Sturm-Liouville Eigenvalue Problems

$$\boxed{-A''(x) = \lambda A(x)}$$

$$\begin{cases} A(0) = 0 \\ A(L) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ \left.\begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0 \end{array}\right\} \text{BCs} \end{cases}$$

I.Cs

→ different BCs are possible

Alway homogeneous

→ What is more of a problem, more complicated differential equation occurs.

Heat Equation

$Q(x,t)$ special

$$\rho(x)\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}(x,t)\right) + \alpha(x)u(x,t)$$

Separate variables: $A(x)\,B(t)$

$$\rho(x)A(x)B'(t) = \left(K_0(x)A'(x)\right)'\underbrace{B(t)} + \alpha(x)A(x)\underbrace{B(t)}$$

$$\frac{B'(t)}{B(t)} = \frac{\left(K_0(x)A'(x)\right)' + \alpha(x)A(x)}{\rho(x)A(x)} = -\lambda$$

and variables are separated ☺

known coeff

$$\left(\underbrace{K_0(x)}_{\text{coeff.}}\underbrace{A'(x)}_{\text{unknow fn}}\right)' + \underbrace{\alpha(x)}_{\text{known c}}\underbrace{A(x)}_{\text{unknw}} = -\lambda\rho(x)A(x)$$

$p(x)$ $\quad$ $y(x) = A(x)$ $\quad$ $q(x)$ $\quad$ $w(x)$

The general Sturm-Liouville eigenvalue problem

$$-\left(p(x)\,y'(x)\right)' + q(x)\,y(x) = \lambda\, w(x)\, y(x)$$

$+$ $\underline{BC}$ $\quad$ $y(0),\ y(L)$

$y'(0),\ y'(L)$

polar coordinates circulary symmetric

Laplacian in polar $\quad \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \underbrace{\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}}_{=0}$

$$\frac{\partial u}{\partial t} = \varkappa \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

$$u(r,t) = A(r)\,B(t)$$

$$A(r)\,B'(t) = \varkappa \frac{1}{r}\left(rA'(r)\right)' B(t)$$

$$\frac{B'(t)}{\varkappa B(t)} = \frac{\frac{1}{r}\left(rA'(r)\right)'}{A(r)} = -\lambda$$

Eigenvalue D.E.

$$\frac{1}{r}\left(rA'(r)\right)' = -\lambda A(r)$$

$$-\left(x\,y'(x)\right)' = \lambda\, x\, y(x)$$

$p(x)$

$q(x)=0$

$w(x)$

weight