Solving the Vibrating String Equation using the Natural Modes of Vibration

The Vibrating String Equation:

$$\frac{\partial^2 u}{\partial t^2} = \underbrace{\frac{T_0}{\rho_0}}_{=c^2} \frac{\partial^2 u}{\partial x^2}, \quad c = \sqrt{\frac{T_0}{\rho_0}}$$

Boundary Conditions

$$\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$$

Dirchlet BCs

→ string fixed at the equilibrium position @ the ends

Other BCs are possible

$h \in \mathbb{R}$

$$u(0,t) + h \frac{\partial u}{\partial x}(0,t) = 0$$

L L

BCs of third kind

Neumann BC : $\frac{\partial u}{\partial x}(0,t) = 0$

special = string float!

Solving by Separation of Variables

$$u(x,t) = A(x)\,B(t)$$

Substitute in VS eq.

$$A(x)\,B''(t) = c^2 A''(x)\,B(t)$$ Ordinary derivatives

Separate independent variables:

$$\frac{B''(t)}{c^2 B(t)} = \frac{A''(x)}{A(x)} = -\lambda$$

must be constant

Don't forget BCs

$$A(0)\,B(t) = 0 \qquad A(0) = 0$$
$$\Rightarrow$$
$$A(L)\,B(t) = 0 \qquad A(L) = 0$$

Since we have no condition on B we can solve it subject to knowing λ

time part

$$B''(t) = -\lambda c^2 B(t)$$

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We expect $\lambda > 0$ so the fundamental solution will be a lin. comb of $\sin(\ t)$ and $\cos(\ t)$

But, if $\lambda = 0$ or scarier $\lambda < 0$ we will encounter $\cosh(\ t)$ & $\sinh(\ t)$ which are unbounded

space part

$$-A''(x) = \lambda A(x)$$

$$A(0) = 0$$

$$A(L) = 0$$

Because of BCs this is NOT 331 problem

This is an eigenvalue problem, it is called a boundary-eigenvalue problem: Solve it below

$\lambda > 0$, set $\lambda = \mu^2$ with $\mu > 0$

The solutions are

$$\underbrace{\lambda = \left(\frac{n\pi}{L}\right)^2}_{\text{eigenvalues}} \text{ with a corresp. eigenfunction } \sin\left(\frac{n\pi}{L}x\right)$$

We obtained the eigenvalues and the corresponding eigenfunctions by using the BCs. Different BCs would lead to different evs & efs.

If we had different BCs we would obtain different eigenvalues and different eigenfunctions

$\lambda = 0$ is NOT an eigenvalue

$\lambda < 0$ cannot be an eigenvalue

FOR THESE BCs.

With different BCs we can have $\lambda = 0$ as an e. value and we can have (but only finitely many) negative eigenvalues.

Now we have ALL the eigenvalues and we go back to $B(t)$:

$$B''(t) = \left(\frac{n\pi}{L}\right)^2 c^2 B(t)$$

The fundamental solution is

$$a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right)$$

So we obtained our sequence of solutions with independent variables separated (in different functions)

$n \in \mathbb{N}$ $\underbrace{\sin\left(\frac{n\pi}{L}x\right)}_{\text{space}} \left(a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right)\right)$

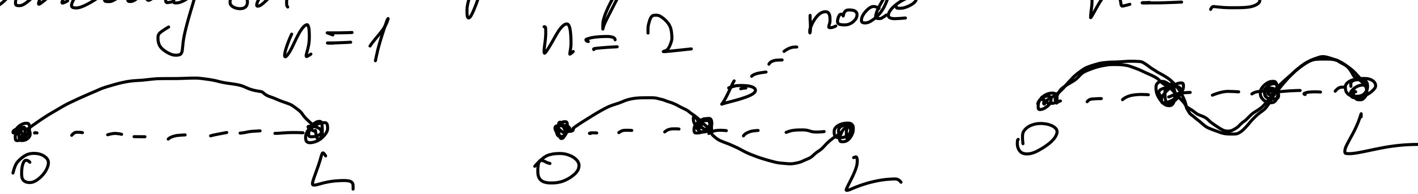
time

$$c = \sqrt{\frac{T_0}{\rho_0}}$$

These are strings vibrating!

Natural modes of vibration. or harmonics first, seco

depending on n space part looks like



Time part:

based on a_n or b_n

$$a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right)$$

$$= \underbrace{\sqrt{a_n^2 + b_n^2}}_{\text{amplitude}} \sin\left(\frac{n\pi c}{L} t + \theta\right)$$

time shift

this part governs frequency of osillations.

— How long does it take to complete one

full oscillation? $0_{start} \xrightarrow{\frac{n\pi c}{L} t} 2\pi$, $t = \dfrac{2\pi}{\frac{n\pi c}{L}} = \underbrace{\dfrac{2L}{nc}}_{\text{sec/one}}$

How many oscillations happen in one second? $\dfrac{nc}{2L}$ oscillations happen in one secon

$$\underbrace{\frac{n}{2L}\sqrt{\frac{T_0}{\rho_0}}}_{\text{frequency of oscillations of n-th harmonic}}$$

$n = 1 \rightarrow$ fundamental frequency

$$\frac{1}{2L}\sqrt{\frac{T_0}{\rho_0}}\ \left.\vphantom{\begin{array}{l}a\\b\end{array}}\right\}\begin{array}{l} L \text{ shorter freq } \uparrow \\ T_0 \text{ higher freq } \uparrow \end{array}$$