In problems that follow we make substitutions

$$\cos\theta = \frac{z^2+1}{2z}, \quad \sin\theta = \frac{z^2-1}{2iz} \quad \text{where} \quad z = e^{i\theta} \quad \text{and} \quad d\theta = \frac{1}{iz}\,dz.$$

Below $C = \{z \in \mathbb{C} \ : \ |z| = 1\}$ (the unit circle) and it is oriented counterclockwise. Symbols C_1 and C_2 denote appropriately chosen circles oriented counterclockwise.

Problem 1 (6.6.1 Ex. 5). Evaluate $\displaystyle\int_0^{\pi} \frac{1}{2-\cos\theta}\, d\theta.$

Solution. Done in class:

$$\int_0^{\pi} \frac{1}{2-\cos\theta}\, d\theta = \frac{\pi}{\sqrt{3}}.$$

□

Problem 2 (6.6.1 Ex. 6). Evaluate $\displaystyle\int_0^{\pi} \frac{1}{1+(\sin\theta)^2}\, d\theta.$

Solution.

$$\begin{aligned}
\int_0^{\pi} \frac{1}{1+(\sin\theta)^2}\, d\theta &= \frac{1}{2}\int_0^{2\pi} \frac{1}{1+(\sin\theta)^2}\, d\theta \\
&= \frac{1}{2}\oint_C \frac{1}{1+\left(\dfrac{z^2-1}{2iz}\right)^2}\, \frac{1}{iz} dz \\
&= \frac{1}{2}\oint_C \frac{-4z^2}{-4z^2+z^4-2z^2+1}\, \frac{1}{iz} dz \\
&= -\frac{2}{i}\oint_C \frac{z}{z^4-6z^2+1}\, dz \\
&= -\frac{2}{i}\oint_C \frac{z}{\big(z^2-(3+2\sqrt{2})\big)\big(z-\sqrt{3-2\sqrt{2}}\big)\big(z+\sqrt{3-2\sqrt{2}}\big)}\, dz \\
&= -\frac{2}{i}\oint_{C_1} \frac{\dfrac{z}{\big(z^2-(3+2\sqrt{2})\big)\big(z-\sqrt{3-2\sqrt{2}}\big)}}{\big(z+\sqrt{3-2\sqrt{2}}\big)}\, dz \\
&\qquad -\frac{2}{i}\oint_{C_2} \frac{\dfrac{z}{\big(z^2-(3+2\sqrt{2})\big)\big(z+\sqrt{3-2\sqrt{2}}\big)}}{\big(z-\sqrt{3-2\sqrt{2}}\big)}\, dz \\
&= -\frac{2}{i}\frac{2\pi i}{0!}\left(\frac{z}{\big(z^2-(3+2\sqrt{2})\big)\big(z-\sqrt{3-2\sqrt{2}}\big)}\right)\Bigg|_{z=-\sqrt{3-2\sqrt{2}}} \\
&\qquad -\frac{2}{i}\frac{2\pi i}{0!}\left(\frac{z}{\big(z^2-(3+2\sqrt{2})\big)\big(z+\sqrt{3-2\sqrt{2}}\big)}\right)\Bigg|_{z=\sqrt{3-2\sqrt{2}}} \\
&= -4\pi\frac{-\sqrt{3-2\sqrt{2}}}{-4\sqrt{2}\big(-2\sqrt{3-2\sqrt{2}}\big)} - 4\pi\frac{\sqrt{3-2\sqrt{2}}}{-8\sqrt{2}\sqrt{3-2\sqrt{2}}} \\
&= 4\pi\frac{1}{8\sqrt{2}} + 4\pi\frac{1}{8\sqrt{2}}
\end{aligned}$$

$$= \frac{\pi}{\sqrt{2}} \qquad \square$$

Problem 3 (6.6.1 Ex. 8). Evaluate $\displaystyle\int_0^{2\pi} \frac{(\cos\theta)^2}{3-\sin\theta}\, d\theta$.

Solution. Done in class:

$$\int_0^{2\pi} \frac{(\cos\theta)^2}{3-\sin\theta}\, d\theta = (6-4\sqrt{2})\pi. \qquad \square$$

Problem 4 (6.6.1 Ex. 10). Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{\cos\theta + 2\sin\theta + 3}\, d\theta$.

Solution.

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{\cos\theta + 2\sin\theta + 3}\, d\theta &= \oint_C \frac{1}{\frac{z^2+1}{2z} + 2\frac{z^2-1}{2iz} + 3}\ \frac{1}{iz} dz \\
&= \oint_C \frac{2iz}{i\,z^2 + i + 2z^2 - 2 + 6iz}\ \frac{1}{iz} dz \\
&= \oint_C \frac{2}{(2+i)z^2 + 6iz + (-2+i)}\ dz \\
&= \oint_C \frac{2}{(2+i)\big(z+(1+2i)\big)\ \big(z+\frac{1}{5}(1+2i)\big)}\ dz \\
&= \oint_C \frac{\dfrac{2}{(2+i)\big(z+(1+2i)\big)}}{z - \left(-\dfrac{1}{5}(1+2i)\right)}\ dz \\
&= 2\pi i \left(\frac{2}{(2+i)\big(z+(1+2i)\big)}\right)\Bigg|_{z=-\frac{1}{5}(1+2i)} \\
&= 2\pi i \frac{2}{(2+i)\big(-\frac{1}{5}(1+2i)+(1+2i)\big)} \\
&= 2\pi i \frac{2}{\frac{4}{5}(2+i)(1+2i)} \\
&= \pi i \frac{5}{2-2+i+4i} \\
&= \pi
\end{aligned} \qquad \square$$

Problem 5 (6.6.1 Ex. 9). Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta}\, d\theta$.

Solution.

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta}\, d\theta &= \oint_C \frac{\dfrac{z^4+1}{2z^2}}{5 - 4\dfrac{z^2+1}{2z}}\ \frac{1}{iz} dz \\
&= \oint_C \frac{\dfrac{z^4+1}{2z^2}}{\dfrac{5z-2z^2-2}{z}}\ \frac{1}{iz} dz
\end{aligned}$$

$$= \oint_C \frac{z^4+1}{(2iz^2)\big(5z-2z^2-2\big)}\ dz$$

$$= \oint_C \frac{-\big(z^4+1\big)}{2(2iz^2)\,(z-2)\,\left(z-\frac{1}{2}\right)}\ dz$$

$$= -\frac{1}{4i}\oint_C \frac{\dfrac{z^4+1}{z-2}}{z^2\left(z-\frac{1}{2}\right)}\ dz$$

$$= -\frac{1}{2i}\oint_{C_1} \frac{\dfrac{z^4+1}{2z^2-5z+2}}{z^2}\ dz - \frac{1}{4i}\oint_{C_2} \frac{\dfrac{z^4+1}{z^2(z-2)}}{z-\frac{1}{2}}\ dz$$

$$= -\frac{1}{2i}\frac{2\pi i}{1!}\left(\frac{d}{dz}\left(\frac{z^4+1}{2z^2-5z+2}\right)\right)\Bigg|_{z=0} - \frac{1}{4i}\frac{2\pi i}{0!}\left(\frac{z^4+1}{z^2(z-2)}\right)\Bigg|_{z=\frac{1}{2}}$$

$$= -\pi\left(\frac{4z^3\big(2z^2-5z+2\big)-\big(z^4+1\big)(4z-5)}{\big(2z^2-5z+2\big)^2}\right)\Bigg|_{z=0} - \frac{\pi}{2}\frac{\frac{1}{16}+1}{\frac{1}{4}\left(\frac{1}{2}-2\right)}$$

$$= -\pi\ \frac{0*2-1*(-5)}{2^2} + \frac{17}{12}\ \pi$$

$$= -\frac{5}{4}\ \pi + \frac{17}{12}\ \pi$$

$$= \frac{-15+17}{12}\ \pi$$

$$= \frac{\pi}{6}$$

□

Problem 6 (6.6.1 Ex. 12). Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\ d\theta$.

Solution.

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\ d\theta = \oint_C \frac{\dfrac{z^6+1}{2z^3}}{5-4\dfrac{z^2+1}{2z}}\ \frac{1}{iz}dz$$

$$= \oint_C \frac{\dfrac{z^6+1}{2z^3}}{\dfrac{5z-2z^2-2}{z}}\ \frac{1}{iz}dz$$

$$= \oint_C \frac{z^6+1}{(2iz^3)\big(5z-2z^2-2\big)}\ dz$$

$$= \oint_C \frac{-\big(z^6+1\big)}{2(2iz^3)\,(z-2)\,\left(z-\frac{1}{2}\right)}\ dz$$

$$= -\frac{1}{4i}\oint_C \frac{\dfrac{z^6+1}{z-2}}{z^3\left(z-\frac{1}{2}\right)}\ dz$$

$$= -\frac{1}{2i}\oint_{C_1} \frac{\dfrac{z^6+1}{2z^2-5z+2}}{z^3}\ dz - \frac{1}{4i}\oint_{C_2} \frac{\dfrac{z^6+1}{z^3(z-2)}}{z-\frac{1}{2}}\ dz$$

$$
\begin{aligned}
&= -\frac{1}{2i}\frac{2\pi i}{2!}\left(\frac{d^2}{dz^2}\left(\frac{z^6+1}{2z^2-5z+2}\right)\right)\Bigg|_{z=0} - \frac{1}{4i}\frac{2\pi i}{0!}\left(\frac{z^6+1}{z^3(z-2)}\right)\Bigg|_{z=\frac{1}{2}} \\
&= -\frac{\pi}{2}\left(\frac{d}{dz}\left(\frac{6z^5\left(2z^2-5z+2\right)-\left(z^6+1\right)(4z-5)}{\left(2z^2-5z+2\right)^2}\right)\right)\Bigg|_{z=0} - \frac{\pi}{2}\frac{\frac{1}{64}+1}{\frac{1}{8}\left(\frac{1}{2}-2\right)} \\
&= -\frac{\pi}{2}\left(\frac{d}{dz}\left(\frac{8z^7-25z^6+12z^5-4z+5}{\left(2z^2-5z+2\right)^2}\right)\right)\Bigg|_{z=0} + \frac{65}{24}\ \pi \\
&= -\frac{\pi}{2}\ \frac{(-4)*2^2-2*5*2*(-5)}{2^4} + \frac{65}{24}\ \pi \\
&= -\frac{21}{8}\ \pi + \frac{65}{24}\ \pi \\
&= \frac{-63+65}{24}\ \pi \\
&= \frac{\pi}{12}
\end{aligned}
$$

□

Problem 7 (6.6.1 Ex. 13). For $a > 1$ evaluate $\displaystyle\int_0^{\pi} \frac{1}{(a+\cos\theta)^2}\ d\theta$.

Solution.

$$
\begin{aligned}
\int_0^{\pi} \frac{1}{(a+\cos\theta)^2}\ d\theta &= \frac{1}{2}\int_0^{2\pi} \frac{1}{(a+\cos\theta)^2}\ d\theta \\
&= \frac{1}{2}\oint_C \frac{1}{\left(a+\frac{z^2+1}{2z}\right)^2}\ \frac{1}{iz}dz \\
&= \frac{1}{2}\oint_C \frac{4z^2}{\left(2az+z^2+1\right)^2}\ \frac{1}{iz}dz \\
&= \frac{2}{i}\oint_C \frac{z}{\left(z+a+\sqrt{a^2-1}\right)^2\left(z-\left(-a+\sqrt{a^2-1}\right)\right)^2}\ dz \\
&= \frac{2}{i}\oint_C \frac{\frac{z}{\left(z+a+\sqrt{a^2-1}\right)^2}}{\left(z-\left(-a+\sqrt{a^2-1}\right)\right)^2}\ dz \\
&= \frac{2}{i}\ \frac{2\pi i}{1!}\left(\frac{d}{dz}\ \frac{z}{\left(z+a+\sqrt{a^2-1}\right)^2}\right)\Bigg|_{z=-a+\sqrt{a^2-1}} \\
&= 4\pi\left(\frac{1*\left(z+a+\sqrt{a^2-1}\right)^2-2z\left(z+a+\sqrt{a^2-1}\right)}{\left(z+a+\sqrt{a^2-1}\right)^4}\right)\Bigg|_{z=-a+\sqrt{a^2-1}} \\
&= 4\pi\left(\frac{-z+a+\sqrt{a^2-1}}{\left(z+a+\sqrt{a^2-1}\right)^3}\right)\Bigg|_{z=-a+\sqrt{a^2-1}} \\
&= 4\pi\ \frac{2a}{\left(2\sqrt{a^2-1}\right)^3} \\
&= \frac{a\pi}{\left(\sqrt{a^2-1}\right)^3}
\end{aligned}
$$

□

Problem 8 (6.6.1 Ex. 14). For $a > b > 0$ evaluate $\displaystyle\int_0^{2\pi} \frac{(\sin\theta)^2}{a+b\cos\theta}\ d\theta$.

Solution.

$$\begin{aligned}
\int_0^{2\pi} \frac{(\sin\theta)^2}{a+b\cos\theta}\,d\theta &= \oint_C \frac{\left(\frac{z^2-1}{2iz}\right)^2}{a+b\frac{z^2+1}{2z}}\,\frac{1}{iz}dz \\
&= \oint_C \frac{\frac{(z^2-1)^2}{-4z^2}}{\frac{2az+bz^2+b}{2z}}\,\frac{1}{iz}dz \\
&= -\frac{1}{2i}\oint_C \frac{(z^2-1)^2}{z^2\left(2az+bz^2+b\right)}\,dz \\
&= -\frac{1}{2i}\oint_C \frac{(z^2-1)^2}{b\,z^2\left(z+\frac{a+\sqrt{a^2-b^2}}{b}\right)\left(z-\frac{-a+\sqrt{a^2-b^2}}{b}\right)}\,dz \\
&= -\frac{1}{2i}\oint_{C_1} \frac{\frac{(z^2-1)^2}{bz^2+2az+b}}{z^2}\,dz \\
&\qquad -\frac{1}{2bi}\oint_{C_2} \frac{\frac{(z^2-1)^2}{z^2\left(z+\frac{a+\sqrt{a^2-b^2}}{b}\right)}}{\left(z-\frac{-a+\sqrt{a^2-b^2}}{b}\right)}\,dz \\
&= -\frac{1}{2i}\frac{2\pi i}{1!}\left(\frac{d}{dz}\,\frac{(z^2-1)^2}{bz^2+2az+b}\right)\Bigg|_{z=0} \\
&\qquad -\frac{1}{2bi}\frac{2\pi i}{0!}\left(\frac{\left(z-\frac{1}{z}\right)^2}{\left(z+\frac{a+\sqrt{a^2-b^2}}{b}\right)}\right)\Bigg|_{z=\frac{-a+\sqrt{a^2-b^2}}{b}} \\
&= -\pi\frac{-2a}{b^2}-\frac{\pi}{b}\frac{\left(\frac{-a+\sqrt{a^2-b^2}}{b}-\frac{b}{-a+\sqrt{a^2-b^2}}\right)^2}{\left(\frac{-a+\sqrt{a^2-b^2}}{b}+\frac{a+\sqrt{a^2-b^2}}{b}\right)} \\
&= \frac{2\pi a}{b^2}-\frac{\pi}{b}\frac{\left(\frac{a^2-2a\sqrt{a^2-b^2}+a^2-b^2-b^2}{b\left(-a+\sqrt{a^2-b^2}\right)}\right)^2}{2\frac{\sqrt{a^2-b^2}}{b}} \\
&= \frac{2\pi a}{b^2}-\frac{\pi}{b}\frac{\left(2\sqrt{a^2-b^2}\frac{\sqrt{a^2-b^2}-a}{b\left(-a+\sqrt{a^2-b^2}\right)}\right)^2}{2\frac{\sqrt{a^2-b^2}}{b}} \\
&= \frac{2\pi a}{b^2}-\frac{\pi}{b}\frac{\left(2\frac{\sqrt{a^2-b^2}}{b}\right)^2}{2\frac{\sqrt{a^2-b^2}}{b}} \\
&= \frac{2\pi a}{b^2}-\frac{\pi}{b}2\frac{\sqrt{a^2-b^2}}{b} \\
&= \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)
\end{aligned}$$

□

Problem 9. For $a > 1$ show $\displaystyle\int_0^\pi \frac{1}{a+\cos\theta}\,d\theta = \frac{\pi}{\sqrt{a^2-1}}$.

Solution.

$$\begin{aligned}
\int_0^\pi \frac{1}{a+\cos\theta}\,d\theta &= \frac{1}{2}\int_0^{2\pi} \frac{1}{a+\cos\theta}\,d\theta \\
&= \frac{1}{2}\oint_C \frac{1}{a+\dfrac{z^2+1}{2z}}\,\frac{1}{iz}dz \\
&= \frac{1}{2}\oint_C \frac{2z}{2az+z^2+1}\,\frac{1}{iz}dz \\
&= \frac{1}{i}\oint_C \frac{\dfrac{1}{z+a+\sqrt{a^2-1}}}{z-(-a+\sqrt{a^2-1})}\,dz \\
&= \frac{1}{i}\frac{2\pi i}{0!}\left(\frac{1}{z+a+\sqrt{a^2-1}}\right)\Bigg|_{z=-a+\sqrt{a^2-1}} \\
&= 2\pi\,\frac{1}{2\sqrt{a^2-1}} \\
&= \frac{\pi}{\sqrt{a^2-1}}
\end{aligned}$$

□

Problem 10. For $a > 0$ show $\displaystyle\int_0^\pi \frac{1}{a+(\cos\theta)^2}\,d\theta = \frac{\pi}{\sqrt{a(a+1)}}$.

Solution.

$$\begin{aligned}
\int_0^\pi \frac{1}{a+(\cos\theta)^2}\,d\theta &= \frac{1}{2}\int_0^{2\pi} \frac{1}{a+(\cos\theta)^2}\,d\theta \\
&= \frac{1}{2}\oint_C \frac{1}{a+\left(\dfrac{z^2+1}{2z}\right)^2}\,\frac{1}{iz}dz \\
&= \frac{1}{2}\oint_C \frac{4z^2}{4az^2+z^4+2z^2+1}\,\frac{1}{iz}dz \\
&= \frac{2}{i}\oint_C \frac{z}{z^4+2(2a+1)z^2+1}\,dz \\
&= \frac{2}{i}\oint_C \frac{\dfrac{z}{z^2+2a+1+2\sqrt{a^2+a}}}{z^2+\left((2a+1)-2\sqrt{a^2+a}\right)}\,dz
\end{aligned}$$

$\left(\text{notice: } 0 < (2a+1)-2\sqrt{a^2+a} < 1\right)$

$$\begin{aligned}
&= \frac{2}{i}\oint_C \frac{\dfrac{z}{z^2+2a+1+2\sqrt{a^2+a}}}{\left(z-i\sqrt{(2a+1)-2\sqrt{a^2+a}}\right)\left(z+i\sqrt{(2a+1)-2\sqrt{a^2+a}}\right)}\,dz \\
&= \frac{2}{i}\oint_{C_1} \frac{\dfrac{z}{\left(z^2+2a+1+2\sqrt{a^2+a}\right)\left(z+i\sqrt{(2a+1)-2\sqrt{a^2+a}}\right)}}{z-i\sqrt{(2a+1)-2\sqrt{a^2+a}}}\,dz
\end{aligned}$$

$$
+\frac{2}{i}\oint_{C_2} \frac{\dfrac{z}{\left(z^2+2a+1+2\sqrt{a^2+a}\right)\left(z-i\sqrt{(2a+1)-2\sqrt{a^2+a}}\right)}}{z+i\sqrt{(2a+1)-2\sqrt{a^2+a}}}\,dz
$$

$$
=\frac{2}{i}\frac{2\pi i}{0!}\left(\frac{z}{\left(z^2+2a+1+2\sqrt{a^2+a}\right)\left(z+i\sqrt{(2a+1)-2\sqrt{a^2+a}}\right)}\right)\Bigg|_{z=i\sqrt{(2a+1)-2\sqrt{a^2+a}}}
$$

$$
+\frac{2}{i}\frac{2\pi i}{0!}\left(\frac{z}{\left(z^2+2a+1+2\sqrt{a^2+a}\right)\left(z-i\sqrt{(2a+1)-2\sqrt{a^2+a}}\right)}\right)\Bigg|_{z=-i\sqrt{(2a+1)-2\sqrt{a^2+a}}}
$$

$$
=4\pi\left(2\;\frac{i\sqrt{(2a+1)-2\sqrt{a^2+a}}}{\left(4\sqrt{a^2+a}\right)\left(2i\sqrt{(2a+1)-2\sqrt{a^2+a}}\right)}\right)
$$

$$
=\frac{\pi}{\sqrt{a^2+a}} \qquad \square
$$