BASES

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Throughout this note $\mathcal{V}$ is a vector space over $\mathbb{F}$ and j, k, l, m, and n are natural numbers.

Definition 1. Vectors $v_1, \dots, v_n \in \mathcal{V}$ are said to be *linearly dependent* if there exist $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ and $k \in \{1, \dots, n\}$ such that $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ and $\alpha_k \neq 0$.

The formal negation of the statement in Definition 1 is:

For all $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ and all $k \in \{1, \dots, n\}$ we have $\alpha_1 v_1 + \cdots + \alpha_n v_n \neq 0$ or $\alpha_k = 0$.

The last statement is equivalent to:

For all $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ and all $k \in \{1, \dots, n\}$ we have $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ implies $\alpha_k = 0$.

The last statement can be restated as:

If $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ and $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$, then $\alpha_k = 0$ for all $k \in \{1, \dots, n\}$.

Definition 2. Vectors $v_1, \dots, v_n \in \mathcal{V}$ are said to be *linearly independent* if $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ and $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ implies $\alpha_k = 0$ for all $k \in \{1, \dots, n\}$.

Lemma 3. *Let $k \leq m$ and let $v_1, \dots, v_m$ be vectors in $\mathcal{V}$. If the vectors $v_1, \dots, v_k$ are linearly dependent, then the vectors $v_1, \dots, v_m$ are linearly dependent.*

Proof. Let the vectors $v_1, \dots, v_k$ be linearly dependent. Then there exist $\alpha_1, \dots, \alpha_k$ in $\mathbb{F}$, not all equal to 0, such that $\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$. Take $\alpha_{k+1} = \cdots = \alpha_m = 0$. Then, not all $\alpha_1, \dots, \alpha_k, \dots, \alpha_m$ are equal to 0 and $\alpha_1 v_1 + \cdots + \alpha_k v_k + \cdots + \alpha_m v_m = 0$. Therefore, $v_1, \dots, v_m$ are linearly dependent. $\square$

The following corollary is the contrapositive of Lemma 3.

Corollary 4. *Let $k \leq m$ and let $v_1, \dots, v_m$ be vectors in $\mathcal{V}$. If the vectors $v_1, \dots, v_m$ are linearly independent, then the vectors $v_1, \dots, v_k$ are linearly independent.*

Lemma 5. *Let $m \geq 2$, let $v_1, \dots, v_m$ be vectors in $\mathcal{V}$. The vectors $v_1, \dots, v_m$ are linearly dependent if and only if there exists $k \in \{1, 2, \dots, m\}$ such that*

$$\operatorname{span}\big\{v_l : l \in \{1, \dots, m\} \setminus \{k\}\big\} = \operatorname{span}\{v_1, \dots, v_m\}. \tag{1}$$

Proof. Assume that $v_1, \dots, v_m$ are linearly dependent. Then there exist $\alpha_1, \dots, \alpha_m \in \mathbb{F}$ such that $\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$ and there exists $k \in \{1, \dots, m\}$ such that $\alpha_k \neq 0$. Now, $\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$ implies

$$v_k = -(1/\alpha_k)\big(\alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1} + \alpha_{k+1} v_{k+1} + \cdots + \alpha_m v_m\big).$$

Thus $v_k \in \operatorname{span}\big\{v_l : l \in \{1, \dots, m\} \setminus \{k\}\big\}$. Consequently

$$\operatorname{span}\{v_1, \dots, v_m\} \subseteq \operatorname{span}\big\{v_l : l \in \{1, \dots, m\} \setminus \{k\}\big\}.$$

Since the converse inclusion is trivial, the "if" part of the lemma is proved.

Assume that there exists $k \in \{1, 2, \dots, m\}$ such that (1) holds. Then $v_k \in \operatorname{span}\big\{v_l : l \in \{1, \dots, m\} \setminus \{k\}\big\}$. Therefore there exist

$$\beta_1, \dots, \beta_{k-1}, \beta_{k+1}, \dots, \beta_m \in \mathbb{F}$$

such that $v_k = \beta_1 v_1 + \cdots + \beta_{k-1} v_{k-1} + \beta_{k+1} v_{k+1} + \cdots + \beta_m v_m$. Consequently,

$$\beta_1 v_1 + \cdots + \beta_{k-1} v_{k-1} + (-1) v_k + \beta_{k+1} v_{k+1} + \cdots + \beta_m v_m = 0.$$

Since $-1 \neq 0$, $v_1, \dots, v_m$ are linearly dependent. □

Lemma 6. *If $\mathcal{V} = \operatorname{span}\{v_1, \dots, v_m\}$ and $w \in \mathcal{V} \setminus \{0\}$, then, after a suitable renumbering of $v_1, \dots, v_m$, we have*

$$\mathcal{V} = \operatorname{span}\{w, v_2, \dots, v_m\}.$$

Proof. Assume that $v_1, \dots, v_m$ span $\mathcal{V}$ and $w \in \mathcal{V} \setminus \{0\}$. Then there exist $\alpha_1, \dots \alpha_m$ in $\mathbb{F}$ such that $w = \alpha_1 v_1 + \cdots \alpha_m v_m$. Since $w \neq 0$ not all $\alpha_1, \dots \alpha_m$ are equal to 0. Renumber $v_1, \dots, v_m$ in such a way that $\alpha_1 \neq 0$. Then

$$v_1 = (1/\alpha_1)(w - \alpha_2 v_2 - \cdots - \alpha_m v_m).$$

Thus $v_1 \in \operatorname{span}\{w, v_2, \dots, v_m\}$. Consequently,

$$\mathcal{V} = \operatorname{span}\{v_1, \dots, v_m\} \subseteq \operatorname{span}\{w, v_2, \dots, v_m\}.$$

Since the converse inclusion is obvious, $\mathcal{V} = \operatorname{span}\{w, v_2, \dots, v_m\}$ is proved. □

Lemma 7. *Let $2 \leq j \leq m$. Let $w_1, \dots, w_j$, and $v_j, v_{j+1}, \dots, v_m$, be vectors in $\mathcal{V}$. If*

$$\mathcal{V} = \operatorname{span}\{w_1, \dots, w_{j-1}, v_j, v_{j+1}, \dots, v_m\} \tag{2}$$

and $w_1, \dots, w_j$ are linearly independent, then, after a suitable renumbering of the vectors $v_j, \dots, v_m$, we have

$$\mathcal{V} = \operatorname{span}\{w_1, \dots, w_{j-1}, w_j, v_{j+1} \dots, v_m\}. \tag{3}$$

Proof. Assume that (2) holds and that $w_1, \dots, w_{j-1}, w_j$ are linearly independent. Then there exist $\beta_1, \dots, \beta_m$ in $\mathbb{F}$ such that

$$w_j = \beta_1 w_1 + \cdots + \beta_{j-1} w_{j-1} + \beta_j v_j + \cdots + \beta_m v_m. \tag{4}$$

Since $w_1, \dots, w_{j-1}, w_j$ are linearly independent we have

$$w_j - \beta_1 w_1 - \cdots - \beta_{j-1} w_{j-1} \neq 0.$$

From (4) we have

$$0 \neq w_j - \beta_1 w_1 - \cdots - \beta_{j-1} w_{j-1} = \beta_j v_j + \cdots + \beta_m v_m.$$

Therefore not all $\beta_j, \ldots, \beta_m$ are equal to 0. Renumber $v_j, \ldots, v_m$ in such a way that $\beta_j \neq 0$. Then

$$v_j = (1/\beta_j)(-\beta_1 w_1 - \cdots - \beta_{j-1} w_{j-1} + w_j - \beta_{j+1} v_{j+1} - \cdots - \beta_m v_m).$$

Thus $v_j \in \operatorname{span}\{w_1, \ldots, w_{j-1}, w_j, \ldots, v_m\}$. Consequently,

$$\mathcal{V} = \operatorname{span}\{w_1, \ldots, w_{j-1}, v_j, \ldots, v_m\} \subseteq \operatorname{span}\{w_1, \ldots, w_j, v_{j+1}, \ldots, v_m\}.$$

Since the converse inclusion is obvious, (3) is proved. □

Theorem 8. *Let $k \leq m$. Let $v_1, \ldots, v_m$, and $w_1, \ldots, w_k$ be vectors in $\mathcal{V}$. If $\mathcal{V} = \operatorname{span}\{v_1, \ldots, v_m\}$ and $w_1, \ldots, w_k$, are linearly independent, then, after a suitable renumbering of $v_1, \ldots, v_m$, we have*

$$\mathcal{V} = \operatorname{span}\{w_1, \ldots, w_k, v_{k+1}, \ldots, v_m\}.$$

Proof. Assume that $\mathcal{V} = \operatorname{span}\{v_1, \ldots, v_m\}$ and that $w_1, \ldots, w_k$ are linearly independent. Then $w_1 \neq 0$. By Lemma 6, after a suitable renumbering of $v_1, \ldots, v_m$, we have $\mathcal{V} = \operatorname{span}\{w_1, v_2, \ldots, v_m\}$. If $k = 1$ the theorem is proved. Let $k \geq 2$ and let $2 \leq j \leq k$. By Corollary 4 the vectors $w_1, \ldots, w_j$ are linearly independent. In particular w_1 and w_2 are linearly independent. Lemma 7 with $j = 2$ yields that, after a suitable renumbering of $v_2, \ldots, v_m$, we have $\mathcal{V} = \operatorname{span}\{w_1, w_2, v_3, \ldots, v_m\}$. Repeated application of Lemma 7 (total of $k-1$ times) yields that, after a suitable renumbering of $v_1, \ldots, v_m$, we have $\mathcal{V} = \operatorname{span}\{w_1, \ldots, w_k, v_{k+1}, \ldots, v_m\}$. □

An important special case of the preceding theorem is when $k = m$. We state it as a corollary.

Corollary 9. *Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ be vectors in $\mathcal{V}$. If $w_1, \ldots, w_m$ are linearly independent and $\mathcal{V} = \operatorname{span}\{v_1, \ldots, v_m\}$, then*

$$\mathcal{V} = \operatorname{span}\{w_1, \ldots, w_m\}.$$

Theorem 10. *Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_k$ be vectors in $\mathcal{V}$. If $w_1, \ldots, w_k$ are linearly independent and $\mathcal{V} = \operatorname{span}\{v_1, \ldots, v_m\}$, then $k \leq m$.*

This theorem has the following logical structure: $P \wedge Q \Rightarrow R$. It is not difficult to show (using the truth tables) that the last implication is equivalent to the implication $P \wedge \neg R \Rightarrow \neg Q$ and also to $\neg R \wedge Q \Rightarrow \neg P$. We state each of these equivalent implications separately. There is no need to number them since these statements are equivalent to Theorem 10.

Statement. *Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_k$ be vectors in $\mathcal{V}$. If $w_1, \ldots, w_k$ are linearly independent and $k > m$, then the vectors $v_1, \ldots, v_m$ do not span $\mathcal{V}$.*

Statement. *Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_k$ be vectors in $\mathcal{V}$. If $k > m$ and $\mathcal{V} = \operatorname{span}\{v_1, \ldots, v_m\}$, then $w_1, \ldots, w_k$ are linearly dependent.*

Proof. We will prove the last statement. Assume that $\mathcal{V} = \operatorname{span}\{v_1, \dots, v_m\}$ and $k > m$. We will consider the following two cases:

Case 1. The vectors $w_1, \dots, w_m$ are linearly dependent.
Case 2. The vectors $w_1, \dots, w_m$ are linearly independent.

In Case 1 by Lemma 3 the vectors $w_1, \dots, w_m, w_{m+1}, \dots, w_k$ are also linearly dependent.

Now consider Case 2. By Corollary 9 we have $\mathcal{V} = \operatorname{span}\{w_1, \dots, w_m\}$. Since $k > m$, we have $k \geq m+1$ and thus, w_{m+1} is a vector in $\mathcal{V}$ which can be written as a linear combination of the vectors $w_1, \dots, w_m$. Thus the vectors $w_1, \dots, w_m, w_{m+1}$ are linearly dependent. Consequently

$$w_1, \dots, w_m, w_{m+1}, \dots, w_k$$

are linearly dependent. □

Definition 11. A vector space $\mathcal{V}$ over $\mathbb{F}$ is *finite dimensional* if there exists $m \in \mathbb{N}$ and vectors $v_1, \dots, v_m \in \mathcal{V}$ such that

$$\mathcal{V} = \operatorname{span}\{v_1, \dots, v_m\}.$$

A vector space which is not finite dimensional is said to be *infinite dimensional*.

Proposition 12. *A vector space $\mathcal{V}$ over $\mathbb{F}$ is infinite dimensional if and only if for every $n \in \mathbb{N}$ there exists linearly independent vectors $v_1, \dots, v_n$ in $\mathcal{V}$.*

Proof. We first prove the "only if" part. Assume that $\mathcal{V}$ is an infinite dimensional vector space over $\mathbb{F}$. For $n \in \mathbb{N}$, denote by $P(n)$ the following statement:

There exist n linearly independent vectors in $\mathcal{V}$.

We will prove that $P(n)$ holds for for every $n \in \mathbb{N}$. Mathematical induction is a natural tool here. Since the space $\{0_{\mathcal{V}}\}$ is finite dimensional, we have $\mathcal{V} \neq \{0_{\mathcal{V}}\}$. Therefore there exists $v \in \mathcal{V}$ such that $v \neq 0_{\mathcal{V}}$. Hence $P(1)$ holds. Let $k \in \mathbb{N}$ and assume that $P(k)$ holds. That is assume that there exists linearly independent vectors $v_1, \dots, v_k$ in $\mathcal{V}$. Since $\mathcal{V}$ is an infinite dimensional, $\operatorname{span}\{v_1, \dots, v_k\}$ is a proper subset of $\mathcal{V}$. Therefore there exists $v \in \mathcal{V}$ such that $v \notin \operatorname{span}\{v_1, \dots, v_k\}$. Now it is not difficult to prove (prove it as an exercise) that $k+1$ vectors $v_1, \dots, v_k, v$ are linearly independent. Hence $P(k+1)$ holds.

We prove the "if" part by proving its contrapositive. Assume that $\mathcal{V}$ is a finite dimensional vector space. Then there exists $m \in \mathbb{N}$ and $v_1, \dots, v_m \in \mathcal{V}$ such that $\mathcal{V} = \operatorname{span}\{v_1, \dots, v_m\}$. Let $w_1, \dots, w_m, w_{m+1}$ be arbitrary vectors in $\mathcal{V}$. By Theorem 10 (more precisely, the second Statement after it) the vectors $w_1, \dots, w_m, w_{m+1}$ are linearly dependent. Thus, for $m+1 \in \mathbb{N}$, any vectors $w_1, \dots, w_m, w_{m+1} \in \mathcal{V}$ are linearly dependent. This completes the proof. □

Proposition 13. *If $\mathcal{V}$ is a finite dimensional vector space over $\mathbb{F}$ and $\mathcal{U}$ is a subspace of $\mathcal{V}$, then $\mathcal{U}$ is a finite dimensional vector space over $\mathbb{F}$.*

Proof. We proceed with a proof by contradiction. So, we make the following three assumptions:

(i) $\mathcal{V}$ is a finite dimensional vector space over $\mathbb{F}$.
(ii) $\mathcal{U}$ is a subspace of $\mathcal{V}$.
(iii) $\mathcal{U}$ is an infinite dimensional vector space over $\mathbb{F}$.

Since $\mathcal{V}$ is finite dimensional there exists $m \in \mathbb{N}$ and $v_1, \ldots, v_m \in \mathcal{V}$ such that $\mathcal{V} = \operatorname{span}\{v_1, \ldots, v_m\}$. By Proposition 12 there exists $u_1, \ldots, u_m \in \mathcal{U}$ which are linearly independent. Since $\mathcal{U} \subseteq \mathcal{V}$, we have $u_1, \ldots, u_m \in \mathcal{V}$. Now, Corollary 9 implies $\operatorname{span}\{u_1, \ldots, u_m\} = \mathcal{V}$. Since $\mathcal{U}$ is infinite dimensional we have

$$\operatorname{span}\{u_1, \ldots, u_m\} \subset \mathcal{U} \quad \text{and} \quad \operatorname{span}\{u_1, \ldots, u_m\} \neq \mathcal{U}.$$

Hence

$$\mathcal{V} \subset \mathcal{U} \subseteq \mathcal{V} \quad \text{and} \quad \mathcal{V} \neq \mathcal{U}.$$

This is a contradiction. The proposition is proved. □

Definition 14. Let $\mathcal{V}$ be a finite dimensional vector space over $\mathbb{F}$. A set $\{v_1, \ldots, v_n\}$ is a *basis of* $\mathcal{V}$ if

$$\mathcal{V} = \operatorname{span}\{v_1, \ldots, v_n\} \quad \text{and} \quad v_1, \ldots, v_n \quad \text{are linearly independent.}$$

The next theorem shows that each nonzero finite dimensional vector space has a basis.

Theorem 15. *Let $\mathcal{V}$ be a vector space over $\mathbb{F}$. If $\mathcal{V} = \operatorname{span}\{v_1, \ldots, v_p\}$ and $\mathcal{V} \neq \{0\}$, then there exist $n \in \mathbb{N}$, $n \leq p$, and $j_1, \ldots, j_n \in \{1, \ldots, p\}$ such that $v_{j_1}, \ldots, v_{j_n}$ is a basis of $\mathcal{V}$.*

Proof. Since $\mathcal{V} \neq \{0\}$ there exists $l \in \{1, \ldots, p\}$ such that $v_l \neq 0$. Put

$$\mathbb{K} = \left\{ k \in \mathbb{N} : \quad k \leq p, \quad \begin{matrix} \exists i_1, \ldots, i_k \in \{1, \ldots, p\} \text{ such that} \\ v_{i_1}, \ldots, v_{i_k} \text{ are linearly independent} \end{matrix} \right\}.$$

The vector v_l is linearly independent. Therefore $1 \in \mathbb{K}$; namely we can choose $i_1 = l$. Thus $\mathbb{K} \neq \emptyset$. Since $\mathbb{K}$ is a subset of $\mathbb{N}$ and it is bounded above by p, $\mathbb{K}$ has a maximum; denote it by n, $n = \max \mathbb{K}$. Since $n \in \mathbb{K}$, there exist $j_1, \ldots, j_n \in \{1, \ldots, p\}$ such that $v_{j_1}, \ldots, v_{j_n}$ are linearly independent. Since $v_{j_1}, \ldots, v_{j_n}$ are linearly independent the indexes $j_1, \ldots, j_n$ are distinct. Therefore, if $n = p$, then $\{j_1, \ldots, j_p\} = \{1, \ldots, p\}$. Consequently, if $n = p$, then the vectors $v_1, \ldots, v_p$ are linearly independent and $\operatorname{span}\{v_1, \ldots, v_p\} = \mathcal{V}$. That is $v_1, \ldots, v_p$ is a basis of $\mathcal{V}$.

If if $n < p$, then $\{j_1, \ldots, j_p\}$ is a proper subset of $\{1, \ldots, p\}$. We shall prove that $\operatorname{span}\{v_{j_1}, \ldots, v_{j_n}\} = \mathcal{V}$. Let

$$k \in \{1, \ldots, p\} \setminus \{j_1, \ldots, j_n\}$$

be arbitrary. Since $n+1 \notin \mathbb{K}$, the vectors ($n+1$ of them) $v_{j_1}, \ldots, v_{j_n}, v_k$ are linearly dependent. Thus there exist $\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \mathbb{F}$ not all zero such that

$$\alpha_1 v_{j_1} + \cdots + \alpha_n v_{j_n} + \alpha_{n+1} v_k = 0.$$

Since the vectors $v_{j_1}, \ldots, v_{j_n}$, are linearly independent, $\alpha_{n+1} = 0$ is not possible. Thus $\alpha_{n+1} \neq 0$. Therefore

$$v_k = -\frac{1}{\alpha_{n+1}} \left(\alpha_1 v_{j_1} + \cdots + \alpha_n v_{j_n} \right). \tag{5}$$

Hence

$$v_k \in \operatorname{span}\{v_{j_1}, \ldots, v_{j_n}\} \quad \text{for each} \quad k \in \{1, \ldots, p\} \setminus \{j_1, \ldots, j_n\}.$$

Consequently

$$\operatorname{span}\{v_1, \ldots, v_p\} \subseteq \operatorname{span}\{v_{j_1}, \ldots, v_{j_n}\}.$$

Since the converse inclusion is obvious, the theorem is proved. □

Theorem 16. *Let $\mathcal{V}$ be a nonzero finite dimensional vector space. Then $\mathcal{V}$ has a basis. If $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ are two basis of $\mathcal{V}$, then $m = n$.*

Proof. The fact that $\mathcal{V}$ has a basis is proved in the proof of Proposition 13. Just set $\mathcal{U} = \mathcal{V}$ in that proof.

Let $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ be two bases of $\mathcal{V}$. Since

$$\mathcal{V} = \operatorname{span}\{v_1, \ldots, v_m\}$$

and $w_1, \ldots, w_n$ are linearly independent, Theorem 10 implies $m \geq n$. Since $\mathcal{V} = \operatorname{span}\{w_1, \ldots, w_n\}$ and $v_1, \ldots, v_m$ are linearly independent Theorem 10 implies $m \leq n$. Thus $m = n$. □

Definition 17. Let $\mathcal{V}$ be a nonzero finite dimensional vector space over $\mathbb{F}$ and let $\{v_1, \ldots, v_n\}$ be a basis of $\mathcal{V}$. The number n is called the *dimension of* $\mathcal{V}$ and it is denoted by $\dim \mathcal{V}$. By definition the dimension of the zero vector space is 0.

Theorem 18. *Let $\mathcal{V}$ be a finite dimensional vector space and let $u_1, \ldots, u_k$ be linearly independent vectors in $\mathcal{V}$. Then there exist vectors $u_{k+1}, \ldots, u_n$ in $\mathcal{V}$ such that $\{u_1, \ldots, u_n\}$ is a basis of $\mathcal{V}$.*

Proof. By Theorem 16 the vector space $\mathcal{V}$ has a basis. Let $\{v_1, \ldots, v_n\}$ be a basis for $\mathcal{V}$. By Theorem 10 we have $k \leq n$. By Theorem 8, after a suitable renumbering of $v_1, \ldots, v_n$, we have

$$\mathcal{V} = \operatorname{span}\{u_1, \ldots, u_k, v_{k+1}, \ldots, v_n\}.$$

Since $v_1, \ldots, v_n$ are linearly independent, by Theorem 10 (see the first Statement) no proper subset of

$$\{u_1, \ldots, u_k, v_{k+1}, \ldots, v_n\}$$

spans $\mathcal{V}$. By Lemma 5 this implies that the vectors $u_1, \ldots, u_k, v_{k+1}, \ldots, v_n$ are linearly independent. □

Proposition 19. *Let $\mathcal{V}$ be a finite dimensional vector space and let $\mathcal{U}$ be a subspace of $\mathcal{V}$. Then* $\dim \mathcal{U} \leq \dim \mathcal{V}$*. Also,* $\mathcal{U} = \mathcal{V}$ *if and only if* $\dim \mathcal{U} = \dim \mathcal{V}$*.*

Proof. Let $m = \dim \mathcal{U}$ and $n = \dim \mathcal{V}$. Let $u_1, \ldots, u_m$ be a basis of $\mathcal{U}$ and let $v_1, \ldots, v_n$ be a basis of $\mathcal{V}$. Since $\mathcal{V} = \operatorname{span}\{v_1, \ldots, v_n\}$ and $u_1, \ldots, u_m$ are linearly independent Theorem 10 implies $m \leq n$.

If $\mathcal{U} = \mathcal{V}$, then clearly $\dim \mathcal{U} = \dim \mathcal{V}$. Now assume that $\mathcal{U}$ is a proper subspace of $\mathcal{V}$. Then there exists $v \in \mathcal{V}$ such that $v \notin \mathcal{U}$. Let again $u_1, \ldots, u_m$ be a basis of $\mathcal{U}$. Then $u_1, \ldots, u_m, v$ are linearly independent vectors in $\mathcal{V}$. By Theorem 10 we have $m + 1 \leq n$. Thus $m < n$. □

Proposition 20. *Let $\mathcal{V}$ be a finite dimensional vector space over $\mathbb{F}$ and let $w_1, \ldots, w_n$ be vectors in $\mathcal{V}$. Then any two of the following three statements imply the remaining one.*

(a) $n = \dim \mathcal{V}$.
(b) $\operatorname{span}\{w_1, \ldots, w_n\} = \mathcal{V}$.
(c) $w_1, \ldots, w_n$ *are linearly independent.*

Proof. Assume (b) and (c). Then (a) follows by the definition of dimension of $\mathcal{V}$.

Notice that (b) and Theorem 15 imply that $n \geq \dim \mathcal{V}$. Therefore, the implication "(a) and (b) imply (c)" is equivalent to the implication: If $\operatorname{span}\{w_1, \ldots, w_n\} = \mathcal{V}$ and $w_1, \ldots, w_n$ are linearly dependent, then $n > \dim \mathcal{V}$. The last implication is an immediate consequence of Lemma 5. Thus (a) and (b) imply (c).

Notice that (c) and Theorem 15 imply that $n \leq \dim \mathcal{V}$. Therefore, the implication "(a) and (c) imply (b)" is equivalent to the implication: If $w_1, \ldots, w_n$ are linearly independent and $\operatorname{span}\{w_1, \ldots, w_n\}$ is a proper subspace of $\mathcal{V}$, then $n < \dim \mathcal{V}$. The last implication is a consequence of Proposition 19. □