4.3 OPTIMIZATION AND MODELING

Finding global maxima and minima is often made possible by having a formula for the function to be maximized or minimized. The process of translating a problem into a function with a known formula is called *mathematical modeling*. The examples that follow give the flavor of modeling.

Example 1 What are the dimensions of an aluminum can that holds 40 in^3 of juice and that uses the least material? Assume that the can is cylindrical, and is capped on both ends.

Solution It is often a good idea to think about a problem in general terms before trying to solve it. Since we're trying to use as little material as possible, why not make the can very small, say, the size of a peanut? We can't, since the can must hold 40 in^3. If we make the can short, to try to use less material in the sides, we'll have to make it fat as well, so that it can hold 40 in^3. See Figure 4.29(a).

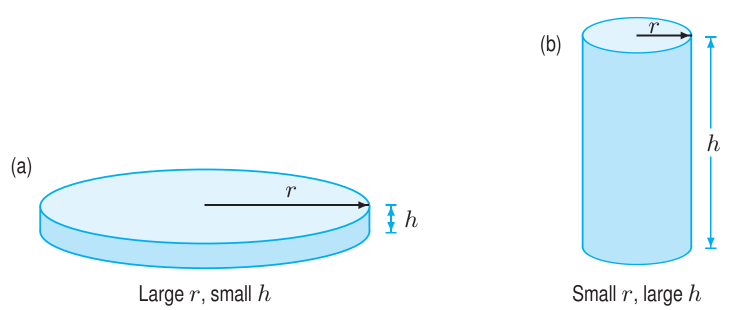


Figure 4.29: Various cylindrical-shaped cans

Table 4.1 *Height, h, and material, M, used in can for various choices of radius, r*

r (in)	h (in)	M (in^2)
0.2	318.31	400.25
1	12.73	86.28
2	3.18	65.13
3	1.41	83.22
4	0.80	120.53
10	0.13	636.32

If we try to save material by making the top and bottom small, the can has to be tall to accommodate the 40 in^3 of juice. So any savings we get by using a small top and bottom may be outweighed by the height of the sides. See Figure 4.29(b).

Table 4.1 gives the height h and amount of material M used in the can for some choices of the radius, r. You can see that r and h change in opposite directions, and that more material is used at the extremes (very large or very small r) than in the middle. It appears that the radius needing the smallest amount of material, M, is somewhere between 1 and 3 inches. Thinking of M as a function of the radius, r, we get the graph in Figure 4.30. The graph shows that the global minimum we want is at a critical point.

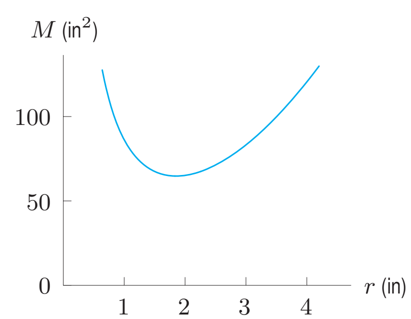


Figure 4.30: Total material used in can, M, as a function of radius, r

Both the table and the graph were obtained from a mathematical model, which in this case is a formula for the material used in making the can. Finding such a formula depends on knowing the geometry of a cylinder, in particular its area and volume. We have

$$M = \text{Material used in the can} = \text{Material in ends} + \text{Material in the side}$$

where

Material in ends $= 2 \cdot$ Area of a circle with radius $r = 2 \cdot \pi r^2$,

Material in the side $=$ Area of curved side of cylinder with height h and radius $r = 2\pi rh$.

We have

$$M = 2\pi r^2 + 2\pi rh.$$

However, h is not independent of r: if r grows, h shrinks, and vice-versa. To find the relationship, we use the fact that the volume of the cylinder, $\pi r^2 h$, is equal to the constant 40 in^3:

$$\text{Volume of can } = \pi r^2 h = 40, \quad \text{giving} \quad h = \frac{40}{\pi r^2}.$$

This means

$$\text{Material in the side} = 2\pi rh = 2\pi r \frac{40}{\pi r^2} = \frac{80}{r}.$$

Thus we obtain the formula for the total material, M, used in a can of radius r if the volume is 40 in^3:

$$M = 2\pi r^2 + \frac{80}{r}.$$

The domain of this function is all $r > 0$ because the radius of the can cannot be negative or zero.

To find the minimum of M, we look for critical points:

$$\frac{dM}{dr} = 4\pi r - \frac{80}{r^2} = 0 \quad \text{at a critical point,} \quad \text{so} \quad 4\pi r = \frac{80}{r^2}.$$

Therefore,

$$\pi r^3 = 20, \quad \text{so} \quad r = \left(\frac{20}{\pi}\right)^{1/3} = 1.85 \text{ inches,}$$

which agrees with the graph in Figure 4.30. We also have

$$h = \frac{40}{\pi r^2} = \frac{40}{\pi (1.85)^2} = 3.7 \text{ inches.}$$

The material used is $M = 2\pi(1.85)^2 + 80/1.85 = 64.7$ in^2.

To confirm that we have found the global minimum, we look at the formula for dM/dr. For small r, the $-80/r^2$ term dominates and for large r, the $4\pi r$ term dominates, so dM/dr is negative for $r < 1.85$ and positive for $r > 1.85$. Thus, M is decreasing for $r < 1.85$ and increasing for $r > 1.85$, so the global minimum occurs at $r = 1.85$.

Practical Tips for Modeling Optimization Problems

1. Make sure that you know what quantity or function is to be optimized.
2. If possible, make several sketches showing how the elements that vary are related. Label your sketches clearly by assigning variables to quantities which change.
3. Try to obtain a formula for the function to be optimized in terms of the variables that you identified in the previous step. If necessary, eliminate from this formula all but one variable. Identify the domain over which this variable varies.
4. Find the critical points and evaluate the function at these points and the endpoints (if relevant) to find the global maxima and/or minima.

The next example, another problem in geometry, illustrates this approach.

Example 2 Alaina wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2000 feet west and 600 feet north of her starting position. Alaina can walk west along the edge of the park on the sidewalk at a speed of 6 ft/sec. She can also travel through the grass in the park, but only at a rate of 4 ft/sec. What path gets her to the bus stop the fastest?

Solution

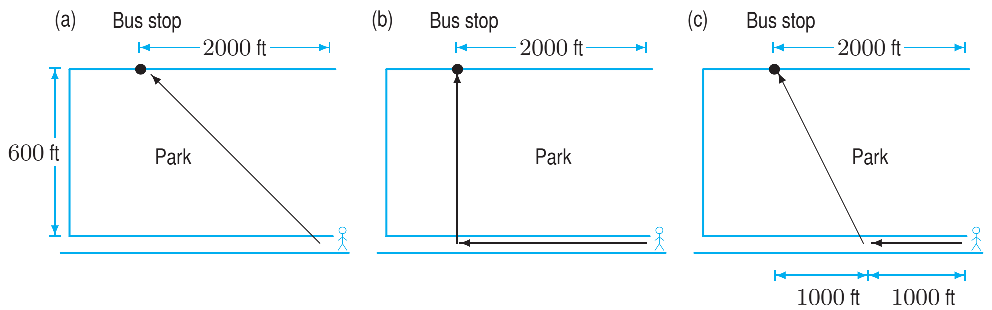


Figure 4.31: Three possible paths to the bus stop

We might first think that she should take a path that is the shortest distance. Unfortunately, the path that follows the shortest distance to the bus stop is entirely in the park, where her speed is slow. (See Figure 4.31(a).) That distance is $\sqrt{2000^2+600^2} = 2088$ feet, which takes her about 522 seconds to traverse. She could instead walk quickly the entire 2000 feet along the sidewalk, which leaves her just the 600-foot northward journey through the park. (See Figure 4.31(b).) This method would take $2000/6 + 600/4 \approx 483$ seconds total walking time.

But can she do even better? Perhaps another combination of sidewalk and park gives a shorter travel time. For example, what is the travel time if she walks 1000 feet west along the sidewalk and the rest of the way through the park? (See Figure 4.31(c).) The answer is about 458 seconds.

We make a model for this problem. We label the distance that Alaina walks west along the sidewalk x and the distance she walks through the park y, as in Figure 4.32. Then the total time, t, is

$$t = t_{\text{sidewalk}} + t_{\text{park}}.$$

Since

$$\text{Time} = \text{Distance}/\text{Speed},$$

and she can walk 6 ft/sec on the sidewalk and 4 ft/sec in the park, we have

$$t = \frac{x}{6} + \frac{y}{4}.$$

Now, by the Pythagorean Theorem, $y = \sqrt{(2000-x)^2 + 600^2}$. Therefore

$$t = \frac{x}{6} + \frac{\sqrt{(2000-x)^2+600^2}}{4} \qquad \text{for } 0 \le x \le 2000.$$

We can find the critical points of this function analytically. (See Problem 15 on page 211.) Alternatively, we can graph the function on a calculator and estimate the critical point, which is $x \approx 1463$ feet. This gives a minimum total time of about 445 seconds.

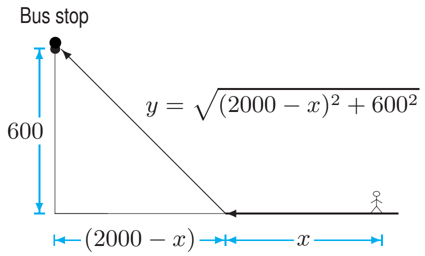


Figure 4.32: Modeling time to bus stop

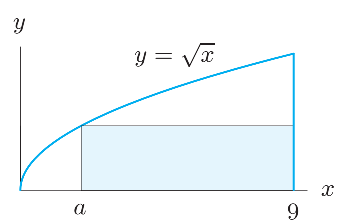


Figure 4.33: Find the rectangle of maximum area with one corner on $y = \sqrt{x}$

Example 3 Figure 4.33 shows the curves $y = \sqrt{x}$, $x = 9$, $y = 0$, and a rectangle with vertical sides at $x = a$ and $x = 9$. Find the dimensions of the rectangle having the maximum possible area.

Solution We want to choose a to maximize the area of the rectangle with corners at $(a, \sqrt{a})$ and $(9, \sqrt{a})$. The area of this rectangle is given by

$$R = \text{Height} \cdot \text{Length} = \sqrt{a}(9-a) = 9a^{1/2} - a^{3/2}.$$

We are restricted to $0 \le a \le 9$. To maximize this area, we first set $dR/da = 0$ to find critical points:

$$\begin{aligned} \frac{dR}{da} = \frac{9}{2}a^{-1/2} - \frac{3}{2}a^{1/2} &= 0 \\ \frac{9}{2\sqrt{a}} &= \frac{3\sqrt{a}}{2} \\ 18 &= 6a \\ a &= 3. \end{aligned}$$

Notice that $R = 0$ at the endpoints $a = 0$ and $a = 9$, and R is positive between these values. Since $a = 3$ is the only critical point, the rectangle with the maximum area has length $9 - 3 = 6$ and height $\sqrt{3}$.

Example 4 A closed box has a fixed surface area A and a square base with side x.

(a) Find a formula for the volume, V, of the box as a function of x. What is the domain of V?
(b) Graph V as a function of x.
(c) Find the maximum value of V.

Solution (a) The height of the box is h, as shown in Figure 4.34. The box has six sides, four with area xh and two, the top and bottom, with area x^2. Thus,

$$4xh + 2x^2 = A.$$

So

$$h = \frac{A - 2x^2}{4x}.$$

Then, the volume, V, is given by

$$V = x^2 h = x^2\left(\frac{A-2x^2}{4x}\right) = \frac{x}{4}\left(A - 2x^2\right) = \frac{A}{4}x - \frac{1}{2}x^3.$$

Since the area of the top and bottom combined must be less than A, we have $2x^2 \le A$. Thus, the domain of V is $0 \le x \le \sqrt{A/2}$.

(b) Figure 4.35 shows the graph for $x \ge 0$. (Note that A is a positive constant.)

(c) To find the maximum, we differentiate, regarding A as a constant:

$$\frac{dV}{dx} = \frac{A}{4} - \frac{3}{2}x^2 = 0$$

so

$$x = \pm\sqrt{\frac{A}{6}}.$$

Since $x \ge 0$ in the domain of V, we use $x = \sqrt{A/6}$. Figure 4.35 indicates that at this value of x, the volume is a maximum.

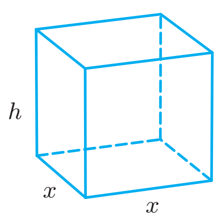


Figure 4.34: Box with base of side x, height h, surface area A, and volume V

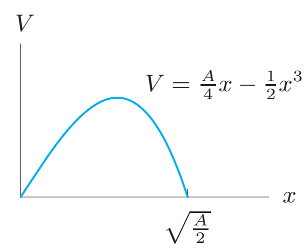


Figure 4.35: Volume, V, against length of side of base, x

From the formula, we see that $dV/dx > 0$ for $x < \sqrt{A/6}$, so V is increasing, and that $dV/dx < 0$ for $x > \sqrt{A/6}$, so V is decreasing. Thus, $x = \sqrt{A/6}$ gives the global maximum.

Evaluating V at $x = \sqrt{A/6}$ and simplifying, we get

$$V = \frac{A}{4}\sqrt{\frac{A}{6}} - \frac{1}{2}\left(\sqrt{\frac{A}{6}}\right)^3 = \left(\frac{A}{6}\right)^{3/2}.$$

Example 5 A light is suspended at a height h above the floor. (See Figure 4.36.) The illumination at the point P is inversely proportional to the square of the distance from the point P to the light and directly proportional to the cosine of the angle θ. How far from the floor should the light be to maximize the illumination at the point P?

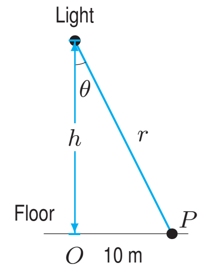


Figure 4.36: How high should the light be?

Solution If the illumination is represented by I and r is the distance from the light to the point P, then we know that for some $k \geq 0$,

$$I = \frac{k\cos\theta}{r^2}.$$

Since $r^2 = h^2 + 10^2$ and $\cos\theta = h/r = h/\sqrt{h^2 + 10^2}$, we have, for $h \geq 0$,

$$I = \frac{kh}{(h^2 + 10^2)^{3/2}}.$$

To find the height at which I is maximized, we differentiate using the quotient rule:

$$\begin{aligned}\frac{dI}{dh} &= \frac{k(h^2+10^2)^{3/2} - kh(\frac{3}{2}(h^2+10^2)^{1/2}(2h))}{[(h^2+10^2)^{3/2}]^2}\\ &= \frac{(h^2+10^2)^{1/2}[k(h^2+10^2) - 3kh^2]}{(h^2+10^2)^3}\\ &= \frac{k(h^2+10^2) - 3kh^2}{(h^2+10^2)^{5/2}}\\ &= \frac{k(10^2 - 2h^2)}{(h^2+10^2)^{5/2}}.\end{aligned}$$

Setting $dI/dh = 0$ for $h \geq 0$ gives

$$\begin{aligned}10^2 - 2h^2 &= 0\\ h &= \sqrt{50} \text{ meters.}\end{aligned}$$

Since $dI/dh > 0$ for $h < \sqrt{50}$ and $dI/dh < 0$ for $h > \sqrt{50}$, there is a local maximum at $h = \sqrt{50}$ meters. There is only one critical point, so the global maximum of I occurs at that point. Thus, the illumination is greatest if the light is suspended at a height of $\sqrt{50} \approx 7$ meters above the floor.

A Graphical Example: Minimizing Gas Consumption

Next we look at an example in which a function is given graphically and the optimum values are read from a graph. We already know how to estimate the optimum values of $f(x)$ from a graph of $f(x)$—read off the highest and lowest values. In this example, we see how to estimate the optimum value of the quantity $f(x)/x$ from a graph of $f(x)$ against x. The question we investigate is how to set driving speeds to maximize fuel efficiency.[2]

Example 6 Gas consumption, g (in gallons/hour), as a function of velocity, v (in mph), is given in Figure 4.37. What velocity minimizes the gas consumption per mile, represented by g/v?

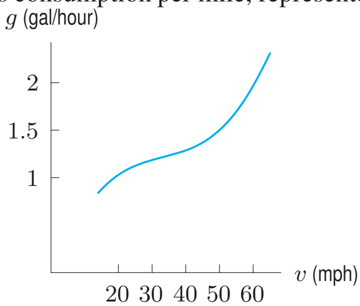


Figure 4.37: Gas consumption versus velocity

Solution Figure 4.38 shows that g/v is the slope of the line from the origin to the point P. Where on the curve should P be to make the slope a minimum? From the possible positions of the line shown in Figure 4.38, we see that the slope of the line is both a local and global minimum when the line is tangent to the curve. From Figure 4.39, we can see that the velocity at this point is about 50 mph. Thus to minimize gas consumption per mile, we should drive about 50 mph.

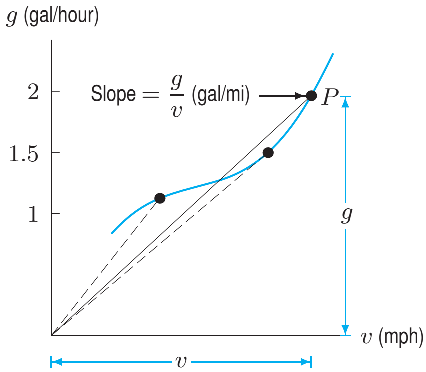


Figure 4.38: Graphical representation of gas consumption per mile, g/v

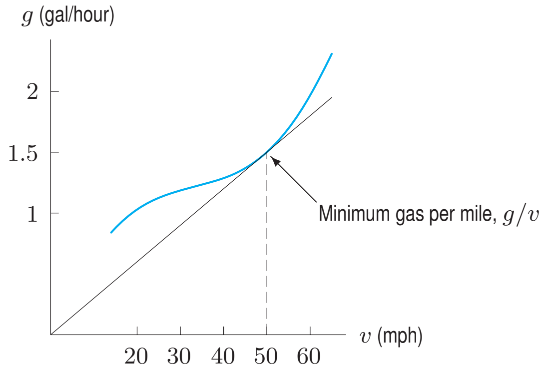


Figure 4.39: Velocity for maximum fuel efficiency

Exercises and Problems for Section 4.3

Exercises

1. The sum of two nonnegative numbers is 100. What is the maximum value of the product of these two numbers?

2. The product of two positive numbers is 784. What is the minimum value of their sum?

3. The sum of three nonnegative numbers is 36, and one of the numbers is twice one of the other numbers. What is the maximum value of the product of these three numbers?

4. The perimeter of a rectangle is 64 cm. Find the lengths of the sides of the rectangle giving the maximum area.

5. If you have 100 feet of fencing and want to enclose a rectangular area up against a long, straight wall, what is the largest area you can enclose?

For the solids in Exercises 6–9, find the dimensions giving the minimum surface area, given that the volume is 8 cm^3.

6. A closed rectangular box, with a square base x by x cm and height h cm.

7. An open-topped rectangular box, with a square base x by x cm and height h cm.

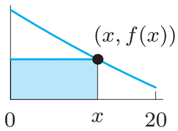
8. A closed cylinder with radius r cm and height h cm.

9. A cylinder open at one end with radius r cm and height h cm.

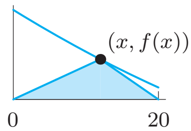
[2] Adapted from Peter D. Taylor, *Calculus: The Analysis of Functions* (Toronto: Wall & Emerson, 1992).

In Exercises 10–11, find the x-value maximizing the shaded area. One vertex is on the graph of $f(x) = x^2/3 - 50x + 1000$, where $0 \le x \le 20$.

10.



11.



12. A rectangle has one side on the x-axis and two vertices on the curve
$$y = \frac{1}{1+x^2}.$$
Find the vertices of the rectangle with maximum area.

13. A right triangle has one vertex at the origin and one vertex on the curve $y = e^{-x/3}$ for $1 \le x \le 5$. One of the two perpendicular sides is along the x-axis; the other is parallel to the y-axis. Find the maximum and minimum areas for such a triangle.

14. A rectangle has one side on the x-axis, one side on the y-axis, one vertex at the origin and one on the curve $y = e^{-2x}$ for $x \ge 0$. Find the

(a) Maximum area **(b)** Minimum perimeter

Problems

15. Find analytically the exact critical point of the function which represents the time, t, to walk to the bus stop in Example 2. Recall that t is given by
$$t = \frac{x}{6} + \frac{\sqrt{(2000-x)^2 + 600^2}}{4}.$$

16. Of all rectangles with given area, A, which has the shortest diagonals?

17. A rectangular beam is cut from a cylindrical log of radius 30 cm. The strength of a beam of width w and height h is proportional to wh^2. (See Figure 4.40.) Find the width and height of the beam of maximum strength.

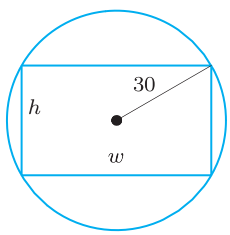
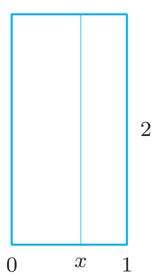


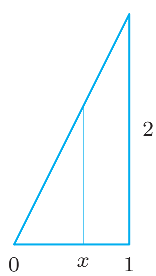
Figure 4.40

In Problems 18–19 a vertical line divides a region into two pieces. Find the value of the coordinate x that maximizes the product of the two areas.

18.



19.



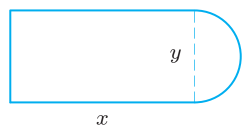
In Problems 20–22 the figures are made of rectangles and semicircles.

(a) Find a formula for the area.

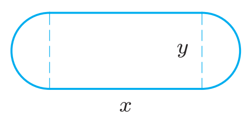
(b) Find a formula for the perimeter.

(c) Find the dimensions x and y that maximize the area given that the perimeter is 100.

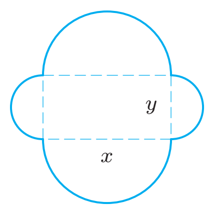
20.



21.



22.



23. A piece of wire of length L cm is cut into two pieces. One piece, of length x cm, is made into a circle; the rest is made into a square.

(a) Find the value of x that makes the sum of the areas of the circle and square a minimum. Find the value of x giving a maximum.

(b) For the values of x found in part (a), show that the ratio of the length of wire in the square to the length of wire in the circle is equal to the ratio of the area of the square to the area of the circle.[3]

(c) Are the values of x found in part (a) the only values of x for which the ratios in part (b) are equal?

[3]From Sally Thomas.

In Problems 24–27, find the minimum and maximum values of the expression where x and y are lengths in Figure 4.41 and $0 \le x \le 10$.

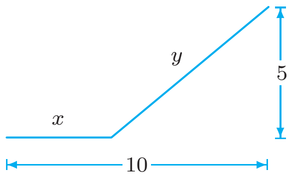


Figure 4.41

24. x **25.** y **26.** $x + 2y$ **27.** $2x + y$

28. Which point on the curve $y = \sqrt{1-x}$ is closest to the origin?

29. Find the point(s) on the ellipse

$$\frac{x^2}{9} + y^2 = 1$$

(a) Closest to the point $(2, 0)$.
(b) Closest to the focus $(\sqrt{8}, 0)$.

[Hint: Minimize the square of the distance—this avoids square roots.]

30. What are the dimensions of the closed cylindrical can that has surface area 280 square centimeters and contains the maximum volume?

31. A hemisphere of radius 1 sits on a horizontal plane. A cylinder stands with its axis vertical, the center of its base at the center of the sphere, and its top circular rim touching the hemisphere. Find the radius and height of the cylinder of maximum volume.

32. In a chemical reaction, substance A combines with substance B to form substance Y. At the start of the reaction, the quantity of A present is a grams, and the quantity of B present is b grams. At time t seconds after the start of the reaction, the quantity of Y present is y grams. Assume $a < b$ and $y \le a$. For certain types of reactions, the rate of the reaction, in grams/sec, is given by

$$\text{Rate } = k(a-y)(b-y), \quad k \text{ is a positive constant.}$$

(a) For what values of y is the rate nonnegative? Graph the rate against y.
(b) Use your graph to find the value of y at which the rate of the reaction is fastest.

33. A smokestack deposits soot on the ground with a concentration inversely proportional to the square of the distance from the stack. With two smokestacks 20 miles apart, the concentration of the combined deposits on the line joining them, at a distance x from one stack, is given by

$$S = \frac{k_1}{x^2} + \frac{k_2}{(20-x)^2}$$

where k_1 and k_2 are positive constants which depend on the quantity of smoke each stack is emitting. If $k_1 = 7k_2$, find the point on the line joining the stacks where the concentration of the deposit is a minimum.

34. A wave of wavelength λ traveling in deep water has speed, v, given for positive constants c and k, by

$$v = k\sqrt{\frac{\lambda}{c} + \frac{c}{\lambda}}$$

As λ varies, does such a wave have a maximum or minimum velocity? If so, what is it? Explain.

35. A circular ring of wire of radius r_0 lies in a plane perpendicular to the x-axis and is centered at the origin. The ring has a positive electric charge spread uniformly over it. The electric field in the x-direction, E, at the point x on the axis is given by

$$E = \frac{kx}{(x^2 + r_0^2)^{3/2}} \quad \text{for} \quad k > 0.$$

At what point on the x-axis is the field greatest? Least?

36. A woman pulls a sled which, together with its load, has a mass of m kg. If her arm makes an angle of θ with her body (assumed vertical) and the coefficient of friction (a positive constant) is μ, the least force, F, she must exert to move the sled is given by

$$F = \frac{mg\mu}{\sin\theta + \mu\cos\theta}.$$

If $\mu = 0.15$, find the maximum and minimum values of F for $0 \le \theta \le \pi/2$. Give answers as multiples of mg.

37. Four equally massive particles can be made to rotate, equally spaced, around a circle of radius r. This is physically possible provided the radius and period T of the rotation are chosen so that the following *action* function is at its global minimum:

$$A(r) = \frac{r^2}{T} + \frac{T}{r}, \quad r > 0.$$

(a) Find the radius r at which $A(r)$ has a global minimum.
(b) If the period of the rotation is doubled, determine whether the radius of the rotation increases or decreases, and by approximately what percentage.

38. You run a small furniture business. You sign a deal with a customer to deliver up to 400 chairs, the exact number to be determined by the customer later. The price will be \$90 per chair up to 300 chairs, and above 300, the price will be reduced by \$0.25 per chair (on the whole order) for every additional chair over 300 ordered. What are the largest and smallest revenues your company can make under this deal?

39. The cost of fuel to propel a boat through the water (in dollars per hour) is proportional to the cube of the speed. A certain ferry boat uses \$100 worth of fuel per hour when cruising at 10 miles per hour. Apart from fuel, the cost of running this ferry (labor, maintenance, and so on) is \$675 per hour. At what speed should it travel so as to minimize the cost *per mile* traveled?

40. A business sells an item at a constant rate of r units per month. It reorders in batches of q units, at a cost of $a+bq$ dollars per order. Storage costs are k dollars per item per month, and, on average, $q/2$ items are in storage, waiting to be sold. [Assume r, a, b, k are positive constants.]

(a) How often does the business reorder?
(b) What is the average monthly cost of reordering?
(c) What is the total monthly cost, C of ordering and storage?
(d) Obtain Wilson's lot size formula, the optimal batch size which minimizes cost.

41. A bird such as a starling feeds worms to its young. To collect worms, the bird flies to a site where worms are to be found, picks up several in its beak, and flies back to its nest. The *loading curve* in Figure 4.42 shows how the number of worms (the load) a starling collects depends on the time it has been searching for them.[4] The curve is concave down because the bird can pick up worms more efficiently when its beak is empty; when its beak is partly full, the bird becomes much less efficient. The traveling time (from nest to site and back) is represented by the distance PO in Figure 4.42. The bird wants to maximize the rate at which it brings worms to the nest, where

$$\text{Rate worms arrive} = \frac{\text{Load}}{\text{Traveling time + Searching time}}$$

(a) Draw a line in Figure 4.42 whose slope is this rate.
(b) Using the graph, estimate the load which maximizes this rate.
(c) If the traveling time is increased, does the optimal load increase or decrease? Why?

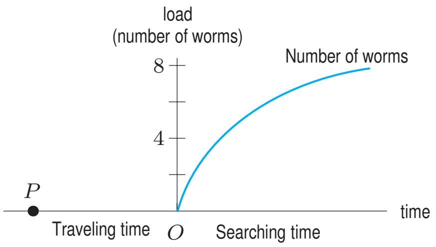


Figure 4.42

42. On the same side of a straight river are two towns, and the townspeople want to build a pumping station, S. See Figure 4.43. The pumping station is to be at the river's edge with pipes extending straight to the two towns. Where should the pumping station be located to minimize the total length of pipe?

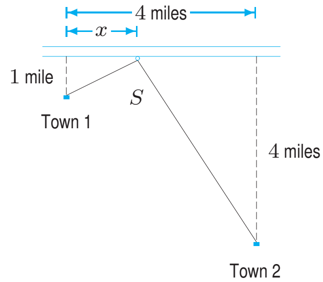


Figure 4.43

43. A pigeon is released from a boat (point B in Figure 4.44) floating on a lake. Because of falling air over the cool water, the energy required to fly one meter over the lake is twice the corresponding energy e required for flying over the bank ($e = 3$ joule/meter). To minimize the energy required to fly from B to the loft, L, the pigeon heads to a point P on the bank and then flies along the bank to L. The distance $\overline{AL}$ is 2000 m, and $\overline{AB}$ is 500 m.

(a) Express the energy required to fly from B to L via P as a function of the angle θ (the angle BPA).
(b) What is the optimal angle θ?
(c) Does your answer change if $\overline{AL}$, $\overline{AB}$, and e have different numerical values?

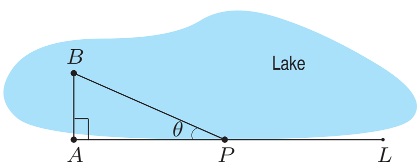


Figure 4.44

44. To get the best view of the Statue of Liberty in Figure 4.45, you should be at the position where θ is a maximum. If the statue stands 92 meters high, including the pedestal, which is 46 meters high, how far from the base should you be? [Hint: Find a formula for θ in terms of your distance from the base. Use this function to maximize θ, noting that $0 \le \theta \le \pi/2$.]

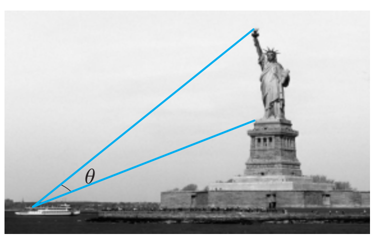


Figure 4.45

[4] Alex Kacelnick (1984). Reported by J. R. Krebs and N. B. Davis, *An Introduction to Behavioural Ecology* (Oxford: Blackwell, 1987).

45. A light ray starts at the origin and is reflected off a mirror along the line $y = 1$ to the point $(2, 0)$. See Figure 4.46. Fermat's Principle says that light's path minimizes the time of travel.[5] The speed of light is a constant.

(a) Using Fermat's Principle, find the optimal position of P.

(b) Using your answer to part (a), derive the Law of Reflection, that $\theta_1 = \theta_2$.

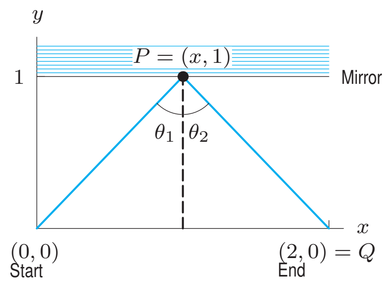


Figure 4.46

46. **(a)** For which positive number x is $x^{1/x}$ largest? Justify your answer.
[Hint: You may want to write $x^{1/x} = e^{\ln(x^{1/x})}$.]

(b) For which positive integer n is $n^{1/n}$ largest? Justify your answer.

(c) Use your answer to parts (a) and (b) to decide which is larger: $3^{1/3}$ or $\pi^{1/\pi}$.

47. The *arithmetic mean* of two numbers a and b is defined as $(a+b)/2$; the *geometric mean* of two positive numbers a and b is defined as $\sqrt{ab}$.

(a) For two positive numbers, which of the two means is larger? Justify your answer.
[Hint: Define $f(x) = (a+x)/2 - \sqrt{ax}$ for fixed a.]

(b) For three positive numbers a, b, c, the arithmetic and geometric mean are $(a+b+c)/3$ and $\sqrt[3]{abc}$, respectively. Which of the two means of three numbers is larger? [Hint: Redefine $f(x)$ for fixed a *and* b.]

48. A line goes through the origin and a point on the curve $y = x^2 e^{-3x}$, for $x \geq 0$. Find the maximum slope of such a line. At what x-value does it occur?

49. The distance, s, traveled by a cyclist, who starts at 1 pm, is given in Figure 4.47. Time, t, is in hours since noon.

(a) Explain why the quantity s/t is represented by the slope of a line from the origin to the point (t, s) on the graph.

(b) Estimate the time at which the quantity s/t is a maximum.

(c) What is the relationship between the quantity s/t and the instantaneous speed of the cyclist at the time you found in part (b)?

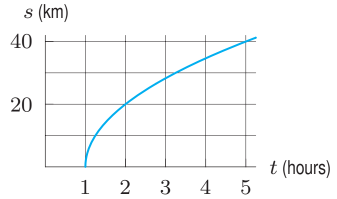


Figure 4.47

50. When birds lay eggs, they do so in clutches of several at a time. When the eggs hatch, each clutch gives rise to a brood of baby birds. We want to determine the clutch size which maximizes the number of birds surviving to adulthood per brood. If the clutch is small, there are few baby birds in the brood; if the clutch is large, there are so many baby birds to feed that most die of starvation. The number of surviving birds per brood as a function of clutch size is shown by the benefit curve in Figure 4.48.[6]

(a) Estimate the clutch size which maximizes the number of survivors per brood.

(b) Suppose also that there is a biological cost to having a larger clutch: the female survival rate is reduced by large clutches. This cost is represented by the dotted line in Figure 4.48. If we take cost into account by assuming that the optimal clutch size in fact maximizes the vertical distance between the curves, what is the new optimal clutch size?

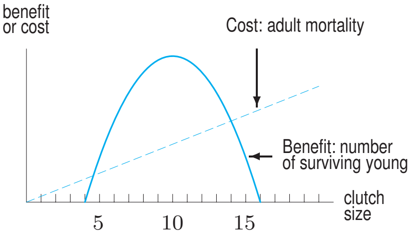


Figure 4.48

51. Let $f(v)$ be the amount of energy consumed by a flying bird, measured in joules per second (a joule is a unit of energy), as a function of its speed v (in meters/sec). Let $a(v)$ be the amount of energy consumed by the same bird, measured in joules per meter.

(a) Suggest a reason in terms of the way birds fly for the shape of the graph of $f(v)$ in Figure 4.49.

(b) What is the relationship between $f(v)$ and $a(v)$?

(c) Where on the graph is $a(v)$ a minimum?

[5]See, for example, D. Halliday, R. Resnik, K. Kane, *Physics*, Vol 2, 4th ed, (New York: Wiley, 1992), p. 909.

[6]Data from C. M. Perrins and D. Lack, reported by J. R. Krebs and N. B. Davies in *An Introduction to Behavioural Ecology* (Oxford: Blackwell, 1987).

(d) Should the bird try to minimize $f(v)$ or $a(v)$ when it is flying? Why?

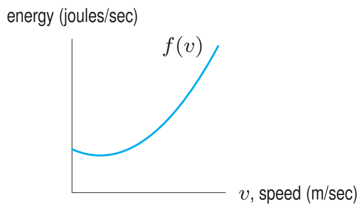


Figure 4.49

52. The forward motion of an aircraft in level flight is reduced by two kinds of forces, known as *induced drag* and *parasite drag*. Induced drag is a consequence of the downward deflection of air as the wings produce lift. Parasite drag results from friction between the air and the entire surface of the aircraft. Induced drag is inversely proportional to the square of speed and parasite drag is directly proportional to the square of speed. The sum of induced drag and parasite drag is called total drag. The graph in Figure 4.50 shows a certain aircraft's induced drag and parasite drag functions.

(a) Sketch the total drag as a function of air speed.
(b) Estimate two different air speeds which each result in a total drag of 1000 pounds. Does the total drag function have an inverse? What about the induced and parasite drag functions?
(c) Fuel consumption (in gallons per hour) is roughly proportional to total drag. Suppose you are low on fuel and the control tower has instructed you to enter a circular holding pattern of indefinite duration to await the passage of a storm at your landing field. At what air speed should you fly the holding pattern? Why?

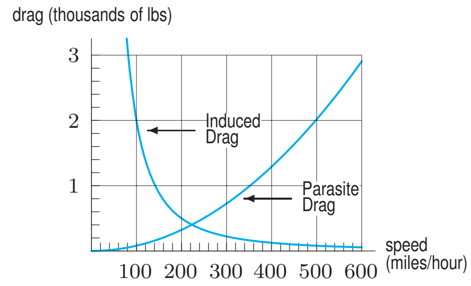


Figure 4.50

53. Let $f(v)$ be the fuel consumption, in gallons per hour, of a certain aircraft as a function of its airspeed, v, in miles per hour. A graph of $f(v)$ is given in Figure 4.51.

(a) Let $g(v)$ be the fuel consumption of the same aircraft, but measured in gallons per mile instead of gallons per hour. What is the relationship between $f(v)$ and $g(v)$?
(b) For what value of v is $f(v)$ minimized?
(c) For what value of v is $g(v)$ minimized?
(d) Should a pilot try to minimize $f(v)$ or $g(v)$?

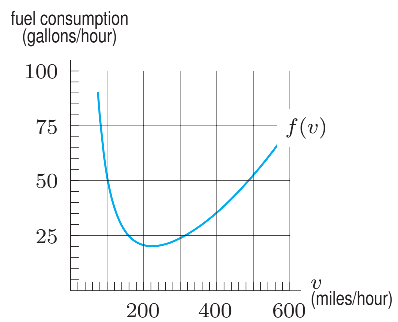


Figure 4.51

Strengthen Your Understanding

In Problems 54–56, explain what is wrong with the statement.

54. If A is the area of a rectangle of sides x and $2x$, for $0 \leq x \leq 10$, the maximum value of A occurs where $dA/dx = 0$.

55. An open box is made from a 20-inch square piece of cardboard by cutting squares of side h from the corners and folding up the edges, giving the box in Figure 4.52. To find the maximum volume of such a box, we work on the domain $h \geq 0$.

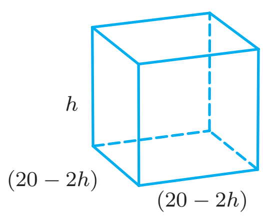


Figure 4.52: Box of volume $V = h(20-2h)^2$

56. The solution of an optimization problem modeled by a quadratic function occurs at the vertex of the quadratic.

In Problems 57–59, give an example of:

57. The sides of a rectangle with perimeter 20 cm and area smaller than 10 cm^2.

58. A context for a modeling problem where you are given that $xy = 120$ and you are minimizing the quantity $2x + 6y$.

59. A modeling problem where you are minimizing the cost of the material in a cylindrical can of volume 250 cubic centimeters.