The next theorem can be stated in English simply as: Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Theorem. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$, $T \in \mathscr{L}(\mathscr{V})$ and $n \in \mathbb{N}$. Assume*

(a) *$\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ are such that $\lambda_i \neq \lambda_j$ for all $i, j \in \{1, \ldots, n\}$ such that $i \neq j$,*

(b) *$v_1, \ldots, v_n \in \mathscr{V}$ are such that $Tv_k = \lambda_k v_k$ and $v_k \neq 0$ for all $k \in \{1, \ldots, n\}$.*

Then $\{v_1, \ldots, v_n\}$ is linearly independent.

Proof. We will prove this by using the mathematical induction on n. For the base case, we will prove the claim for $n = 1$. Let $\lambda_1 \in \mathbb{F}$ and let $v_1 \in \mathscr{V}$ be such that $v_1 \neq 0$ and $Tv_1 = \lambda_1 v_1$. Since $v_1 \neq 0$, we conclude that $\{v_1\}$ is linearly independent.

Next we prove the inductive step. Let $m \in \mathbb{N}$ be arbitrary. The inductive hypothesis is the assumption that the following implication holds.

If the following two conditions are satisfied:

(i) $\mu_1, \ldots, \mu_m \in \mathbb{F}$ are such that $\mu_i \neq \mu_j$ for all $i, j \in \{1, \ldots, m\}$ such that $i \neq j$,

(ii) $w_1, \ldots, w_m \in \mathscr{V}$ are such that $Tw_k = \mu_k w_k$ and $w_k \neq 0$ for all $k \in \{1, \ldots, m\}$,

then $\{w_1, \ldots, w_m\}$ is linearly independent.

We need to prove the following implication

If the following two conditions are satisfied:

(I) $\lambda_1, \ldots, \lambda_{m+1} \in \mathbb{F}$ are such that $\lambda_i \neq \lambda_j$ for all $i, j \in \{1, \ldots, m+1\}$ such that $i \neq j$,

(II) $v_1, \ldots, v_{m+1} \in \mathscr{V}$ are such that $Tv_k = \lambda_k v_k$ and $v_k \neq 0$ for all $k \in \{1, \ldots, m+1\}$,

then $\{v_1, \ldots, v_{m+1}\}$ is linearly independent.

Assume (I) and (II) in the red box. We need to prove that $\{v_1, \ldots, v_{m+1}\}$ is linearly independent.

Let $\alpha_1, \ldots, \alpha_{m+1} \in \mathbb{F}$ be such that

$$\alpha_1 v_1 + \cdots + \alpha_m v_m + \alpha_{m+1} v_{m+1} = 0. \tag{1}$$

Applying $T \in \mathscr{L}(\mathscr{V})$ to both sides of (1), using the linearity of T and assumption (II) we get

$$\alpha_1 \lambda_1 v_1 + \cdots + \alpha_m \lambda_m v_m + \alpha_{m+1} \lambda_{m+1} v_{m+1} = 0. \tag{2}$$

Multiplying both sides of (1) by λ_{m+1} we get

$$\alpha_1 \lambda_{m+1} v_1 + \cdots + \alpha_m \lambda_{m+1} v_m + \alpha_{m+1} \lambda_{m+1} v_{m+1} = 0. \tag{3}$$

Subtracting (3) from (2) we get

$$\alpha_1(\lambda_1 - \lambda_{m+1}) v_1 + \cdots + \alpha_m(\lambda_m - \lambda_{m+1}) v_m = 0.$$

Since by assumption (I) we have $\lambda_j - \lambda_{m+1} \neq 0$ for all $j \in \{1, \ldots, m\}$, setting

$$w_j = (\lambda_j - \lambda_{m+1}) v_j, \qquad j \in \{1, \ldots, m\},$$

and taking into account (II) we have

$$w_j \neq 0 \qquad \text{and} \qquad Tw_j = \lambda_j w_j \qquad \text{for all} \qquad j \in \{1, \ldots, m\}. \tag{4}$$

Thus, by (I) and (4), the scalars $\lambda_1, \ldots, \lambda_m$ and vectors $w_1, \ldots, w_m$ satisfy assumptions (i) and (ii) of the inductive hypothesis (the green box). Consequently, the vectors $w_1, \ldots, w_m$ are linearly independent. Since by (4) we have

$$\alpha_1 w_1 + \cdots + \alpha_m w_m = 0,$$

it follows that $\alpha_1 = \cdots = \alpha_m = 0$. Substituting these values in (1) we get $\alpha_{m+1} v_{m+1} = 0$. Since by (II), $v_{m+1} \neq 0$ we conclude that $\alpha_{m+1} = 0$. This completes the proof of the linear independence of $v_1, \ldots, v_{m+1}$. $\square$