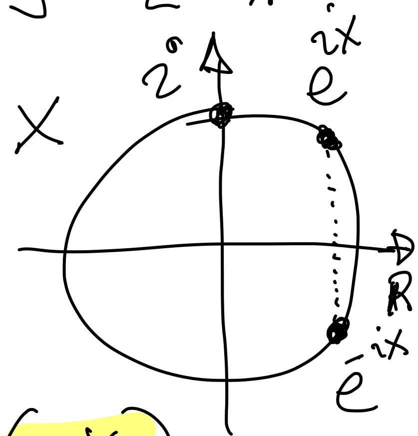
Complex Fourier Series

The Derivation of Vibrating String

Euler's Formula

Now we restrict to $[-\pi, \pi]$ $L = \pi$

$$e^{ix} = \cos x + i \sin x$$



$1, \cos(nx), \sin(nx), n \in \mathbb{N}$

complex valued function $\rightarrow$

$$\underbrace{e^{inx}}_{\text{complex exponential}} = \cos(nx) + i \sin(nx)$$

$e^{inx} = (e^{ix})^n$

The idea is to expand $f : [-\pi, \pi] \to \mathbb{C}$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx} \quad n \in \mathbb{Z}$$

$z \in \mathbb{C}$ $\quad |z|^2 = z\bar{z}$ $\quad |i|^2 = 1$

working with vectors in $\mathbb{C}^n$ appropriate

dot product is $\vec{x}, \vec{y} \in \mathbb{C}^n$

$= x_1, \dots, x_n$ $\quad \to y_1, \dots, y_n$

complex conjugate

$$\vec{x} \cdot \vec{y} = \sum_{k=1}^{n} x_k \bar{y}_k$$

length of $\vec{x}$, $\|\vec{x}\|$ is $\|\vec{x}\|^2 = \sum_{k=1}^{n} x_k \bar{x}_k$

$$= \sum_{k=1}^{n} |x_k|^2$$

$$\overline{e^{-inx}} = \overline{(\cos(-nx) + i\sin(-nx))} =$$

$$= (\cos(nx) - i\sin(nx)) =$$

$$= \cos(nx) + i\sin(nx) = e^{inx}$$

$\overline{(e^{it})} = e^{-it}$

cos even

sin odd

For complex functions $f, g : [-\pi, \pi] \to \mathbb{C}$ the orthogonality relation is

$$\int_{-\pi}^{\pi} f(\xi)\overline{g(\xi)}\, d\xi \qquad n \in \mathbb{Z}$$

Now consider the function $e^{-inx} : [-\pi, \pi] \to \mathbb{C}$

$k, m \in \mathbb{Z}$, $k \neq m$, e^{-ikx} is orthogonal to e^{-imx}

$$\int_{-\pi}^{\pi} e^{-ikx}\, \overline{e^{-imx}}\, dx = \int_{-\pi}^{\pi} e^{-ikx} e^{imx}\, dx = \int_{-\pi}^{\pi} e^{i(m-k)x}\, dx$$

$$= \frac{1}{i(m-k)}\, e^{i\underbrace{(m-k)}_{\in \mathbb{Z}}x}\Big|_{-\pi}^{\pi} = \frac{1}{i(m-k)}\Big(\underbrace{e^{i(m-k)\pi}}_{(-1)^{m-k}} - \underbrace{e^{i(m-k)(-\pi)}}_{(-1)^{m-k}}\Big) = 0$$

$$\int_{-\pi}^{\pi} e^{-ikx}\,\overline{e^{-ikx}}\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi$$

$$\boxed{f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}}$$

e^{-ikx} and integrate on $[-\pi,\pi]$

$$\boxed{\int_{-\pi}^{\pi} f(\xi)e^{ik\xi}\,d\xi = c_k \int_{-\pi}^{\pi} \underbrace{e^{-ikx}\,e^{ikx}}_{1}\,dx}$$

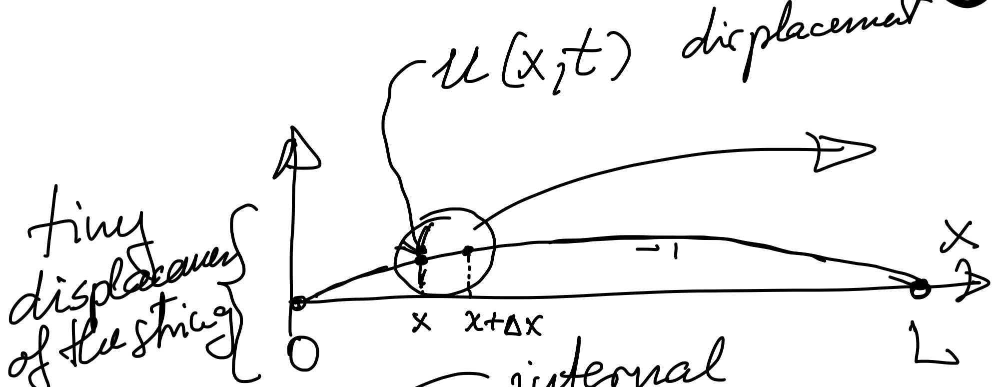
$c_k \cdot 2\pi$

true even for $\underline{k=0}$

$$\boxed{c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)e^{ik\xi}\,d\xi}$$

all $k \in \mathbb{Z}$

Vibrating String Equation

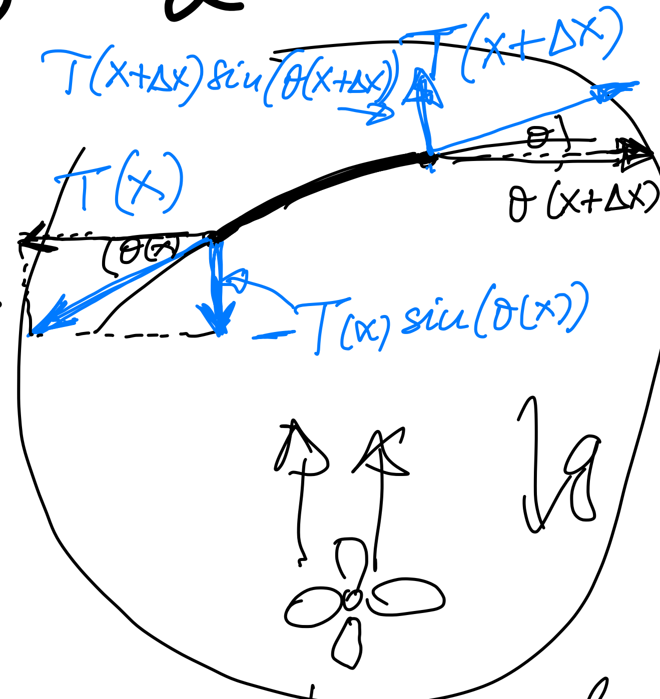


internal

The only force in the string is $\boxed{\text{Tension}}$ $T(x)$

The force in the string is

$T(x+\Delta x)\sin(\theta(x+\Delta x)) - T(x)\sin(\theta(x))$



θ is tiny angle

$\cos\theta(x+\Delta x) \doteq \cos(\theta(x))$

$F = ma$

the mass of this tiny piece of string is

$$\underbrace{\rho(x)}_{\text{linearity density of the sl}} \Delta x \underbrace{\frac{\partial^2 u}{\partial t^2}(x,t)}_{a} \simeq T(x+\Delta x)\sin(\theta(x+\Delta x)) - T(x)\sin\theta(x)$$

$$+ \underbrace{\rho(x)\Delta x}_{\text{mass}} \underbrace{Q(x,t)}_{\text{acceleration}}$$

divide by Δx and let $\Delta x \to 0$

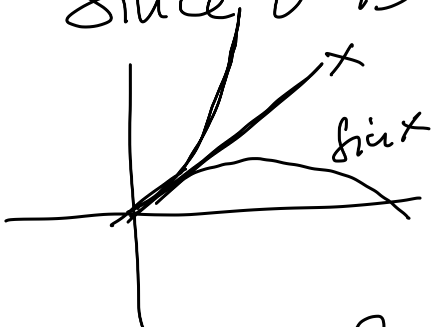
$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\underbrace{T(x)\sin(\theta(x))}\right) + \rho(x)Q(x,t)$$

We want everything in terms of $u(x,t)$

where is u

The tensile force is TANGENT to the string $\underbrace{\tan\theta(x)}_{\approx\ \sin\theta(x)} = \frac{\partial u}{\partial x}(x,t)$

Since θ is small



$$\rho(x)\,\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial}{\partial x}\left(T(x)\,\frac{\partial u}{\partial x}(x,t)\right) + \rho(x)Q(x,t)$$

If ρ, T constant.

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \rho_0 Q(x,t)$$

$$\frac{\partial^2 u}{\partial t^2} = \underset{\text{"}T_0/\rho_0\text{"}}{k} \frac{\partial^2 u}{\partial x^2} + \underset{\overset{\shortparallel}{0}}{Q(x,t)}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = k \frac{\partial^2 u}{\partial x^2}}$$

Boundary Conditions tomorrow.