Let us calculate the circumference of the ellipse with axes a and b where a>0 and b>0. The following command does not work.

In[72]:=
```
(* Integrate[√(a² (Sin[t])²+b²(Cos[t])²),{t,0,2 Pi}] *)
```

The command below works but gives a conditional expression since *Mathematica* does not know that a > 0 and b > 0.

In[73]:=
```
4 Integrate[√(a² (Sin[t])² + b² (Cos[t])²), {t, 0, Pi / 2}]
```

Out[73]=
```
ConditionalExpression[4 √(b²) EllipticE[1 - a²/b²],
  Re[b²] > 0 && ((a ∈ Reals && (Re[a] == 0 || Re[b] ≠ 0)) ||
     (Re[a] == 0 && b ∉ Reals) || (a ∉ Reals && Re[a] ≠ 0 && Im[b] + (Re[a] Re[b])/Im[a] ≠ 0))]
```

In[74]:=
```
Options[Integrate]
```

Out[74]=
```
{Assumptions :→ $Assumptions, GenerateConditions → Automatic, PrincipalValue → False}
```

The following command is faster.

In[75]:=
```
4 Integrate[√(a² (Sin[t])² + b² (Cos[t])²), {t, 0, Pi / 2}, Assumptions :→ And[a > 0, b > 0]]
```

Out[75]=
```
4 b EllipticE[1 - a²/b²]
```

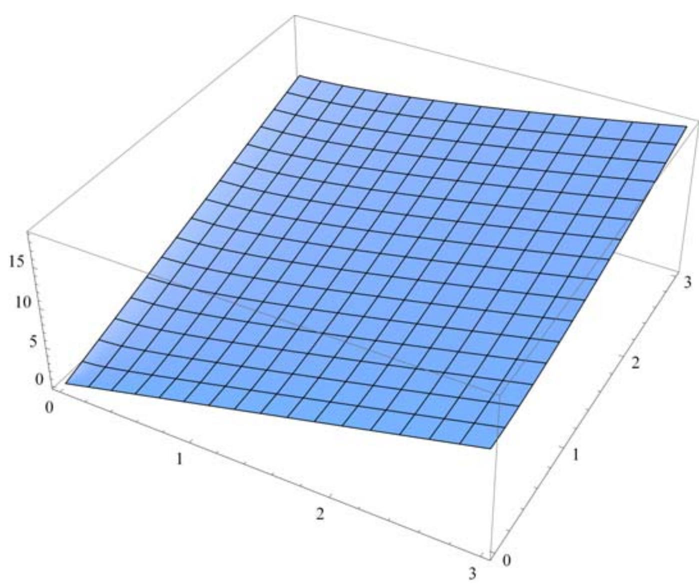
Integrating over $[0, 2\pi]$ gives. It is slow, but it does evaluate.

```
(* Integrate[√(a²(Sin[t])²+b²(Cos[t])²),{t,0,2Pi},Assumptions:→And[a>0,b>0]] *)
```

Out[76]=
```
2 (b EllipticE[1 - a²/b²] + a EllipticE[1 - b²/a²])
```

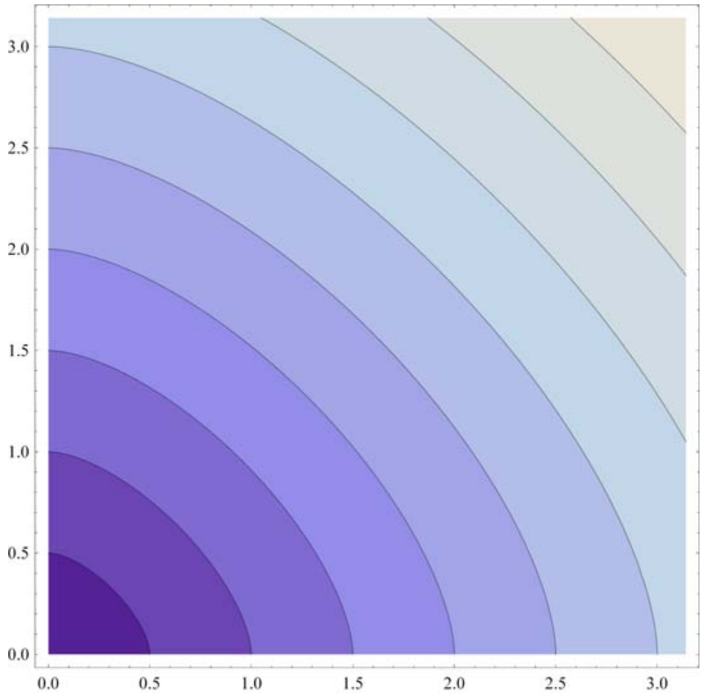
In[77]:=
```
Plot3D[2 (b EllipticE[1 - a²/b²] + a EllipticE[1 - b²/a²]), {a, 0, 3}, {b, 0, 3}]
```

Out[77]=



In[78]:=
```
ContourPlot[2 (b EllipticE[1 - a^2/b^2] + a EllipticE[1 - b^2/a^2]), {a, 0, Pi}, {b, 0, Pi}]
```

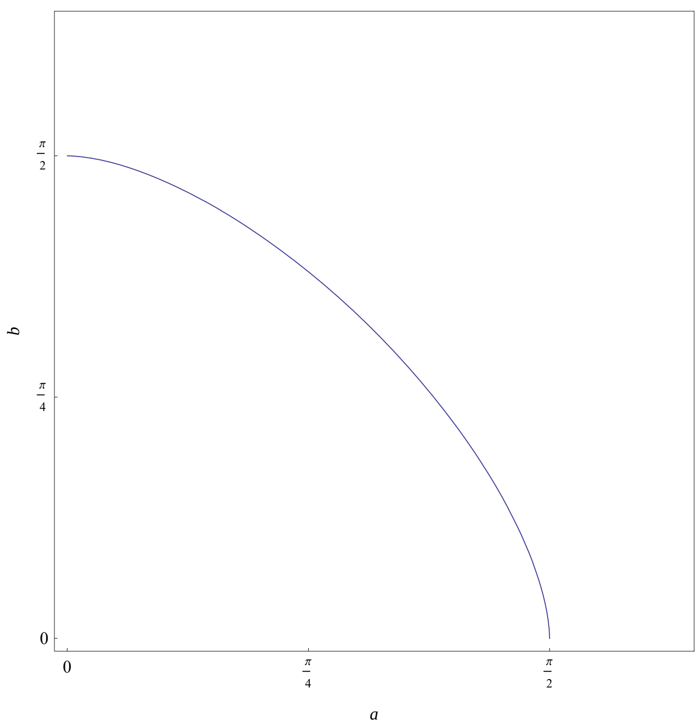
Out[78]=



The following graph plots the values of a and b which give ellipses whose circumference is 2π.

In[79]:=
```
ContourPlot[2 (b EllipticE[1 - a^2/b^2] + a EllipticE[1 - b^2/a^2]) == 2 Pi, {a, 0, 2}, {b, 0, 2},
 FrameTicks → {Range[0, Pi, Pi/4], Range[0, Pi, Pi/4], {}, {}}, FrameLabel → {a, b}]
```

Out[79]=



The following solve command does not work.

In[80]:= $$\text{Solve}\left[2\left(b\,\text{EllipticE}\left[1-\frac{a^2}{b^2}\right]+a\,\text{EllipticE}\left[1-\frac{b^2}{a^2}\right]\right)==2\,\text{Pi},\ b\right]$$

Solve::nsmet : This system cannot be solved with the methods available to Solve. ≫

Out[80]= $$\text{Solve}\left[2\left(b\,\text{EllipticE}\left[1-\frac{a^2}{b^2}\right]+a\,\text{EllipticE}\left[1-\frac{b^2}{a^2}\right]\right)==2\,\pi,\ b\right]$$

But, using a different formula for the circumference does work.

In[81]:= $$4\,b\,\text{EllipticE}\left[1-\frac{a^2}{b^2}\right]$$

Out[81]= $$4\,b\,\text{EllipticE}\left[1-\frac{a^2}{b^2}\right]$$

In[82]:= $$\text{Solve}\left[4\,b\,\text{EllipticE}\left[1-\frac{a^2}{b^2}\right]==2\,\text{Pi},\ a\right]$$

Solve::ifun :

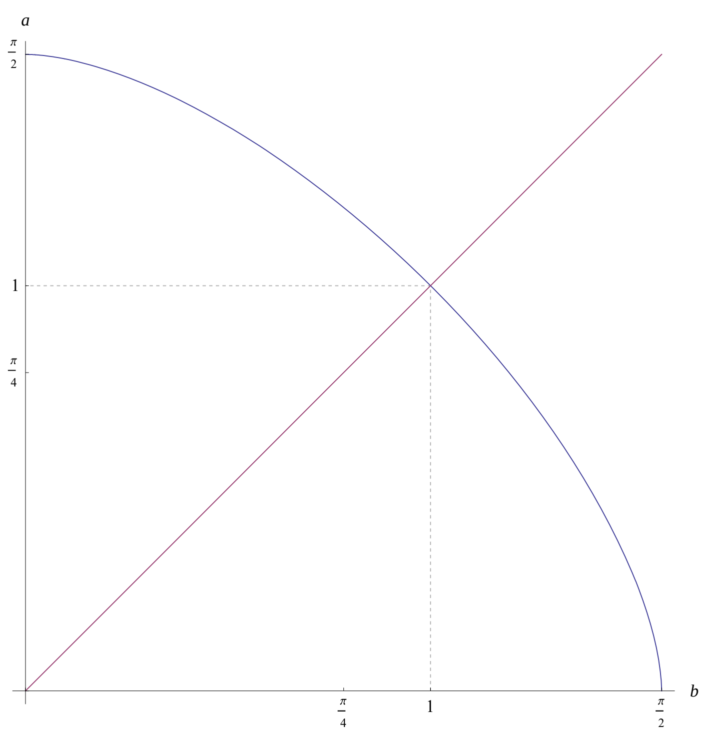
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[82]= $$\left\{\left\{a\to-\sqrt{b^2-b^2\,\text{EllipticE}^{(-1)}\left[\frac{\pi}{2\,b}\right]}\right\},\left\{a\to\sqrt{b^2-b^2\,\text{EllipticE}^{(-1)}\left[\frac{\pi}{2\,b}\right]}\right\}\right\}$$

In[83]:= $$\text{Plot}\left[\left\{\sqrt{b^2-b^2\,\text{EllipticE}^{(-1)}\left[\frac{\pi}{2\,b}\right]},\ b\right\},\ \{b,\ 0,\ \text{Pi}/2\},\right.$$

```
Prolog → {{Dashing[{0.005, .005}], GrayLevel[0.7], Line[{{1, 0}, {1, 1}, {0, 1}}]}},
AspectRatio → Automatic,
Ticks → {Join[Range[0, Pi, Pi/4], {1}], Join[Range[0, Pi, Pi/4], {1}]}, AxesLabel → {b, a}]
```

Out[83]=



Notice that for b=1 we have

In[84]:= $\sqrt{1^2 - 1^2 \,\text{EllipticE}^{(-1)}\left[\frac{\pi}{2\times 1}\right]}$

Out[84]= 1

Although the formula $\sqrt{b^2 - b^2 \,\text{EllipticE}^{(-1)}\left[\frac{\pi}{2\,b}\right]}$ for a does work, it is slow when we use it in the plot.

Further notice that the function $\sqrt{b^2 - b^2 \,\text{EllipticE}^{(-1)}\left[\frac{\pi}{2\,b}\right]}$ is symmetric with respect to b=a line. That is, this function is its own inverse. Therefore we need to calculate only values for a when b is in the interval [0,1]

In[85]:= Myabs = Table$\left[\text{N}\left[\left\{\sqrt{b^2 - b^2\,\text{EllipticE}^{(-1)}\left[\frac{\pi}{2\,b}\right]},\ b\right\}\right],\ \left\{b,\ \frac{1}{20},\ 1,\ \frac{1}{20}\right\}\right]$;

In[86]:= Myabs[[1]]

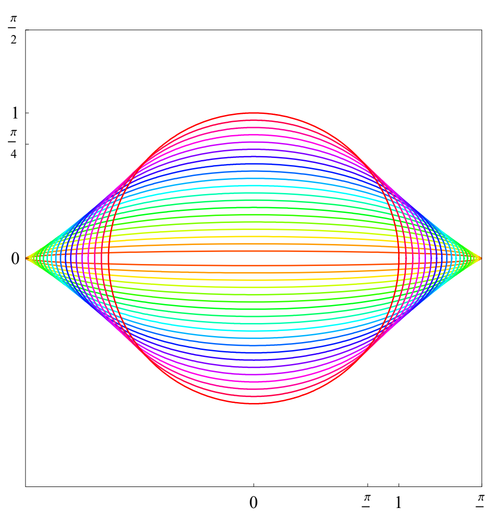
Out[86]= {1.56734, 0.05}

In[87]:= Myabs[[1]][[1]]

Out[87]= 1.56734

In[88]:= Show[Table[ParametricPlot[{(Myabs[[k]][[1]]) Cos[t], (Myabs[[k]][[2]]) Sin[t]},
{t, 0, 2 Pi}, PlotStyle → {{Thickness[0.003], Hue[Myabs[[k]][[2]]]}},
PlotRange → $\left\{\left\{-\frac{Pi}{2}, \frac{Pi}{2}\right\}, \left\{-\frac{Pi}{2}, \frac{Pi}{2}\right\}\right\}$, Axes → False, Frame → True,
FrameTicks → $\left\{\text{Join}\left[\text{Range}\left[0, Pi, \frac{Pi}{4}\right], \{1\}\right], \text{Join}\left[\text{Range}\left[0, Pi, \frac{Pi}{4}\right], \{1\}\right], \{\}, \{\}\right\}$,
ImageSize → 250], {k, 1, Length[Myabs]}]]

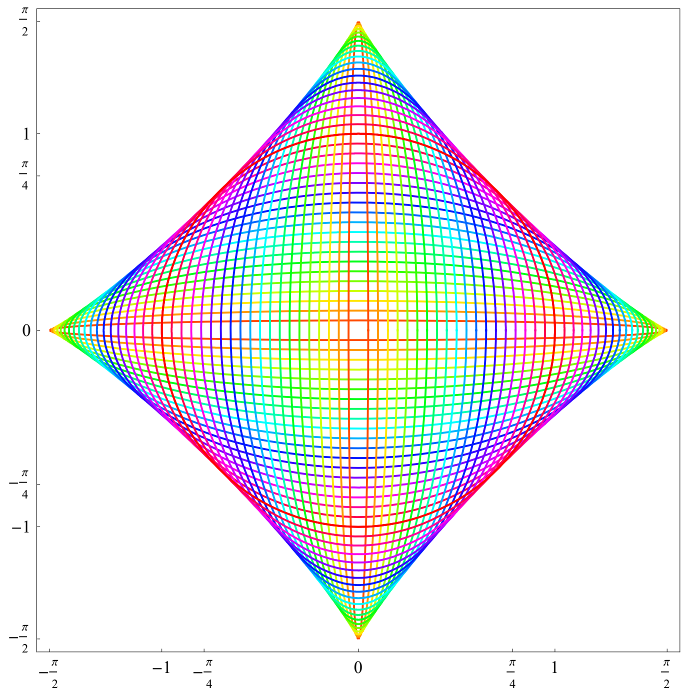
Out[88]=



To get the complete picture we have to swap the roles of a and b.

In[89]:=
```
Show[Table[ParametricPlot[{(Myabs[[k]][[1]]) Cos[t], (Myabs[[k]][[2]]) Sin[t]}, {t, 0, 2 Pi},
    PlotStyle → {{Thickness[0.003], Hue[Myabs[[k]][[2]]]}}], {k, 1, Length[Myabs]}],
  Table[ParametricPlot[{(Myabs[[k]][[2]]) Cos[t], (Myabs[[k]][[1]]) Sin[t]}, {t, 0, 2 Pi},
    PlotStyle → {{Thickness[0.003], Hue[Myabs[[k]][[2]]]}}], {k, 1, Length[Myabs]}],
  PlotRange → {{-Pi/2, Pi/2}, {-Pi/2, Pi/2}}, Axes → False, Frame → True,
  FrameTicks → {Join[Range[-Pi, Pi, Pi/4], {-1, 1}],
    Join[Range[-Pi, Pi, Pi/4], {-1, 1}], {}, {}}, ImageSize → 350]
```

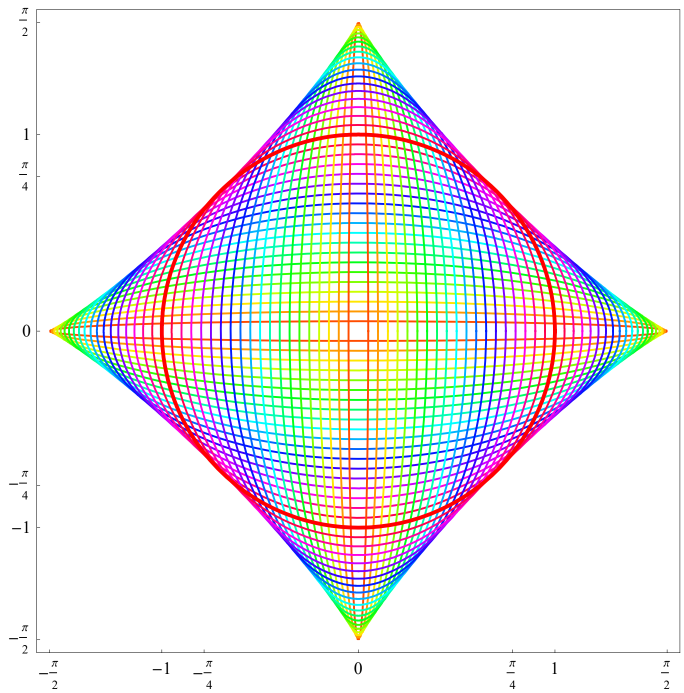
Out[89]=



To emphasize the unit circle

```
In[90]:= Show[Table[ParametricPlot[{(Myabs[[k]][[1]]) Cos[t], (Myabs[[k]][[2]]) Sin[t]}, {t, 0, 2 Pi},
    PlotStyle → {{Thickness[0.003], Hue[Myabs[[k]][[2]]]}}], {k, 1, Length[Myabs]}],
  Table[ParametricPlot[{(Myabs[[k]][[2]]) Cos[t], (Myabs[[k]][[1]]) Sin[t]}, {t, 0, 2 Pi},
    PlotStyle → {{Thickness[0.003], Hue[Myabs[[k]][[2]]]}}], {k, 1, Length[Myabs]}],
  ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2 Pi}, PlotStyle → {{Thickness[0.006], Red}}],
  PlotRange → {{-Pi/2, Pi/2}, {-Pi/2, Pi/2}}, Axes → False,
  Frame → True, FrameTicks → {Join[Range[-Pi, Pi, Pi/4], {-1, 1}],
    Join[Range[-Pi, Pi, Pi/4], {-1, 1}], {}, {}}, ImageSize → 350]
```

Out[90]=

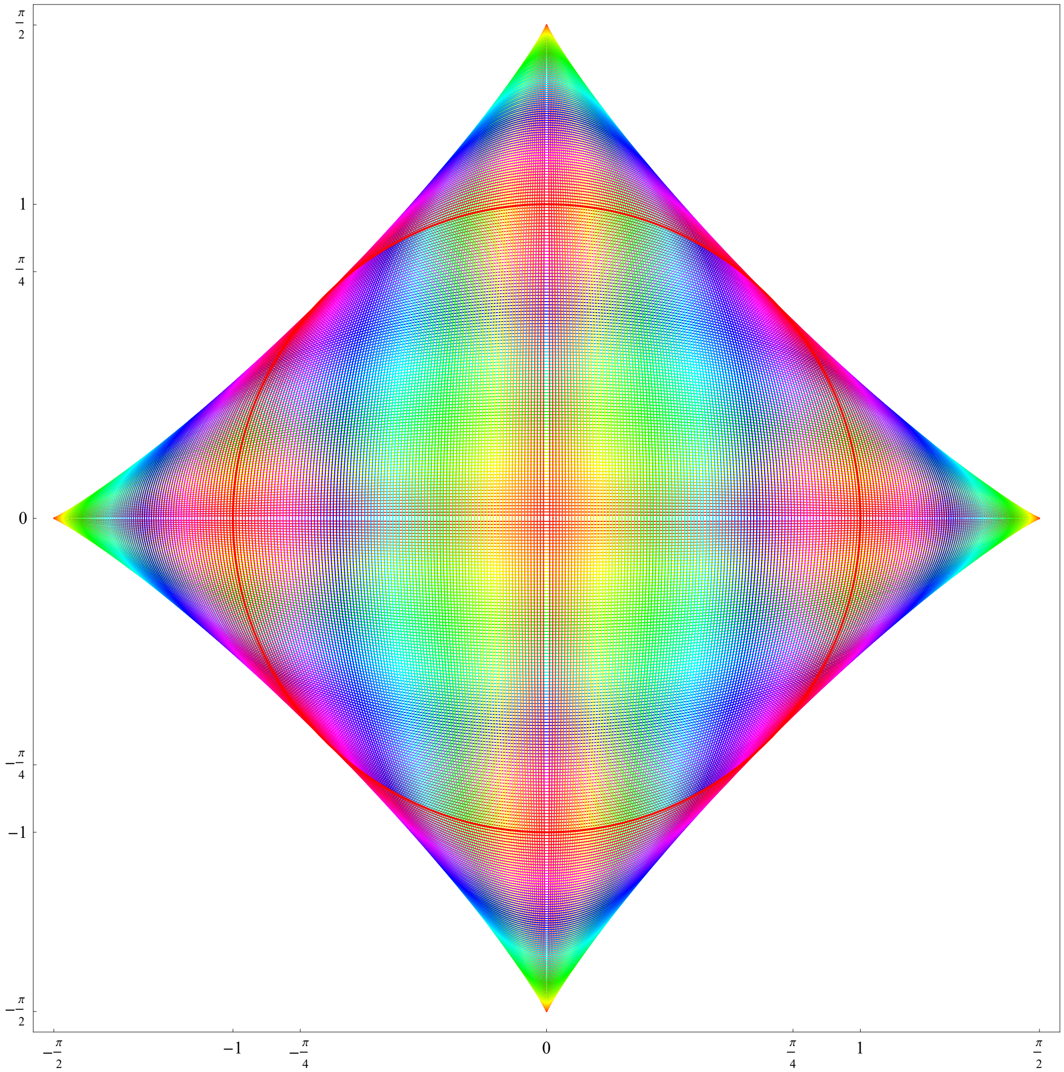


To experiment with the number of ellipses we use

```
In[99]:= st = 1/100; ET = Table[N[{b Sqrt[1 - EllipticE^(-1)[π/(2 b)]], b}], {b, st, 1 - st, st}];
```

```
(* thickness for ellipses *) the = 0.001;
(* thickness for the unit circle *) thc = 0.002;

Show[
 (* wide ellipses *)

 Table[ParametricPlot[{(ET[[k]][[1]]) Cos[t], (ET[[k]][[2]]) Sin[t]}, {t, 0, 2 Pi},
   PlotStyle → {{Thickness[the], Hue[ET[[k]][[2]]]}}], {k, 1, Length[ET]}],

 (* toll ellipses *)
 Table[ParametricPlot[{(ET[[k]][[2]]) Cos[t], (ET[[k]][[1]]) Sin[t]}, {t, 0, 2 Pi},
   PlotStyle → {{Thickness[the], Hue[ET[[k]][[2]]]}}], {k, 1, Length[ET]}],

 (* unit circle *)
 ParametricPlot[{ Cos[t], Sin[t]}, {t, 0, 2 Pi}, PlotStyle → {{Thickness[thc], Red}}],

 PlotRange → {{-Pi/2, Pi/2}, {-Pi/2, Pi/2}}, Axes → False,
 Frame → True, FrameTicks → {Join[Range[-Pi, Pi, Pi/4], {-1, 1}],
   Join[Range[-Pi, Pi, Pi/4], {-1, 1}], {}, {}}, ImageSize → 550]
```

Out[102]=



$\frac{\pi}{2}$
1
$\frac{\pi}{4}$
0
$-\frac{\pi}{4}$
−1
$-\frac{\pi}{2}$
$-\frac{\pi}{2}$
−1
$-\frac{\pi}{4}$
0
$\frac{\pi}{4}$
1
$\frac{\pi}{2}$