VECTOR SPACES

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1. Axioms

Definition 1.1. A subset $\mathbb{F}$ of $\mathbb{C}$ is called a *scalar field* if the following five statements hold.

SF1 $0, 1 \in \mathbb{F}$.
SF2 If $\alpha, \beta \in \mathbb{F}$, then $\alpha + \beta \in \mathbb{F}$ and $\alpha\beta \in \mathbb{F}$.
SF3 If $\alpha \in \mathbb{F}$, then $-\alpha \in \mathbb{F}$.
SF4 If $\alpha \in \mathbb{F}$ and $\alpha \neq 0$, then $\dfrac{1}{\alpha} \in \mathbb{F}$.
SF5 If $\alpha \in \mathbb{F}$, then $\overline{\alpha} \in \mathbb{F}$.

Definition 1.2. Let $\mathcal{V}$ be a set and let $\mathbb{F}$ be a scalar field. The set $\mathcal{V}$ is called a *vector space over* $\mathbb{F}$ if the following ten conditions are satisfied.

AE There exists a function $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$.

(The mapping in **AE** is called *addition* and its value on a pair $(u, v) \in \mathcal{V} \times \mathcal{V}$ is denoted by $u + v$.)

AA For all $u, v, w \in \mathcal{V}$ we have $u + (v + w) = (u + v) + w$.
AC For all $u, v \in \mathcal{V}$ we have $u + v = v + u$.
AZ There exists an element $0_{\mathcal{V}} \in \mathcal{V}$ such that $v + 0_{\mathcal{V}} = v$ for all $v \in \mathcal{V}$.
AO For each $v \in \mathcal{V}$ there exists $w \in \mathcal{V}$ such that $v + w = 0_{\mathcal{V}}$.

SE There exists a function $\cdot : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$.

(The mapping in **SE** is called *scaling* and its value on a pair $(\alpha, v) \in \mathbb{F} \times \mathcal{V}$ is denoted by $\alpha \cdot v$, or simply αv.)

SA For all $\alpha, \beta \in \mathbb{F}$ and all $v \in \mathcal{V}$ we have $\alpha(\beta v) = (\alpha\beta)v$.
SD For all $\alpha \in \mathbb{F}$ and all $u, v \in \mathcal{V}$ we have $\alpha(u + v) = \alpha u + \alpha v$.
SD For all $\alpha, \beta \in \mathbb{F}$ and all $v \in \mathcal{V}$ we have $(\alpha + \beta)v = \alpha v + \beta v$.
SO For all $v \in \mathcal{V}$ we have $1v = v$.

2. Basic propositions

Few immediate consequences of Definitions 1.2 and 1.1 are collected in the following propositions.

Proposition 2.1. *Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$. Then for every $v \in \mathcal{V}$ we have $0\, v = 0_{\mathcal{V}}$.*

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Proof. Let $v \in \mathcal{V}$ be arbitrary. Then by **SE** we have that $0v \in \mathcal{V}$. By **AO** there exists $w \in \mathcal{V}$ such that $0v + w = 0_{\mathcal{V}}$. Then

$$\begin{aligned}
0_{\mathcal{V}} &= 0v + w && \text{by the choice of } w \\
&= (0+0)v + w && \text{since } 0+0=0 \text{ in } \mathbb{C} \text{ and ...} \\
&= (0v + 0v) + w && \text{by } \mathbf{SD} \\
&= 0v + (0v + w) && \text{by } \mathbf{AA} \\
&= 0v + 0_{\mathcal{V}} && \text{by the choice of } w \\
&= 0v && \text{by } \mathbf{AZ}.
\end{aligned}$$

The presented sequence of equalities proves the proposition. □

The proof of the next proposition is similar.

Proposition 2.2. *Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$. Then for every $\alpha \in \mathbb{F}$ we have $\alpha 0_{\mathcal{V}} = 0_{\mathcal{V}}$.*

Proposition 2.3. *Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$. For every $v \in \mathcal{V}$ the equation $v + x = 0_{\mathcal{V}}$ has a unique solution.*

Proof. Let $v \in \mathcal{V}$ be arbitrary. Assume that $u, w \in \mathcal{V}$ are such that $v + u = v + w = 0_{\mathcal{V}}$. Then

$$\begin{aligned}
u &= u + 0_{\mathcal{V}} && \text{by } \mathbf{AZ} \\
&= u + (v + w) && \text{by the assumption and ...} \\
&= (u + v) + w && \text{by } \mathbf{AA} \\
&= (v + u) + w && \text{by } \mathbf{AC} \text{ and ...} \\
&= 0_{\mathcal{V}} + w && \text{by the assumption and ...} \\
&= w + 0_{\mathcal{V}} && \text{by } \mathbf{AC} \\
&= w && \text{by } \mathbf{AZ}.
\end{aligned}$$

The presented sequence of equalities proves the proposition. □

Definition 2.4. Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$ and let $v \in \mathcal{V}$. The unique solution of equation $v + x = 0_{\mathcal{V}}$ is denoted by $-v$ and it is called the *opposite* of v.

Proposition 2.5. *Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$. For every $v \in \mathcal{V}$ we have $-v = (-1)v$.*

Proof. □

3. Examples

Example 3.1. Let $\mathbb{F}$ be a scalar field. Then $\mathcal{V} = \mathbb{F}$ is a vector space over $\mathbb{F}$. The addition in $\mathcal{V} = \mathbb{F}$ is the addition of complex numbers in $\mathbb{F}$ and the scaling in $\mathcal{V} = \mathbb{F}$ is just the multiplication of complex numbers. The axioms of the vector space then follow from the axioms of the scalar field and the properties of the complex numbers.

The next example is a generalization of the previous one.

Example 3.2. Let $\mathbb{F}$ and $\mathbb{K}$ be scalar fields such that $\mathbb{F} \subseteq \mathbb{K}$. Then $\mathcal{V} = \mathbb{K}$ is a vector space over $\mathbb{F}$. The addition in $\mathcal{V} = \mathbb{K}$ is the addition of complex numbers in $\mathbb{K}$ and the scaling in $\mathcal{V} = \mathbb{K}$ is just the multiplication of complex numbers. The axioms of the vector space then follow from the axioms of the scalar field and the properties of the complex numbers.

Example 3.3. This is the quintessential example of a vector space. Many other vector spaces are special cases of this example. Let D be an arbitrary nonempty set and let $\mathbb{F}$ be a scalar field. Let $\mathcal{V}$ be the set of all functions from D to $\mathbb{F}$. This set is denoted by $\mathbb{F}^D$. The addition in $\mathbb{F}^D$ is defined as follows: let $f, g \in \mathbb{F}^D$, the function $f + g$ is defined by

$$(f+g)(t) := f(t) + g(t) \quad \text{for all} \quad t \in D.$$

The scaling in $\mathbb{F}^D$ is defined as follows: let $\alpha \in \mathbb{F}$ and $f \in \mathbb{F}^D$, the function $\alpha\, f$ is defined by

$$(\alpha\, f)(t) := \alpha f(t) \quad \text{for all} \quad t \in D.$$

The above definitions of addition and scaling of functions are called *pointwise* definitions. As an exercise you should go through the proofs of all the axioms of the vector space for this specific case.

Example 3.4. This is a special case of Example 3.3. Let $n \in \mathbb{N}$ and

$$D = \{t \in \mathbb{N} : t \le n\}.$$

Sometimes this set is written simply as $D = \{1, \ldots, n\}$. Then the vector space $\mathbb{F}^D$ can be identified with the space $\mathbb{F}^n$ of all n-tuples of elements of $\mathbb{F}$.

Example 3.5. This is another special case of Example 3.3. Let $m, n \in \mathbb{N}$ and

$$D = \big\{(s,t) : s, t \in \mathbb{N},\ s \le m, t \le n\big\};$$

that is $D = \{1, \ldots, m\} \times \{1, \ldots, n\}$. Then $\mathbb{F}^D$ can be identified with the space $\mathbb{F}^{m\times n}$ of all $m \times n$ matrices with entries in $\mathbb{F}$.

Example 3.6. Let $\mathbb{F}$ be a scalar field. By $\mathbb{F}[z]$ we denote the set of all polynomials in variable z with coefficients from the scalar field $\mathbb{F}$. Then $\mathbb{F}[z]$ is a vector space with addition and scalar multiplication defined pointwise.

The next example is a generalization of Example 3.3,

Example 3.7. Let D be an arbitrary nonempty set and let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$. Let $\mathcal{W}$ be the set of all functions from D to $\mathcal{V}$; that is $\mathcal{W} = \mathcal{V}^D$. With the addition and scaling of functions defined pointwise, $\mathcal{W}$ is a vector space over $\mathbb{F}$. The functions in $\mathcal{V}^D$ are said to be *vector valued* functions.

4. Set operations in a vector space

In set theory class we learned about set operations. For two sets A and B we defined $A \cap B$, $A \cup B$, $A \setminus B$ and $A \Delta B$. In a vector space $\mathcal{V}$ over a field $\mathbb{F}$ fun with subsets is enriched by two more set operations: the addition of sets and scaling of sets.

Definition 4.1. Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$ and let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of $\mathcal{V}$. We define the sum of $\mathcal{A} + \mathcal{B}$ by

$$\mathcal{A} + \mathcal{B} = \bigl\{ u + v : u \in \mathcal{A}, v \in \mathcal{B} \bigr\}.$$

For $\alpha \in \mathbb{F}$ we define $\alpha\mathcal{A}$ by

$$\alpha\mathcal{A} = \bigl\{ \alpha u : u \in \mathcal{A} \bigr\}.$$

Let $n \in \mathbb{N}$ and let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be subsets of $\mathcal{V}$. By recursion we define

$$\mathcal{A}_1 + \cdots + \mathcal{A}_k := \bigl(\mathcal{A}_1 + \cdots + \mathcal{A}_{k-1} \bigr) + \mathcal{A}_k, \quad k = 2, \ldots, n.$$

By **AA**, the set $\mathcal{A}_1 + \cdots + \mathcal{A}_n$ consists of all the sums $v_1 + \cdots + v_n$ where $v_j \in \mathcal{A}_j$ where $j \in \{1, \ldots, n\}$.

5. Special subsets of a vector space

The following definition distinguishes important subsets of a vector space $\mathcal{V}$ over a field $\mathbb{F}$.

Definition 5.1. Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$. A subset $\mathcal{U}$ of $\mathcal{V}$ is said to be a *subspace* of $\mathcal{V}$ if the following three conditions are satisfied:

SuZ $0_{\mathcal{V}} \in \mathcal{U}$.
SuA $\mathcal{U} + \mathcal{U} \subseteq \mathcal{U}$.
SuS For every $\alpha \in \mathbb{F}$ we have $\alpha\mathcal{U} \subseteq \mathcal{U}$

Proposition 5.2. *An intersection of subspaces of a vector space is also a subspace.*

Proposition 5.3. *A sum of subspaces of a vector space is also a subspace.*

A union of subspaces of a vector space is not necessarily a vector space. Problems 7.3 and 7.5 deal with this question.

Definition 5.4. Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. A nonempty subset $\mathcal{C}$ of $\mathcal{V}$ is said to be a *cone* in $\mathcal{V}$ if $\alpha\mathcal{C} \subseteq \mathcal{C}$ for all $\alpha > 0$.

Definition 5.5. Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. A nonempty subset $\mathcal{S}$ of $\mathcal{V}$ is said to be a *convex set* in $\mathcal{V}$ if $\alpha u + (1-\alpha)v \in \mathcal{S}$ for all $\alpha \in [0,1]$.

Exercise 5.6. Let $\mathcal{V}$ be a vector space over $\mathbb{R}$ and let $\mathcal{C}$ be a cone in $\mathcal{V}$. Prove that $\mathcal{C}$ is a convex set if and only if $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$.

6. Direct sums of subspaces

Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$. Let $\mathcal{U}$ and $\mathcal{W}$ be subspaces of $\mathcal{V}$. Recall that $v \in \mathcal{U}+\mathcal{V}$ if and only if there exist $u \in \mathcal{U}$ and $w \in \mathcal{W}$ such that $v = u+w$. A stronger version of the last statement is in the following definition.

Definition 6.1. Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$ and let $\mathcal{U}$ and $\mathcal{W}$ be subspaces of $\mathcal{V}$. The sum $\mathcal{U}+\mathcal{V}$ is called a *direct sum* if for every $v \in \mathcal{U}+\mathcal{V}$ there exist unique $u \in \mathcal{U}$ and $w \in \mathcal{W}$ such that $v = u+w$. The direct sum is denoted by $\mathcal{U} \oplus \mathcal{V}$.

For example, let $\mathbb{F} = \mathbb{R}$, $\mathcal{V} = \mathbb{R}^4$,

$$\mathcal{U} = \bigl\{(s_1, s_2, s_3, 0) : s_1, s_2, s_3 \in \mathbb{R}\bigr\} \text{ and } \mathcal{W} = \bigl\{(0, t_1, t_2, t_3) : t_1, t_2, t_3 \in \mathbb{R}\bigr\}.$$

Then $\mathbb{R}^4 = \mathcal{U} + \mathcal{W}$. However, this sum is not a direct sum. For $v = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ we can take $u = (x_1, s_2, s_3, 0) \in \mathcal{U}$ and $w = (0, x_2 - s_2, x_3 - s_3, x_4) \in \mathcal{W}$ with $s_2, s_3 \in \mathbb{R}$ arbitrary.

Setting

$$\mathcal{U} = \bigl\{(s_1, s_2, s_2, 0) : s_1, s_2 \in \mathbb{R}\bigr\} \quad \text{and} \quad \mathcal{W} = \bigl\{(0, -t_1, t_1, t_2) : t_1, t_2 \in \mathbb{R}\bigr\},$$

we have $\mathbb{R}^4 = \mathcal{U} \oplus \mathcal{W}$. Prove this as an exercise.

Proposition 6.2. *Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$ and let $\mathcal{U}$ and $\mathcal{W}$ be subspaces of $\mathcal{V}$. The following statements are equivalent:*

(a) *The sum $\mathcal{U}+\mathcal{W}$ is direct.*
(b) *If $u \in \mathcal{U}$ and $w \in \mathcal{W}$ and $u + w = 0_{\mathcal{V}}$, then $u = w = o_{\mathcal{V}}$.*
(c) $\mathcal{U} \cap \mathcal{W} = \{0_{\mathcal{V}}\}$.

Proof. □

Definition 6.3. Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$, let $n \in \mathbb{N}$ and let $\mathcal{U}_1, \dots, \mathcal{U}_n$ be subspaces of $\mathcal{V}$. The sum $\mathcal{U}_1 + \cdots + \mathcal{U}_n$ is called a *direct sum* if for every $v \in \mathcal{U}_1 + \cdots + \mathcal{U}_n$ there exist unique $u_j \in \mathcal{U}_j$, $j \in \{1, \dots, n\}$, such that $v = u_1 + \cdots + u_n$. The direct sum is denoted by $\mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_n$.

Proposition 6.4. *Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$, let $n \in \mathbb{N}$ and let $\mathcal{U}_1, \dots, \mathcal{U}_n$ be subspaces of $\mathcal{V}$. The following statements are equivalent:*

(a) *The sum $\mathcal{U}_1 + \cdots + \mathcal{U}_n$ is direct.*
(b) *If $u_j \in \mathcal{U}_j$ for all $j \in \{1, \dots, n\}$ and $u_1 + \cdots + u_n = 0_{\mathcal{V}}$, then $u_j = o_{\mathcal{V}}$ for all $j \in \{1, \dots, n\}$.*

Proof. □

7. Problems

Problem 7.1. Consider the vector space $\mathbb{R}^{\mathbb{R}}$ of all real valued functions defined on $\mathbb{R}$. This vector space is considered over the field $\mathbb{R}$. The purpose of this exercise is to study some special subspaces of the vector space $\mathbb{R}^{\mathbb{R}}$. Let γ be an arbitrary real number. Consider the set

$$\mathcal{S}_\gamma := \Bigl\{ f \in \mathbb{R}^{\mathbb{R}} : \exists\ a, b \in \mathbb{R} \ \text{ such that } \ f(t) = a \sin(\gamma t + b), \ t \in \mathbb{R} \Bigr\}.$$

(a) Do you see exceptional values for γ for which the set $\mathcal{S}_\gamma$ is particularly simple?
(b) Prove that for every $\gamma \in \mathbb{R}$ the set $\mathcal{S}_\gamma$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.
(c) For each $\gamma \in \mathbb{R}$ find a basis for $\mathcal{S}_\gamma$. Plot the function $\gamma \mapsto \dim \mathcal{S}_\gamma$.

Problem 7.2. Let D be a nonempty set and $\mathbb{F}$ a scalar field. Let $\mathbb{F}^D$ be a vector space introduced in Example 3.3. Let $\varphi : D \to D$ be a bijection. Set

$$\begin{aligned} \mathcal{O} &= \bigl\{ f \in \mathbb{F}^D : f\bigl(\varphi(t)\bigr) = -f(t) \ \forall t \in D \bigr\}, \\ \mathcal{E} &= \bigl\{ f \in \mathbb{F}^D : f\bigl(\varphi(t)\bigr) = f(t) \ \forall t \in D \bigr\}. \end{aligned}$$

(a) Prove that $\mathcal{O}$ and $\mathcal{E}$ are subspaces of $\mathbb{F}^D$.
(b) Prove $\mathcal{O} \cap \mathcal{E} = \bigl\{ 0_{\mathbb{F}^D} \bigr\}$.
(c) Characterize the functions in the set $\mathcal{O} + \mathcal{E}$.
(d) Find a necessary and sufficient condition on $\varphi : D \to D$ for the equality $\mathbb{F}^D = \mathcal{O} + \mathcal{E}$.

Note: This problem is inspired by the concepts of odd and even functions encountered in a precalculus class. In this precalculus setting $D = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$ and $\varphi(t) = -t, t \in \mathbb{R}$. It would be helpful to work out this problem for this particular case first.

Problem 7.3. Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$. Let $\mathcal{U}$ and $\mathcal{W}$ be subspaces of $\mathcal{V}$. Prove that $\mathcal{U} \cup \mathcal{W}$ is a subspace of $\mathcal{V}$ if and only if $\mathcal{U} \subseteq \mathcal{W}$ or $\mathcal{W} \subseteq \mathcal{U}$.

Problem 7.4. Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$ and let $n \in \mathbb{N}$, $n > 2$. Let $\mathcal{U}_1, \ldots, \mathcal{U}_n$ be subspaces of $\mathcal{V}$. If the union $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ is a subspace, then

$$\mathcal{U}_1 \subseteq \mathcal{U}_2 \cup \cdots \cup \mathcal{U}_n \quad \text{or} \quad \mathcal{U}_n \subseteq \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_{n-1}. \tag{7.1}$$

Proof. We will prove the contrapositive. Assume that (7.1) is not true. Then there exist $u_1 \in \mathcal{U}_1$ such that $u_1 \notin \mathcal{U}_j$ for all $j \in \{2, \ldots, n\}$ and there exist $u_n \in \mathcal{U}_n$ such that $u_n \notin \mathcal{U}_j$ for all $j \in \{1, \ldots, n-1\}$.

Let $\alpha \in \mathbb{F} \setminus \{0\}$. Then $\alpha u_n \in \mathcal{U}_n$ since $\mathcal{U}_n$ is a subspace and, since $\alpha \neq 0$, $\alpha u_n \notin \mathcal{U}_j$ for all $j \in \{1, \ldots, n-1\}$.

Since $u_1 \in \mathcal{U}_1$ and $\alpha u_n \notin \mathcal{U}_1$ we have $u_1 + \alpha u_n \notin \mathcal{U}_1$ for all $\alpha \in \mathbb{F} \setminus \{0\}$.

Since $u_1 \notin \mathcal{U}_n$ and $\alpha u_n \in \mathcal{U}_n$ we have $u_1 + \alpha u_n \notin \mathcal{U}_n$ for all $\alpha \in \mathbb{F}$.

Let $m \in \mathbb{N}$ be such that $1 < m < n$. (Since $n > 2$ such m exists.) By the choice of u_1 and u_n we have $u_1 \notin \mathcal{U}_m$ and $\alpha u_n \notin \mathcal{U}_m$ for all $\alpha \in \mathbb{F} \setminus \{0\}$. Therefore, for at most one $\alpha \in \mathbb{F} \setminus \{0\}$ we can have $u_1 + \alpha u_n \in \mathcal{U}_m$. (If $u_1 + \alpha u_n \in \mathcal{U}_m$ and $u_1 + \beta u_n \in \mathcal{U}_m$ with $\alpha - \beta \neq 0$, then $(u_1 + \alpha u_n) - (u_1 + \beta u_n) = (\alpha - \beta) u_n \in \mathcal{U}_m$ with $\alpha - \beta \neq 0$ and $u_n \notin \mathcal{U}_m$ which is a contradiction.)

Thus, for at most $n-2$ numbers $\alpha \in \mathbb{F} \setminus \{0\}$ we have

$$u_1 + \alpha u_n \in \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n.$$

Since the set $\mathbb{F} \setminus \{0\}$ is infinite, there exists $\alpha \in \mathbb{F} \setminus \{0\}$ such that

$$u_1 + \alpha u_n \notin \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n.$$

Recall that

$$u_1, u_n \in \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n.$$

The last two displayed relations show that $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ is not a subspace of $\mathcal{V}$. $\square$

Problem 7.5. Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$ and let $n \in \mathbb{N}$. Let $\mathcal{U}_1, \ldots, \mathcal{U}_n$ be subspaces of $\mathcal{V}$. Prove that the union $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ is a subspace if and only if there exists $m \in \{1, \ldots, n\}$ such that $\mathcal{U}_k \subseteq \mathcal{U}_m$ for all $k \in \{1, \ldots, n\}$.

Problem 7.6 (Samantha Smith)**.** Let $\mathcal{V}$ be a vector space over a scalar field $\mathbb{F}$. Let $\mathcal{P}(\mathcal{V})$ be the power set of $\mathcal{V}$, that is the set of all subsets of $\mathcal{V}$. Set $\mathcal{W} = \mathcal{P}(\mathcal{V}) \setminus \{\emptyset\}$. Let the addition and scaling in $\mathcal{W}$ be defined as in Section 4. Is $\mathcal{W}$ with these two operations a vector space over $\mathbb{F}$.