Sturm-Liouville

Problems - Properties & How to Solve

Regular S-L Proplem:

$[0,L]$ Let $a, b \in \mathbb{R}$, $a<b$, let

$p, q, w : [a,b] \to \mathbb{R}$

SL eq: $-(py')' + qy = \lambda w y$

+ BCs at a and at b

two

$y(a) = 0$

$y'(a) = 0$

$y'(a) - hy(a) = 0$

same at b

$y(b) = 0$

$y'(b) = 0$

$\vdots$

OR BCs involving both points $y(a) = y(b)$

periodic BCs

In general

$$\underbrace{\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \end{bmatrix}}_{B} \begin{bmatrix} y(a) \\ (py')(a) \\ y(b) \\ (py')(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Periodic $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

Must satisfy a special symmetry property.

The theory of S-L eigenvalue problem is

- All eigenvalues are Real
- there exists the smallest eigenvalue but not the largest

the eigenvalues form an unbounded sequence

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$$

* To each eigenvalue there corresponds an eigenfunction φ_n

λ_n corres. e. fn $\varphi_n(x) \neq 0$

* The eigenfunction corr. to <u>distinct</u> eigenvalues are orthogonal. meaning

$m \neq n$ $\displaystyle\int_a^b \varphi_n(x)\varphi_m(x)\,w(x)\,dx = 0$

* Similar to Fourier series

f piecewise smooth

$$f(x) \sim \sum_{n=1}^{\infty} c_n \varphi_n(x)$$

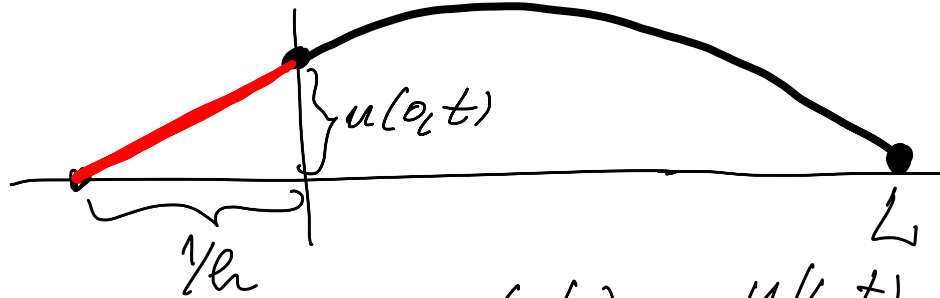
with similar convergence properties to F.S.

$$c_n = \frac{1}{\int_a^b (\varphi_n(x))^2\,dx} \int_a^b f(x)\,\varphi_n(x)\,dx$$

Example

Separate variables

$$\boxed{\begin{aligned} -y''(x) &= \lambda y(x) \\ y'(0) - h\,y(0) &= 0 \\ y(L) &= 0 \end{aligned}}$$



$$\frac{\partial u}{\partial x}(0,t) \approx \frac{u(0,t)}{1/h}$$

$u(L,t)=0$ $\forall t>0$

$$\frac{\partial u}{\partial x}(0,t) - h\,u(0,t) = 0$$

$h>0$

S-L.

Case 1 $\lambda>0$. Set $\lambda=\mu^2$, $\mu>0$

The Fundamental sol. of

$$y''(x)+\mu^2 y(x)=0$$

$$\text{"}y(x)=\text{"}\, C_1 \cos(\mu x) + C_2 \sin(\mu x)$$

$$y'(x) = -\mu C_1\, s(\mu x) + \mu C_2\, c(\mu x)$$

We want those μ-s for which a nonzero $y(x)$ exists satisfying Both BCs

$$y(0) = C_1$$

$$y'(0) = \mu C_2$$

$$y(L) = C_1\, c(\mu L) + C_2\, s(\mu L)$$

BCs are: $y'(0) - h\, y(0) = 0$

$y(L) = 0$

that is:

$$\boxed{\begin{aligned} \mu C_2 - h C_1 &= 0 \\ C_1\, c(\mu L) + C_2\, s(\mu L) &= 0 \end{aligned}}$$

unknowns are C_1 & C_2

nontrivial sol.

$$-h C_1 + \quad C_2 = 0$$
$$c(\mu L) C_1 + s(\mu L) C_2 = 0$$

$$\begin{vmatrix} -h & 1 \\ c(\mu L) & s(\mu L) \end{vmatrix} = 0$$

$$-h\, s(\mu L) - c(\mu L) = 0$$

Find μ-s s.t.