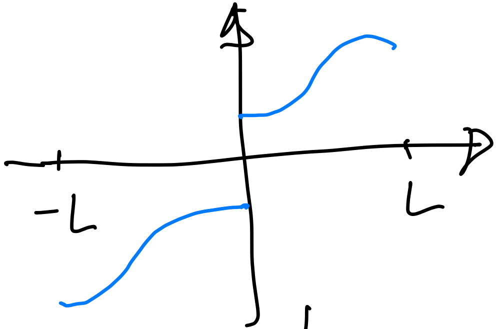
cos is an even function
sin is an odd function
f is an even function

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(\xi)\,d\xi = 2\,\frac{1}{2L}\int_{0}^{L} f(\xi)\,d\xi = \frac{1}{L}\int_{0}^{L} f(\xi)\,d\xi$$

$$a_k = \frac{2}{L}\int_{0}^{L} f(\xi)\cos\left(\frac{k\pi}{L}\xi\right)d\xi$$

$$b_k = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(\xi)}_{\text{even}}\,\underbrace{\sin\left(\frac{k\pi}{L}\xi\right)}_{\text{odd}}}_{\text{odd}}\,d\xi = 0$$

Thus, the Fourier Series of an even function consist of only cosine terms.

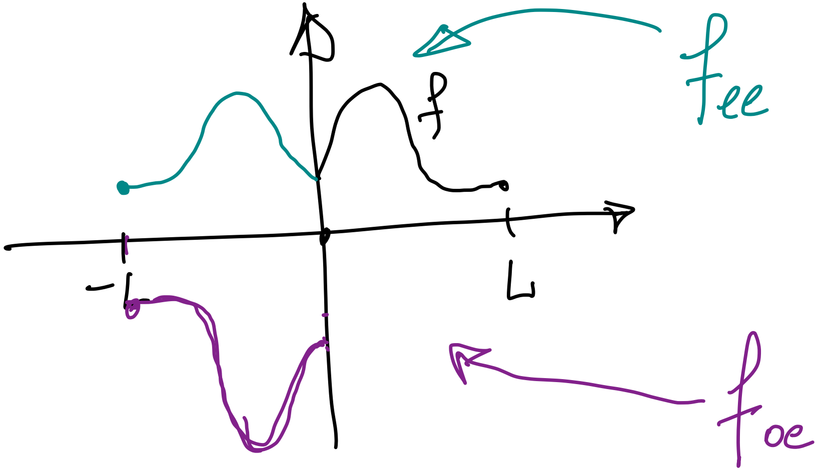


Similarly if $f(x)$ is an odd function

then $a_0 = 0$
$a_k = 0 \quad \forall k \in \mathbb{N}$

$$b_k = \frac{2}{L}\int_0^L \underbrace{\underbrace{f(\xi)}_{\text{odd}}\,\underbrace{\sin\left(\frac{k\pi}{L}\xi\right)}_{\text{odd}}}_{\text{even}} d\xi$$

If a function f is defined on $[0,L]$ only. $f:[0,L]\to\mathbb{R}$. We can extend f to $[-L,0]$ so that the resulting function is even or odd



if I calculate the F.S. of f_{ee} I will get only cosine terms. This will give us a cosine series for f

if I calculate the F.S. of f_{oe}, I will get only sine terms. This results in a representation for f with a sin series.

$$f_{ee} + f_{oe} = \begin{cases} 0 & \text{on } [-L,0] \\ 2f & \text{on } [0,L] \end{cases}$$

$\frac{1}{2}(f_{ee} + f_{oe})$ gives the extension of f by 0 on $[-L,0]$

$f : [-L, L] \to \mathbb{R}$ neither odd nor even $\mathcal{F}_{\text{Fourier}}(x)$

$$\boxed{f(x) \sim \underbrace{a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{L} x\right)}_{\substack{\text{even function} \\ f_e(x)}} + \underbrace{\sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L} x\right)}_{\substack{\text{odd function} \\ f_o(x)}}}$$

associated with

is the F.S. of $f_e(x)$

the F.S. of $f_o(x)$

$$+\begin{cases} \boxed{f(x) = f_e(x) + f_o(x)} & \forall x \in [-L, L] \\ f(-x) = f_e(-x) + f_o(-x) & \forall x \in [-L, L] \end{cases}$$

opposites

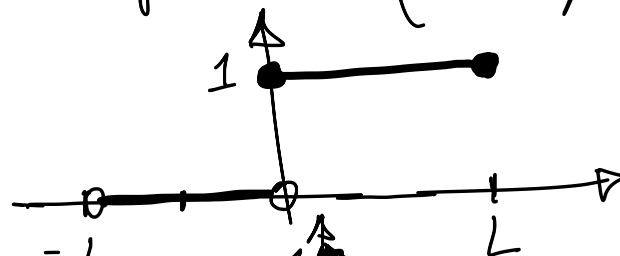
$$f(x) + f(-x) = 2 f_e(x) \Rightarrow \boxed{f_e(x) = \frac{f(x) + f(-x)}{2}}$$

$$f(x) - f(-x) = 2 f_o(x) \Rightarrow \boxed{f_o(x) = \frac{f(x) - f(-x)}{2}}$$

$$\underbrace{\cosh(x)}_{\text{even}} = \frac{e^{x}+e^{-x}}{2} \leftarrow \text{definition} \left(\begin{array}{c}\text{cosh is the}\\ \text{even part of exp}\end{array}\right)$$

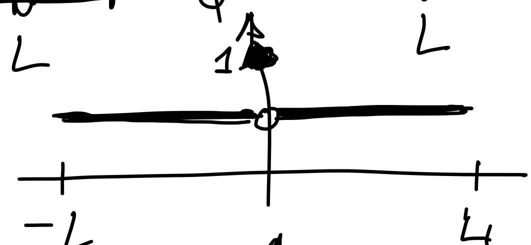
$$\sinh(x) = \frac{e^{x}-e^{-x}}{2} \leftarrow \text{definition} \left(\begin{array}{c}\text{sinh is the}\\ \text{even part of exp}\end{array}\right)$$

$$f(x) = us(x)\Big|_{[-L,L]}$$



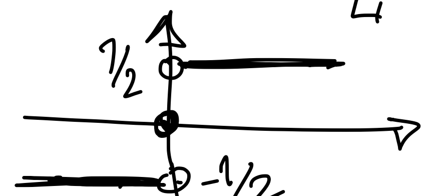
Fourier series

$$f_e(x) = \frac{f(x)+f(-x)}{2} = \begin{cases} 1/2 & x>0 \\ 1 & x=0 \\ 1/2 & \end{cases}$$



1/2

$$f_o(x) = \begin{Bmatrix} 1/2 & \\ 0 & x=0 \\ -1/2 & \end{Bmatrix} = \frac{1}{2}\text{sign}(x)$$



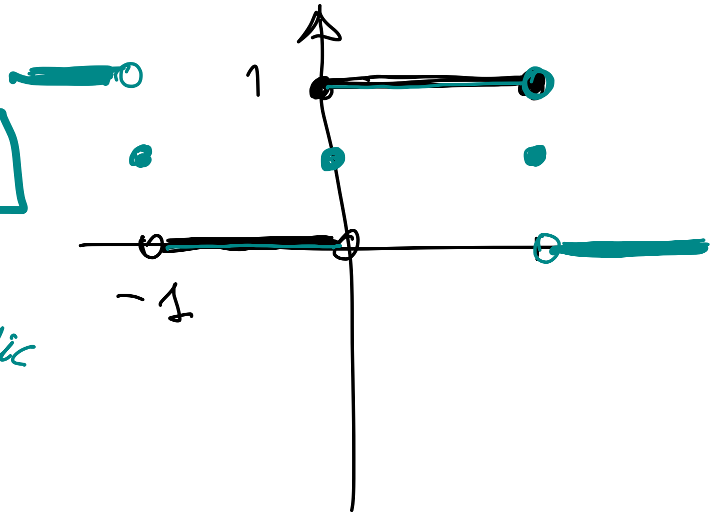
$L = 1$ F.S. of $f(x)$ is

$$\boxed{\frac{1}{2}\,\frac{}{4} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\sin\big((2k-1)\pi x\big)}$$

converges to

teal functio

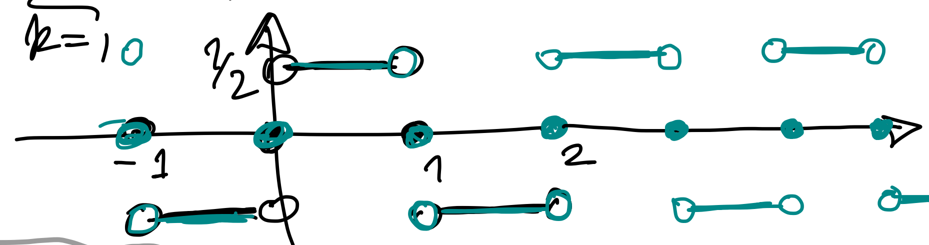
the Fourier periodic ext.



$$f_e(x) = \begin{cases} 1/2 & x \neq 0 \\ 1 & x = 0 \end{cases} \sim 1/2$$

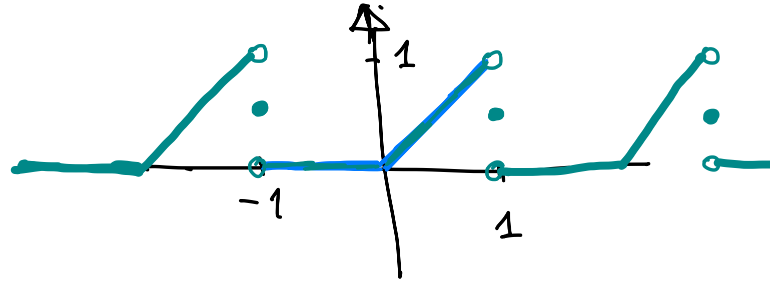
$$f_o(x) = \frac{1}{2}\,\mathrm{sign}(x) \sim \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\big((2k-1)\pi x\big)$$

$$\therefore\ \underbrace{\tilde{f}_0}_{\text{Fourier}}(x) = \tilde{f}_0(x)$$



Added after class ↓↓↓↓

Example $f(x) = \begin{cases} 0, & x \in [-1,0) \\ x, & x \in [0,1] \end{cases}$



Fourier coefficients:

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(\xi)\,d\xi = \frac{1}{4}$$

$$a_k = \frac{1}{1}\int_0^1 \xi \cos(k\pi\xi)\,d\xi = \left(\frac{1}{k\pi}\xi \sin(k\pi\xi) + \frac{1}{(k\pi)^2}\cos(k\pi\xi)\right)\Bigg|_0^1$$

$$= \frac{1}{k\pi}\underbrace{\sin(k\pi)}_{=0} + \frac{1}{(k\pi)^2}\cos(k\pi) - \frac{1}{(k\pi)^2} = \frac{(-1)^k - 1}{(k\pi)^2} = \begin{cases} 0 & k \text{ even} \\ -\dfrac{2}{k^2\pi^2} \end{cases}$$

$$b_k = \int_0^1 \xi \sin(k\pi\xi)\,d\xi = \left(-\frac{1}{k\pi}\xi\cos(k\pi\xi) + \frac{1}{(k\pi)^2}\sin(k\pi\xi)\right)\Bigg|_0^1$$

$$= -\frac{1}{k\pi}\cos(k\pi) + \frac{1}{k\pi} = \frac{1-(-1)^k}{k\pi} = \begin{cases} 0 & k \text{ even} \\ \dfrac{2}{k\pi} & k \text{ odd} \end{cases}$$

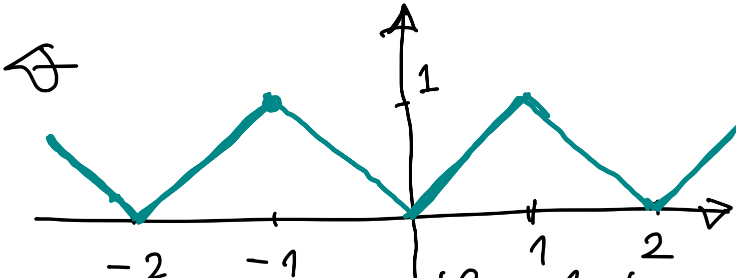
The Fourier Series

$$\boxed{\frac{1}{4} - \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos((2k-1)\pi x) + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin((2k-1)\pi x)}$$

The Fourier Series of $\frac{1}{2}|x|$ | The Fourier series of $\frac{1}{2}x$

Thus the Fourier Series of $|x|$

$$\frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos\big((2k-1)\pi x\big)$$



Since the periodic extension of $|x|$ with the period 2 is continuous the F.S. converges uniformly to the periodic extension. Hence, with $x = 1$

$$\frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\underbrace{\cos\big((2k-1)\pi\big)}_{-1} = 1.$$

Always interesting to calculate a specific numerical sequence

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{odd} \frac{1}{k^2} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}$

Hence $$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3}\,\frac{\pi^2}{8} = \frac{\pi^2}{6}}$$