MATH 504 Final Examination November 11, 2008

Name Key

Problem 1. Let $\mathcal{V}$ be a vector space over $\mathbb{F}$ and let $u, v \in \mathcal{V}$. Let $\langle \cdot , \cdot \rangle$ be an inner product on $\mathcal{V}$. Prove that $\langle u, v\rangle\langle v, u\rangle = \langle u, u\rangle\langle v, v\rangle$ if and only if the vectors u and v are linearly dependent.

Problem 2. Assume:

(a) $k \in \mathbb{N}$, (b) $\mathcal{V}$ is a vector space over $\mathbb{F}$, (c) $T \in \mathcal{L}(\mathcal{V})$,
(d) $\mathcal{W}$ is a subspace of $\mathcal{V}$ which is invariant under T, (e) $v_1, \ldots, v_k \in \mathcal{V}$,
(f) $\lambda_1, \ldots \lambda_k$ are mutually distinct scalars in $\mathbb{F}$, (g) $Tv_j = \lambda_j v_j$, $j = 1, \ldots, k$.

Prove the following implication: If $v_1 + \cdots + v_k \in \mathcal{W}$, then $v_j \in \mathcal{W}$ for all $j \in \{1, \ldots, k\}$.

Problem 3. Let $\mathcal{V}$ be a finite dimensional vector space over a field $\mathbb{F}$. Let $\langle \cdot , \cdot \rangle$ be a positive definite inner product on $\mathcal{V}$. Let $\mathcal{L}(\mathcal{V}, \mathbb{F})$ be the space of all linear functionals on $\mathcal{V}$. Define the mapping

$$\Phi \ : \ \mathcal{V} \to \mathcal{L}(\mathcal{V}, \mathbb{F})$$

by the formula

$$(\Phi v)(u) = \langle u, v\rangle \quad \text{for each} \quad u \in \mathcal{V}.$$

Here v is an arbitrary vector in $\mathcal{V}$. Prove that Φ is a bijection.

Problem 4. Let $\mathcal{V}$ be a finite dimensional vector spaces over $\mathbb{F}$ and let $\langle \cdot , \cdot \rangle$ be an inner product on $\mathcal{V}$. Let $u, v \in \mathcal{V}$. Prove that $\langle u, v\rangle = 0$ if and only if

$$\|u\| \leq \|u + \alpha\, v\| \qquad \text{for all} \qquad \alpha \in \mathbb{F}.$$

(One direction is easy. The other direction is easier if you assume $\mathbb{F} \subseteq \mathbb{R}$. This will earn you partial credit.)

Problem 5. Let $\mathcal{V}$ be a finite dimensional vector spaces over $\mathbb{F}$ and let $\langle \cdot , \cdot \rangle$ be an inner product on $\mathcal{V}$. Let $T : \mathcal{V} \to \mathcal{V}$ be a self-adjoint mapping. Prove the following implications.

(a) If $\alpha, \beta \in \mathbb{R}$ are such that $\alpha^2 < 4\beta$, then $T^2 + \alpha T + \beta I$ is an invertible mapping.

(b) If $\mathbb{F} = \mathbb{R}$, then T has an eigenvalue.

Problem 6. Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces over $\mathbb{F}$. Let $\mathcal{U}$ be a subspace of $\mathcal{V}$. Set $k = \dim \mathcal{U}$, $m = \dim \mathcal{V}$ and $n = \dim \mathcal{W}$. For $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ we define the mapping $T|_{\mathcal{U}}$ (the restriction of T to $\mathcal{U}$) by

$$(T|_{\mathcal{U}})x = Tx \quad \text{for all} \quad x \in \mathcal{U}.$$

Clearly $T|_{\mathcal{U}} \in \mathcal{L}(\mathcal{U}, \mathcal{W})$ whenever $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. Next, we define the mapping

$$\Psi : \mathcal{L}(\mathcal{V}, \mathcal{W}) \to \mathcal{L}(\mathcal{U}, \mathcal{W})$$

by

$$\Psi(T) = T|_{\mathcal{U}} \quad \text{for all} \quad T \in \mathcal{L}(\mathcal{V}, \mathcal{W}).$$

Prove that Ψ is linear. (This is easy, but do it right.) Describe $\mathcal{N}(\Psi)$ and $\mathcal{R}(\Psi)$. Calculate $\dim \mathcal{N}(\Psi)$ and $\dim \mathcal{R}(\Psi)$. Prove your claims.

Problem 7. Let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{C}$. Suppose $S, T \in \mathcal{L}(\mathcal{V})$ be such that $ST = TS$. Then S and T have a common 1-dimensional invariant subspace $\mathcal{W}$, that is S and T have a common eigenvector w.

P1 Discover a universal identity. ▽ $\boxed{1}$

$$\left\langle \langle u,u\rangle v - \langle v,u\rangle u \,,\, \langle u,u\rangle v - \langle v,u\rangle u \right\rangle$$

$$= \underline{\langle u,u\rangle}^2 \langle v,v\rangle - \langle v,u\rangle\langle u,v\rangle \underline{\langle u,u\rangle}$$

$$+ \underline{\langle u,u\rangle}\langle u,v\rangle\langle v,u\rangle - \langle v,u\rangle\langle u,v\rangle\underline{\langle u,u\rangle}$$

$$= \langle u,u\rangle \left(\langle u,u\rangle\langle v,v\rangle - \langle v,u\rangle\langle u,v\rangle + \cancel{\langle u,v\rangle\langle v,u\rangle} - \cancel{\langle v,u\rangle\langle u,v\rangle} \right)$$

Hence

$$\| \langle u,u\rangle v - \langle v,u\rangle u \|^2 =$$

$$= \|u\|^2 \left(\langle u,u\rangle\langle v,v\rangle - \langle v,u\rangle\langle u,v\rangle \right)$$

Assume $u \neq 0$

~~Hence~~ then $\langle u,u\rangle\langle v,v\rangle = \langle v,u\rangle\langle u,v\rangle$ iff

$$\langle u,u\rangle \mathbf{v} - \langle v,u\rangle \mathbf{u} = 0$$

$\Rightarrow v, u$ linearly dependent

Assume lin. dep $v = \alpha u$ so

$\langle u,u\rangle\langle v,v\rangle = |\alpha|^2 \|u\|^4$

$\langle u,v\rangle\langle v,u\rangle = |\alpha|^2 \|u\|^4$) so ok! ▽

(P2) Name the statement $P(k)$ $\boxed{2}$

$P(1)$ is trivialy true.

Assume $P(k)$.

Let $v_1 + \cdots + v_k + v_{k+1} \in W$.

Since W invariant under T

$T(v_1 + \cdots + v_{k+1}) \in W$

so $\lambda_1 v_1 + \cdots + \lambda_k v_k + \lambda_{k+1} v_{k+1} \in W$

But $\lambda_{k+1} v_1 + \cdots + \lambda_{k+1} v_k + \lambda_{k+1} v_{k+1} \in W$,

as well. Hence

$$\underbrace{\underbrace{(\lambda_1 - \lambda_{k+1})}_{\neq 0} v_1}_{\substack{\text{eigenvector} \\ \text{corresp. to } \lambda_1}} + \cdots + \underbrace{\underbrace{(\lambda_k - \lambda_{k+1})}_{\neq 0} v_k}_{\substack{\text{eigenvector} \\ \text{corresp. to } \lambda_k}} \in W$$

Apply $P(k)$ to this to conclude

$\underbrace{(\lambda_j - \lambda_{k+1})}_{\neq 0} v_j \in W \quad j = 1, \ldots, k$

Hence $v_1, \ldots, v_k \in W$

Since $v_1 + \cdots + v_k + v_{k+1} \in W$, clearly $v_{k+1} \in W$.

P3 $\boxed{3}$

Φ is 1-to-1.

$\Phi(v_1) = \Phi(v_2) \Rightarrow$

$\langle u, v_1 \rangle = \langle u, v_2 \rangle \quad \forall u \in V$

then $\langle u, v_1 - v_2 \rangle = 0 \quad \forall u \in V$

Hence $v_1 - v_2 = 0$. This proves 1-to-1.

Φ is onto. Let $e_1, \dots, e_n$ be ONB

Let $\varphi \in \mathcal{L}(V, \mathbb{F})$.

for V. Let $v = \overline{\varphi(e_1)}\, e_1 + \dots + \overline{\varphi(e_n)}\, e_n$

Then $\big(\Phi(v)\big)(u) = \langle u, v \rangle = \langle u, \sum_{j=1}^{n} \overline{\varphi(e_j)}\, e_j \rangle$

$= \sum_{j=1}^{n} \varphi(e_j) \langle u, e_j \rangle =$

$= \varphi\left(\sum_{j=1}^{n} \langle u, e_j \rangle e_j \right) = \varphi(u).$

This holds for all $u \in V$. So $\Phi(v) = \varphi$. This proves <u>onto</u>.

(P4) $v=0$ trivial - so assume $v \neq 0$.

Assume $\|u\| \leq \|u+\alpha v\| \ \forall \alpha \in \mathbb{F}$. $\boxed{4}$

Then $\|u\|^2 \leq \|u+\alpha v\|^2 \ \forall \alpha \in \mathbb{F}$

Then $\alpha \langle v, u\rangle + \bar{\alpha}\langle u, v\rangle + |\alpha|^2 \langle v, v\rangle \geq 0$ for all $\alpha \in \mathbb{F}$

set $\alpha = t\langle u, v\rangle, t \in \mathbb{R} \cap \mathbb{F}$

then $t\,|\langle u,v\rangle|^2 + t|\langle u,v\rangle|^2 + t^2 |\langle u,v\rangle|^2 \langle v,v\rangle \geq 0$

$$|\langle u,v\rangle|^2 \left(2t + \langle v,v\rangle t^2\right) \geq 0$$

set $t = -\frac{1}{\langle v,v\rangle}$

$$|\langle u,v\rangle|^2 \left(-\frac{2}{\langle v,v\rangle} + \frac{1}{\langle v,v\rangle}\right) \geq 0$$

$$-\frac{|\langle u,v\rangle|^2}{\langle v,v\rangle} \geq 0$$

thus $\langle u,v\rangle = 0$.

P5 Lemma If $S \in \mathcal{L}(V)$, $\langle Sv, v\rangle > 0$ $\boxed{5}$

(a) $\forall v \in V \setminus \{0\}$ then S is invertible.

The contrapositive is CLEAR!

Let $v \in V \setminus \{0\}$ and $T = T^*$.

$$\langle (T^2 + \alpha T + \beta I)v, v\rangle = \langle T^2 v, v\rangle + \alpha\langle Tv, v\rangle + \beta\|v\|^2$$

$$\overset{\Delta}{\geq} \|Tv\|^2 - |\alpha| |\langle Tv, v\rangle| + \beta\|v\|^2$$

$$\underset{\text{CBS}}{\geq} \|Tv\|^2 - |\alpha| \, \|Tv\| \, \|v\| + \beta\|v\|^2$$

$$= \|v\|^2 \Bigg(\underbrace{\left(\frac{\|Tv\|}{\|v\|}\right)^2}_{s} - |\alpha| \underbrace{\frac{\|Tv\|}{\|v\|}}_{s} + \beta \Bigg) =$$

$$= \|v\|^2 (s^2 - |\alpha| s + \beta) = \|v\|^2 \left(s^2 - |\alpha| s + \frac{|\alpha|^2}{4} - \frac{|\alpha|^2}{4} + \beta \right)$$

$$= \|v\|^2 \left(\left(s - \frac{|\alpha|}{2}\right)^2 + \frac{4\beta - \alpha^2}{4} \right) > 0.$$

Hence $T^2 + \alpha T + \beta$ is invertible.

(b) . . . .

(P6) $\boxed{6}$

By definition for $S, T \in \mathcal{L}(V,W)$

$$\left.\begin{aligned}(\Psi(S))x &= Sx \quad \forall x \in \mathcal{U}\\ (\Psi(T))x &= Tx \quad \forall x \in \mathcal{U}.\end{aligned}\right\} \circledast$$

$\underline{\underline{x \in \mathcal{U}}}$

$$\Big(\Psi(\alpha S+\beta T)\Big)x \underset{\text{def. of } \Psi}{=} (\alpha S+\beta T)(x) \overset{\text{def of } \downarrow \text{ lin. comb of mappings}}{=}$$

$$\overset{\circledast}{=} \alpha Sx + \beta Tx =$$

$$= \alpha(\Psi(S))x + \beta(\Psi(T))x$$

$$= \Big(\alpha\Psi(S) + \beta\Psi(T)\Big)x$$

Hence $\Psi(\alpha S+\beta T) = \alpha\Psi(S)+\beta\Psi(T).$

$\mathcal{N}(\Psi) = \{T \in \mathcal{L}(V,W) : \mathcal{U} \subseteq \mathcal{N}(T)\}$

$T|_{\mathcal{U}} = 0 \iff \mathcal{U} \subseteq \mathcal{N}(T).$

$\mathcal{R}(\Psi) = \mathcal{L}(\mathcal{U},W)$. To prove this let $\mathcal{Z} \subseteq V$ be such that $V = \mathcal{Z} \oplus \mathcal{U}$. Let $S \in \mathcal{L}(\mathcal{U},W)$

Pf Set $\boxed{7}$

$Tv = Su$ whenever $v = z + u$ where $z \in Z$, $u \in U$ are unique ~~elemen~~ vectors.

Then $T|_U = S$, so $\Psi(T) = S$.

Thus Ψ is ONTO. $\triangledown$

We know

$$\dim \mathcal{L}(V, W) = \dim N(\Psi) + \dim R(\Psi)$$

$$\dim \mathcal{L}(V, W) = m \cdot n$$

$$\dim R(\Psi) = \dim (U, W) = k \cdot n$$

Hence $\dim R(\Psi) = k \cdot n$

$\dim N(\Psi) = (m-k) n$.

(P7) since $\mathbb{F} = \mathbb{C}$, S has an eigenvalue, say $\lambda \in \mathbb{C}$. Then $\boxed{8}$

$\mathcal{U} = \mathcal{N}(S - \lambda I) \neq \{0_v\}$

$u \in \mathcal{U} \iff Su = \lambda u$

But $TSu = STu$ and

$TSu = \lambda Tu$.

Thus $(S - \lambda I)Tu = 0$. Hence $Tu \in \mathcal{U}$

That is $T\mathcal{U} \subseteq \mathcal{U}$. $T|_{\mathcal{U}} \in \mathcal{L}(\mathcal{U})$

$\mathcal{U} \neq \{0\}$ over $\mathbb{C}$ $\exists$ eigenvalue μ for $T|_{\mathcal{U}}$. That is $\exists y \in \mathcal{U}$ $y \neq 0$ s.t. $Ty = \mu y$.

But $y \in \mathcal{U}$ so $Sy = \lambda y$. DONE!