

$\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable

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The symbol \mathbb{Z}^+ denotes the set of positive integers.

Definition 1. A set S is *countably infinite* if there exists a bijection $f : \mathbb{Z}^+ \rightarrow S$.

The goal of this note is to provide a rigorous proof that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countably infinite. That is to provide a specific bijection from \mathbb{Z}^+ to $\mathbb{Z}^+ \times \mathbb{Z}^+$. Below such a bijection is called f .

First some preliminaries. Recall that the sequence of triangular numbers is given by

$$T_n = \frac{n(n+1)}{2}, \quad n \in \mathbb{Z}^+.$$

It is convenient to also define $T_0 = 0$.

Exercise 2. Prove that for every $n \in \mathbb{Z}^+$ we have $n \leq T_n$.

Solution. Let $n \in \mathbb{Z}^+$ be arbitrary. Multiplying each side of the inequality $1 \leq n$ by $n > 0$ we get $n \leq n^2$. Adding n to each side of the last inequality yields $2n \leq n^2 + n$, that is, $2n \leq n(n+1)$. Dividing by 2 yields $n \leq T_n$. □

To get an idea how triangular numbers are spaced among positive integers we present the following table. The **triangular numbers** are in bold face.

	T_1		T_2		T_3		T_4				T_5				T_6								
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
R_n	1	2	2	3	3	3	4	4	4	4	5	5	5	5	5	6	6	6	6	6	6	7	7

The table above indicates that the following sequence

$$R_1 = 1, R_2 = 2, R_3 = 2, R_4 = 3, R_5 = 3, R_6 = 3, R_7 = 4, R_8 = 4, R_9 = 4, R_{10} = 4, R_{11} = 5, R_{12} = 5, \dots$$

is closely related to the sequence of triangular numbers. For a given $n \in \mathbb{Z}^+$ R_n is the index of the smallest triangular number which is larger or equal than n . Formally, we define the sequence $R : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ by

$$R_n = \min\{k \in \mathbb{Z}^+ : n \leq T_k\}, \quad n \in \mathbb{Z}^+.$$

The above definition uses the concept of minimum. To make this definition rigorous, we need to prove that the above minimum exists. By Exercise 2 for arbitrary $n \in \mathbb{Z}^+$ we have $n \leq T_n$. Therefore

$$n \in \{k \in \mathbb{Z}^+ : n \leq T_k\};$$

that is, the set $\{k \in \mathbb{Z}^+ : n \leq T_k\}$ is a nonempty set of positive integers. By the well ordering axiom this set has a minimum. This justifies the definition of R_n .

By the definition of minimum, R_n belongs to the set $\{k \in \mathbb{Z}^+ : n \leq T_k\}$. Therefore $n \leq T_{R_n}$. Also, by the definition of minimum $R_n - 1$ does not belong to the set $\{k \in \mathbb{Z}^+ : n \leq T_k\}$. Therefore $T_{R_n-1} < n$. Thus, for every $n \in \mathbb{Z}^+$ we have

$$T_{R_n-1} < n \leq T_{R_n}. \tag{1}$$

In other words, for an arbitrary $n \in \mathbb{Z}^+$, the integer R_n provides the index of the smallest triangular number which is $\geq n$. Notice that (1) also claims that n is larger than the triangular number with index $R_n - 1$.

Remark 3. There are several other formulas for the sequence R . For example, for $n \in \mathbb{Z}^+$,

$$R_n = \left\lfloor \frac{1}{2} + \sqrt{2n} \right\rfloor = \left\lceil -\frac{1}{2} + \sqrt{2n} \right\rceil.$$

Here $\lfloor \cdot \rfloor$ is the floor function, $\lceil \cdot \rceil$ is the ceiling function and $\sqrt{\cdot}$ is the square root function.

Recall that

$$\mathbb{Z}^+ \times \mathbb{Z}^+ := \{(s, t) : s, t \in \mathbb{Z}^+\}.$$

The set $\mathbb{Z}^+ \times \mathbb{Z}^+$ is illustrated by the following infinite table:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	...
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	...
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	...
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	...
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	...
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

Rearranging the pairs we can enumerate them with positive integers. This enumeration is demonstrated in the table below. Each pair is enumerated by a positive integer placed in a small circle. Usually the table below is considered to be a proof of the countability of $\mathbb{Z}^+ \times \mathbb{Z}^+$.

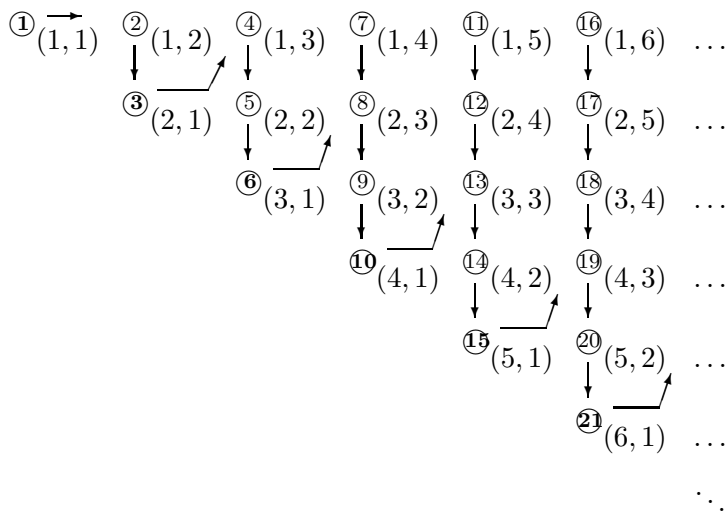


Table 1: Labeled $\mathbb{Z}^+ \times \mathbb{Z}^+$

If one accepts the above enumeration table as a proof, then one would never know which pair is associated with the positive integer 321, or, which circled positive integer is used to enumerate the pair (21, 5). Furthermore, the enumeration table above poses an interesting challenge: find a *formula* for the function $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$ which is indicated by the table. Since we expect such f to be a bijection, we also need to find a formula for its inverse, call it $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$, such that

$$f(g(s, t)) = (s, t) \quad \forall (s, t) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \quad \text{and} \quad g(f(n)) = n \quad \forall n \in \mathbb{Z}^+. \quad (2)$$

Two identities in (2) are equivalent to the statement: f is a bijection.

Notice that the circled labels along the diagonal in Table 1 are triangular numbers. The pattern is clear: the label for $(s, 1)$ is the triangular number T_s . The sum of the entries of each pair in the same column as $(s, 1)$ is $s + 1$. The labels decrease as we climb up the column, that is as t increases. This gives us the function $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$:

$$g(s, t) = \frac{(s+t-1)(s+t)}{2} + 1 - t = \frac{(s+t-2)(s+t-1)}{2} + s, \quad s, t \in \mathbb{Z}^+.$$

Next we have to figure out a pair associated with $n \in \mathbb{Z}^+$. As we have noticed before triangular numbers play an important role in the labeling. As we can see from Table 1 the numbers s and t are related to how far n is from the previous and the following triangular number. We already know from (1) that

$$\frac{(R_n - 1)R_n}{2} < n \leq \frac{R_n(R_n + 1)}{2}.$$

Now it is not difficult to see from Table 1 that $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$ is given by

$$f(n) = \left(\underbrace{n - \frac{(R_n - 1)R_n}{2}}_{\text{distance to the preceding triangular number}}, \underbrace{\frac{R_n(R_n + 1)}{2} - n + 1}_{\text{distance to the following triangular number}} \right), \quad n \in \mathbb{Z}^+.$$

By the definition of triangular numbers the formulas for g and f can be written as

$$\begin{aligned} g(s, t) &= T_{(s+t-1)} + 1 - t, & s, t \in \mathbb{Z}^+, \\ f(n) &= \left(n - T_{(R_n-1)}, T_{R_n} - n + 1 \right), & n \in \mathbb{Z}^+. \end{aligned}$$

Let $s, t \in \mathbb{Z}^+$. We evaluate $R_{(T_{(s+t-1)}+1-t)}$ first. Since $0 < s$ and $0 \leq t - 1$ we have

$$T_{(s+t-2)} = T_{(s+t-1)} - (s+t-1) < T_{(s+t-1)} + 1 - t \leq T_{(s+t-1)}.$$

In the first equality above we used the identity

$$T_k = T_{(k-1)} + k, \tag{3}$$

which follows from

$$T_k - T_{(k-1)} = \frac{k(k+1)}{2} - \frac{(k-1)k}{2} = \frac{k^2 + k - k^2 + k}{2} = k.$$

Hence, the integer $T_{(s+t-1)} + 1 - t$ is squeezed between two consecutive triangular numbers:

$$T_{(s+t-2)} < T_{(s+t-2)} + s = T_{(s+t-2)} + s + t - 1 + 1 - t = T_{(s+t-1)} + 1 - t \leq T_{(s+t-1)},$$

so, by (1),

$$R_{(T_{(s+t-1)}+1-t)} = s + t - 1.$$

We have thus calculated that

$$R_{g(s,t)} = s + t - 1.$$

Next we use the last identity, the definitions of g and f and (3) to calculate

$$\begin{aligned} f(g(s, t)) &= \left(g(s, t) - \frac{(R_{g(s,t)} - 1)R_{g(s,t)}}{2}, \frac{R_{g(s,t)}(R_{g(s,t)} + 1)}{2} - g(s, t) + 1 \right) \\ &= \left(g(s, t) - \frac{(s+t-1-1)(s+t-1)}{2}, \frac{(s+t-1)(s+t-1+1)}{2} - g(s, t) + 1 \right) \\ &= \left(g(s, t) - T_{(s+t-2)}, T_{(s+t-1)} - g(s, t) + 1 \right) \\ &= \left(T_{(s+t-1)} + 1 - t - T_{(s+t-2)}, T_{(s+t-1)} - (T_{(s+t-1)} + 1 - t) + 1 \right) \\ &= (s + t - 1 + 1 - t, t) \\ &= (s, t). \end{aligned}$$

This proves $f(g(s, t)) = (s, t)$ for all $s, t \in \mathbb{Z}^+$.

Let $n \in \mathbb{Z}^+$ be arbitrary. Before proceeding with the proof $g(f(n)) = n$, notice that by (3) the sum of entries in the pair $f(n)$ is

$$n - \frac{(R_n - 1)R_n}{2} + \frac{R_n(R_n + 1)}{2} - n + 1 = R_n + 1.$$

We use this and the definitions of g and f to calculate

$$\begin{aligned} g(f(n)) &= g\left(n - \frac{(R_n - 1)R_n}{2}, \frac{R_n(R_n + 1)}{2} - n + 1\right) \\ &= \frac{(R_n + 1 - 1)(R_n + 1)}{2} + 1 - \left(\frac{R_n(R_n + 1)}{2} - n + 1\right) \\ &= n. \end{aligned}$$

This proves $g(f(n)) = n$ for all $n \in \mathbb{Z}^+$.

Thus (2) is proved, implying the $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$ is a bijection. This completes our rigorous proof that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.