Membranes

(Negative eigenvalues)
and consequences

vibrating string equation $\quad \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le L, \quad L > 0$

BC,

I.C. $\quad u(x,0) = f(x) \qquad 0 \le x \le L$

$\dfrac{\partial u}{\partial t}(x,0) = g(x) \qquad 0 \le x \le L$

$$u(x,t) = A(x)B(t)$$

$$\frac{B''(t)}{c^2 B(t)} = \frac{A''(x)}{A(x)} = -\lambda$$

Assume we found positive eigenvalues $\lambda_1 \le \lambda_2 \le \dots$ with a corresponding eigenfunctio $\varphi_1, \varphi_2, \dots$

Assume we found one negative eigenvalue λ_0 with a corresp. e.f. Y_0

Go back to $B(t)$

$$B''(t) = -\lambda c^2 B(t)$$

Case 1 $\lambda \in \{\lambda_1, \lambda_2, \ldots\}$ that is $\lambda > 0$

$$\lambda_j = \mu_j^2, \quad \mu_j > 0$$

$$B''(t) + \mu_j^2 c^2 B(t) = 0$$

Fund. Solution

$$B(t) = C_1 \cos(\mu_j c t) + C_2 \sin(\mu_j c t)$$

Case 2 $\lambda = \lambda_0 < 0$

$$\lambda_0 = -\mu_0^2, \ \mu_0 > 0$$

the $B''(t) - \mu_0^2 c^2 B(t) = 0$

Fundamental Solution is:

$$B(t) = c_1 \, ch(\mu_0 c t) + c_2 \, sh(\mu_0 c t)$$

Why is all of this done? Since now we have a "general solution" of the PDE:

$$u(x,t) = a_0 y_0(x)\,ch(\mu_0 c t) + b_0 y_0(x)\,sh(\mu_0 c t) + \sum_{n=1}^{\infty}\left(a_n y_n(x)\cos(\mu_n c t) + b_n y_n(x)\sin(\mu_n c t)\right)$$

~~Now we can bring ICs. $g(x)=0$, $u(x,0)=f(x)$~~

I should have mentioned product solutions

$$a_0 y_0(x)\,ch(\mu_0 c t) + b_0 y_0(x)\,sh(\mu_0 c t) = \sqrt{a_0^2+b_0^2}\; y_0(x)\,sh(\mu_0 c t + \omega_0)$$

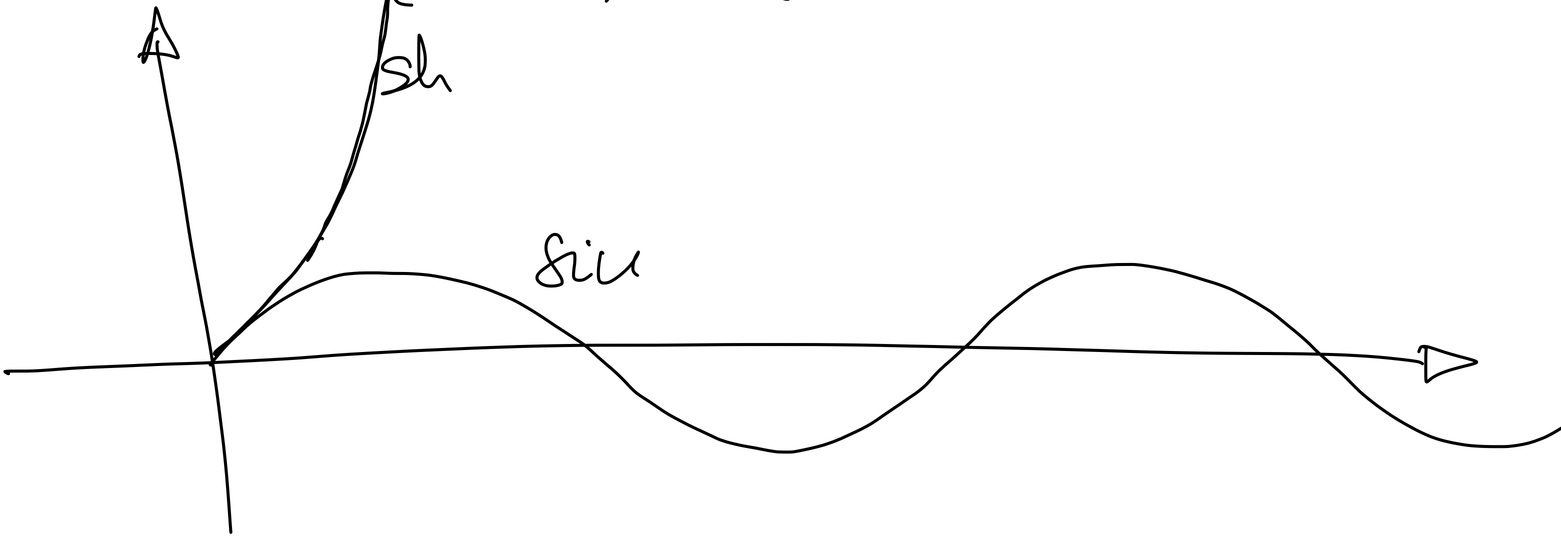
This sol. does not vibrate

$$a_n y_n(x)\cos(\mu_n c t) + b_n y_n(x)\sin(\mu_n c t) = \sqrt{a_0^2+b_0^2}\; y_n(x)\sin(\mu_n c t + \omega_n)$$

vibrations

For the Dirichlet BCs

$$\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}t\right) \qquad \underline{\underline{n \in \mathbb{N}}}$$



given ICs: $u(x,0) = f(x)$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

$$u(x,t) = a_0 y_0(x)\, ch(\mu_0 c t) + b_0 y_0(x)\, sh(\mu_0 c t) + \sum_{n=1}^{\infty} \left(a_n y_n(x) \cos(\mu_n c t) + b_n y_n(x) \sin(\mu_n c t) \right)$$

Now we can bring ICs. $g(x) = 0$

$u(x,0) = f(x)$

$t = 0$

$$f(x) = u(x,0) = a_0 y_0(x) + \sum_{n=1}^{\infty} a_n y_n(x) \quad / * \, y_k(x)$$

We recall that all $y_0, y_1, \ldots, y_n, \ldots$ are mutually orthogonal, so we calculate $a_0, \ldots$

$$\int_0^L f(\xi)\, y_k(\xi)\, d\xi = a_k \int_0^L y_k(\xi)^2\, d\xi$$

$$a_k = \frac{1}{\int_0^L y_k(\xi)^2\, d\xi} \int_0^L f(\xi)\, y_k(\xi)\, d\xi$$

$$\frac{\partial u}{\partial t}(x,t) = \mu_0 c a_0\, y_0(x)\, \operatorname{sh}(\mu_0 c t) + \mu_0 c b_0\, y_0(x)\, \operatorname{ch}(\mu_0 c t)$$
$$+ \sum a_n \mu_n c\, y_n(x) \sin(\mu_n c t) +$$
$$+ b_n \mu_n c\, y_n(x) \cos(\mu_n c t)$$

$$g(x) = \mu_0 c b_0\, y_0(x) + \sum_{n=1}^{\infty} b_n \mu_n c\, y_n(x)$$

$$b_n = \frac{1}{\mu_n c \int_0^L y_n(\xi)^2 d\xi} \int_0^L g(\xi) y_n(\xi) d\xi$$

$n \in \mathbb{N} \cup \{0\}$

All this becomes truly alive after reading my Mathematica notebook:
Super_glued_ends_solution_v12.nb
to be found on K-drive and in
430_Files on Dropbox