MATH 101 Assignment 1
January 1, 2011

Name ___ Branko Ćurgus

Problem 1. Prove that for all $a \in (-1,1)$ and all $b \in [-1,1]$ we have

$$\frac{a+b}{1+ab} \in [-1,1]. \tag{1}$$

Solution. Assume that $a \in (-1,1)$ and $b \in [-1,1]$. Then $|a| \in [0,1)$ and $|b| \in [0,1]$. Consequently, $|ab| \in [0,1)$. That is $ab \in (-1,1)$. Therefore $1+ab > 0$.

Since $a \in (-1,1)$, we have $1+a > 0$ and $1-a > 0$. Similarly, since $b \in [-1,1]$, we have $1+b \geq 0$ and $1-b \geq 0$.

Since $1+a > 0$ and $1+b \geq 0$, we have $(1+a)(1+b) \geq 0$. Consequently, $1+a+b+ab \geq 0$ and hence

$$-1-ab \leq a+b.$$

As $1+ab > 0$, dividing both sides of the last inequality by $1+ab$ we get

$$-1 \leq \frac{a+b}{1+ab}. \tag{2}$$

Since $1-a > 0$ and $1-b \geq 0$, we have $(1-a)(1-b) \geq 0$. Consequently, $1-a-b+ab \geq 0$ and hence

$$a+b \leq 1+ab.$$

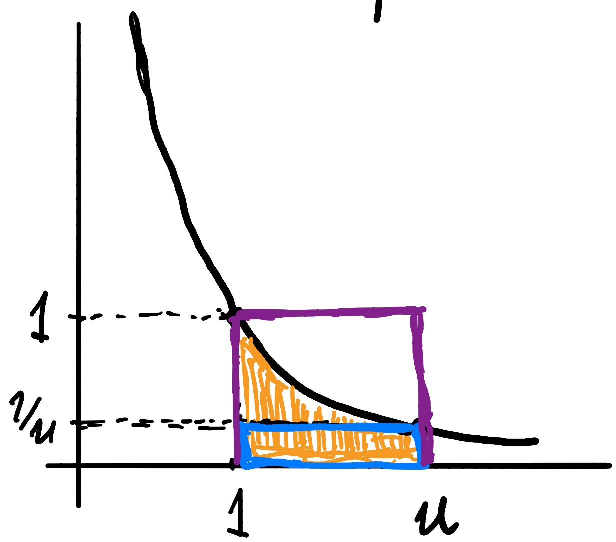
As $1+ab > 0$, dividing both sides of the last inequality by $1+ab$ we get

$$\frac{a+b}{1+ab} \leq 1. \tag{3}$$

Inequalities (2) and (3) prove inequality (1). □

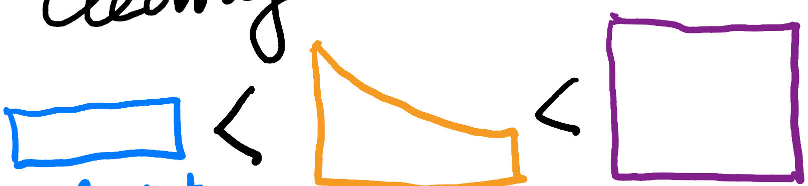
From calculus we know that for $u > 0$

$$\ln(u) = \int_1^u \frac{1}{t}\,dt.$$



$\ln(u) =$ orange area

Clearly



calculate ↓ function of u $<$ $\ln(u)$ $<$ calculate function of u

① apply to $u = 1 + \frac{1}{x}$

② deduce an inequality for $\ln\left(1+\frac{1}{x}\right)^x$

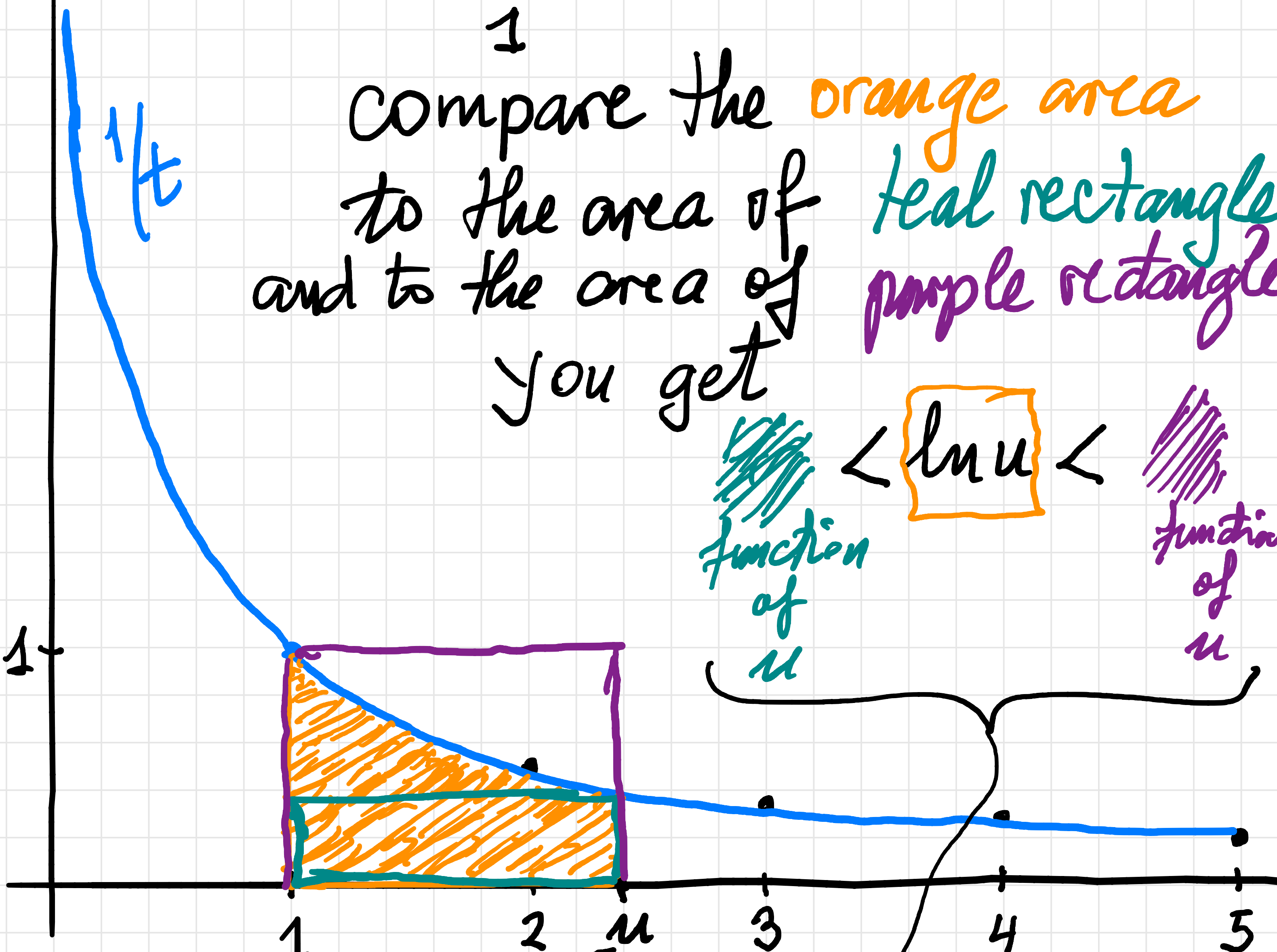
I hope this helps!

$\ln u = \int_1^u \frac{1}{t}\,dt$ .... orange area

compare the orange area
to the area of teal rectangle
and to the area of purple rectangle
you get

$\square < \boxed{\ln u} < \square$

function of u — function of u



$u = 1 + \frac{1}{x}$ apply to

fun of x

$\bigcirc < \ln\left(1+\frac{1}{x}\right) < \bigcirc$

finaly

$\bigcirc < \ln\left(1+\frac{1}{x}\right)^x < \bigcirc$