Solving S-L

Eigenvalue Problem: Negative Eigenvalues

$$-y''(x) = \lambda y(x) \qquad 0 \le x \le L$$

$$y'(0) + h y(0) = 0$$

$$y(L) = 0$$

Negative eigenvalues:

$\lambda = -\mu^2$, $\boxed{\mu > 0}$

The key step here is KNOWING

the fundamental solution of

$$y''(x) - \mu^2 y(x) = 0$$

$$y(x) = C_1 \cosh(\mu x) + C_2 \sinh(\mu x)$$

$$y'(x) = \mu C_1 \, sh(\mu x) + \mu C_2 \, ch(\mu x)$$

$$y(0) = C_1 \qquad y(L) = C_1 \, ch(\mu L) + C_2 \, sh(\mu L)$$

$$y'(0) = \mu C_2$$

So, applying BCs we get the system

$$\boxed{\begin{aligned} h\,C_1 + \mu\,C_2 &= 0 \\ ch(\mu L)\,C_1 + sh(\mu L)\,C_2 &= 0 \end{aligned}}$$

Seek μ-s for which this system has nontrivial solutions.

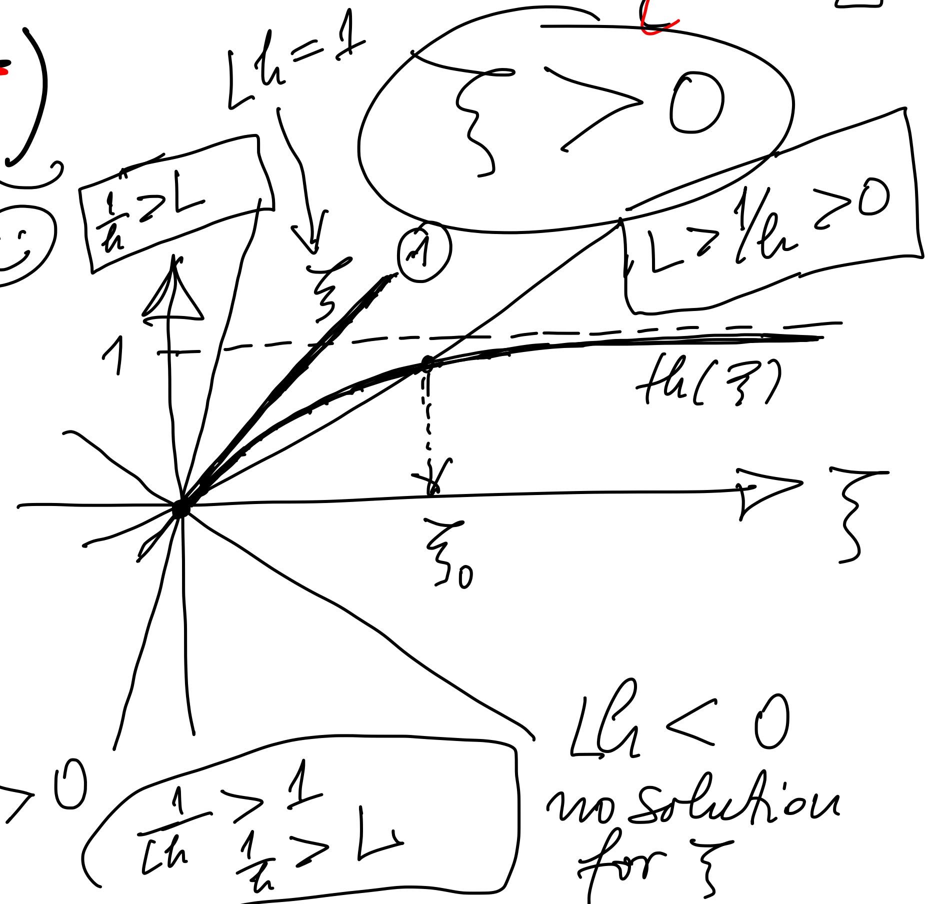
$$\begin{vmatrix} h & \mu \\ ch(\mu L) & sh(\mu L) \end{vmatrix} = 0 \qquad h\,sh(\mu L) - \mu\,ch(\mu L) = 0$$

$$\boxed{\mu = h\,th(\mu L)}$$

Introduce new variable $\xi = \mu L$, $\mu = \frac{\xi}{L}$

$$\underbrace{\xi}_{\text{☺}} = Lh\,\underbrace{th(\xi)}_{\text{☺}}$$

$$\underbrace{\frac{1}{Lh}}_{h \neq 0}\xi = th(\xi) \quad \text{"}\frac{sh(\xi)}{ch(\xi)}\text{"}$$



$0 < \frac{1}{Lh} < 1$

$Lh > 1$

$L > 1/h > 0$

there exists one negative eigenvalue $\iff$

$$\iff 0 < \frac{1}{h} < L.$$

Done with eigenvalues. Now find a corresponding eigenfunction.

Assume $0 < \frac{1}{h} < L$

Let $\mu_0 > 0$ be such that

$$\mu_0 = h \, \mathrm{th}(\mu_0 L)$$

(we can find an approx in Mathematica)

$$\boxed{\begin{aligned} h\, c_1 + \mu_0 c_2 &= 0 \\ \operatorname{ch}(\mu_0 L)\, c_1 + \operatorname{sh}(\mu_0 L)\, c_2 &= 0 \end{aligned}}$$

We know that this system has a nontrivial solution

We need only one such, we choose an easy one $c_1 = \mu_0 \quad c_2 = -h$.

thus an eigenfunction is :

$$y_0(x) = \mu_0 \operatorname{ch}(\mu_0 x) - h \operatorname{sh}(\mu_0 x)$$

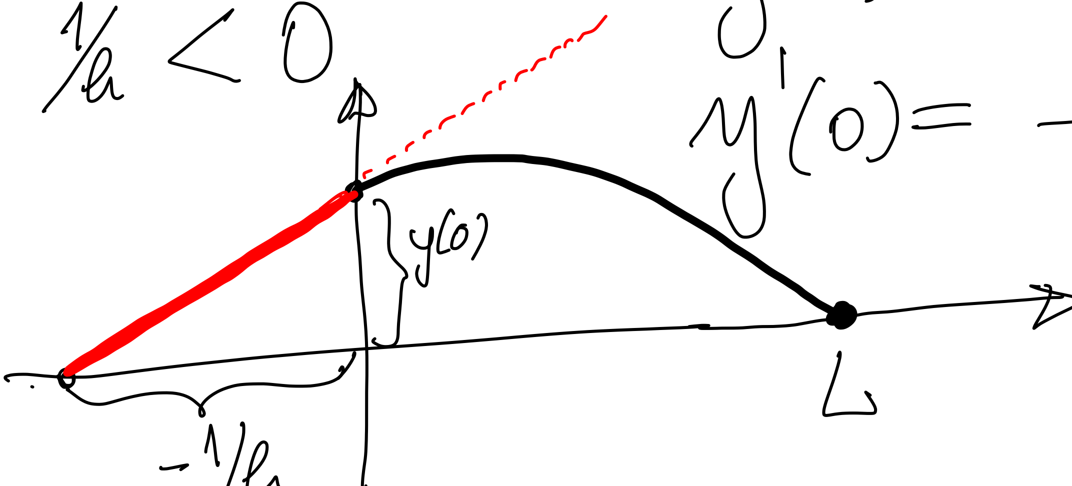
This eigenfunction corresponds to the eigenvalue $-\mu_0^2 < 0$.

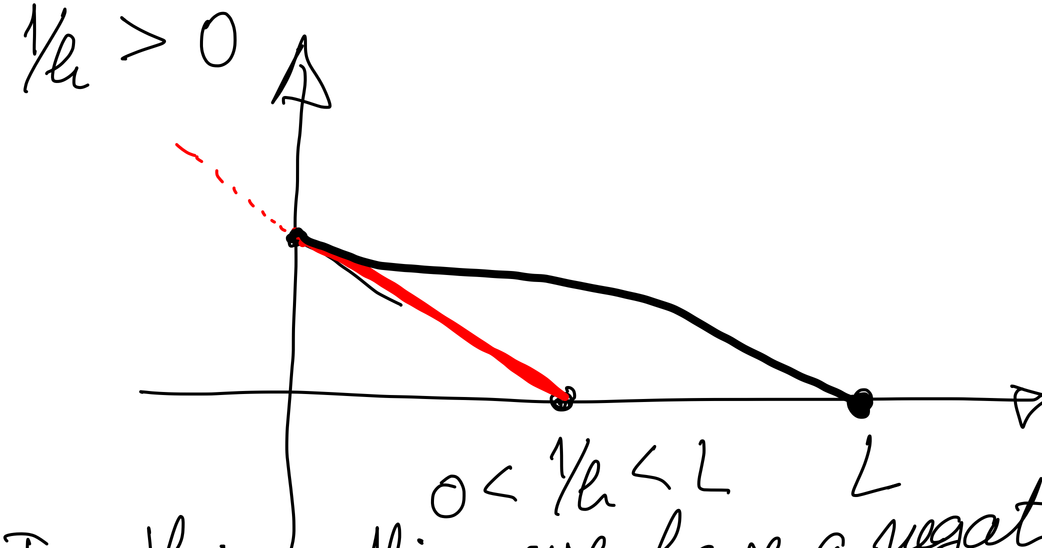
the geometric representation of the BCs

$$y'(0) = -h\,y(0)$$

$$y'(0) = \frac{y(0)}{-1/h}$$

$1/h < 0$





In this setting we have a negative e.value