```
va = {a1, a2, a3};
vb = {b1, b2, b3};
vc = {c1, c2, c3};
vd = {d1, d2, d3};
```

Some properties of the cross product and dot product

■ Mixed product a.(b × c)

The product **a**.(**b** x **c**) is called the mixed product. The geometric meaning of the mixed product is the volume of the parallelepiped spanned by the vectors **a**, **b**, **c**, provided that they follow the right hand rule. This fact is consistent with the above identities.

It turns out that the mixed product equals the 3×3 determinant whose rows are the coordinates of the vectors in the order of the mixed product.

$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

The most important property of the mixed product is that it is “circular”

$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \mathbf{c}.(\mathbf{a} \times \mathbf{b}) = \mathbf{b}.(\mathbf{c} \times \mathbf{a})$

Here is a *Mathematica* “proof” of this property

```
Simplify[va.Cross[vb, vc] - vc.Cross[va, vb]]
```

```
0
```

```
Simplify[va.Cross[vb, vc] - vb.Cross[vc, va]]
```

```
0
```

■ Double cross product a×(b×c) = (a.c) b - (a.b) c

The following identity holds for the double cross product of three vectors

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}.\mathbf{c})\, \mathbf{b} - (\mathbf{a}.\mathbf{b})\, \mathbf{c}$

Here is a *Mathematica* proof of this identity:

```
Simplify[Cross[va, Cross[vb, vc]] - ((va.vc) vb - (va.vb) vc)]
```

```
{0, 0, 0}
```

How to prove this Identity?

■ Here is a proof

First we prove the identity with the unit mutually orthogonal vectors **u** and **v**. Let **u** and **v** be unit vectors such that **u** and **v** are orthogonal. Then

$\mathbf{a}\times(\mathbf{u}\times\mathbf{v}) = (\mathbf{a}.\mathbf{v})\, \mathbf{u} - (\mathbf{a}.\mathbf{u})\, \mathbf{v}$

To prove this identity we notice that the vector $\mathbf{a}\times(\mathbf{u}\times\mathbf{v})$ is orthogonal to $\mathbf{u}\times\mathbf{v}$ and therefore must be in the plane spanned by **u** and **v**. Therefore $\mathbf{a}\times(\mathbf{u}\times\mathbf{v})$ must be a linear combination of **u** and **v**. That is

$\mathbf{a}\times(\mathbf{u}\times\mathbf{v}) = \alpha\, \mathbf{u} + \beta\, \mathbf{v}$

Since **u** and **v** are unit vectors and $\mathbf{u}\perp\mathbf{v}$, by the definition of the cross product we have $(\mathbf{u}\times\mathbf{v})\times\mathbf{u} = \mathbf{v}$ and $(\mathbf{u}\times\mathbf{v})\times\mathbf{v} = -\mathbf{u}$. Further, since $\mathbf{u}\perp\mathbf{v}$, we have $\mathbf{u}.\mathbf{v} = \mathbf{0}$. Therefore

$\alpha = \mathbf{u}.(\mathbf{a}\times(\mathbf{u}\times\mathbf{v})) = \mathbf{a}.((\mathbf{u}\times\mathbf{v})\times\mathbf{u}) = \mathbf{a}.\mathbf{v}$

and

β = **v.**(**a**×(**u**×**v**)) = **a**.((**u**×**v**)×**v**) = -(**a.u**)

This proves

a×(**u**×**v**) = (**a.v**) **u** - (**a.u**) **v**

whenever **u** and **v** are unit vectors and **u**⊥**v**.

Now we choose any orthogonal unit vectors **u** and **v** in the plane spanned by **b** and **c** and write **b** = α **u** + β **v** and **c** = γ **u** + δ **v**. Then, using the linearity of the cross product and the facts **u**×**u** = **v**×**v** = **0** and **v**×**u** = **-u**×**v**, we have

a×(**b**×**c**) = **a** × ((α **u** + β **v**)×(γ **u** + δ **v**))
= $\alpha\gamma$ (**a**×(**u**×**u**)) + $\alpha\delta$ (**a**×(**u**×**v**)) + $\beta\gamma$ (**a**×(**v**×**u**)) + $\beta\delta$ (**a**×(**v**×**v**))
= ($\alpha\delta$-$\beta\gamma$) (**a**×(**u**×**v**))

Using the linearity of the dot product we get

(**a.c**) **b** - (**a.b**) **c** = (**a.**(γ **u** + δ **v**)) (α **u** + β **v**) - (**a.**(α **u** + β **v**)) (γ **u** + δ **v**)
= ($\alpha\gamma$ - $\alpha\gamma$) (**a.u**) **u** + ($\alpha\delta$ - $\beta\gamma$) (**a.v**) **u** +($\beta\gamma$ - $\alpha\delta$) (**a.u**) **v** +($\beta\delta$ - $\beta\delta$) (**a.v**) **v**
= ($\alpha\delta$-$\beta\gamma$) ((**a.v**) **u** - (**a.u**) **v**)

Together with

a×(**u**×**v**) = (**a.v**) **u** - (**a.u**) **v**

the preceding two identifies prove that

a×(**b**×**c**) = (**a.c**) **b** - (**a.b**) **c**

- **Dot product of two cross products (a x b).(c x d) = (a.c) (b.d) - (a.d)(b.c)**

Here is the *Mathematica* proof of the identity in the title.

```
Simplify[(va.vc) (vb.vd) - (va.vd) (vb.vc) - Cross[va, vb].Cross[vc, vd]]
```

```
0
```

```
Cross[va, vb]
```

```
{-a3 b2 + a2 b3, a3 b1 - a1 b3, -a2 b1 + a1 b2}
```

```
Cross[vc, vd]
```

```
{-c3 d2 + c2 d3, c3 d1 - c1 d3, -c2 d1 + c1 d2}
```

The identity in the title is deduced by using first the “circularity” property of the mixed product and then the identity for the double cross product of three vectors:

(**a** × **b**).(**c** × **d**)) = **c**.(**d** × (**a** × **b**)) = **c.**((**b**.**d**) **a** - (**a**.**d**) **b**) = (**b**.**d**) (**a.c**) - (**a**.**d**) (**b.c**)

A natural derivation of the algebraic formula for Curl

The last identity proved in the preceding section, when applied to an abstract version of Green’s theorem, yields the defining property of the Curl. First rewrite

(**a** × **b**).(**c** × **d**) = (**a.c**) (**b**.**d**) - (**a**.**d**) (**b**.**c**)

as

(**a** × **b**).(**c** × **d**) = (**a.c**) (**d**.**b**) - (**b**.**c**)(**d**.**a**)

In the formulas below we use the formal Nabla operator vector $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$. Applied to a scalar function f the formal Nabla operator vector gives the gradient ∇f of that function. The expression ∇f is a vector function. One thinks of the last expression as the Nabla vector being scaled by a scalar function. The dot product $\nabla . \boldsymbol{F}$ of the Nabla operator vector and a vector function $\boldsymbol{F}$ is the divergence of $\boldsymbol{F}$.

An abstract version of Green's theorem is as follows: Let **p** and **q** be unit vectors and let C be a simple, closed, piecewise smooth curve in the plane determined by **p** and **q**. Then the circulation along C of a vector field **F** is given as the double integral of the difference of the directional derivative in direction **p** of the projection of **F** onto **q** and the directional derivative in direction **q** of the projection of **F** onto **p**. That is

$$\oint_C \mathbf{F} . d\mathbf{r} = \int\int_R (\mathbf{p} . \nabla (\mathbf{F} . \mathbf{q}) - \mathbf{q} . \nabla (\mathbf{F} . \mathbf{p}))\, dA$$

This formula tells us that the circulation density of $\mathbf{F}$ in direction $(\mathbf{p} \times \mathbf{q})$ is given by

$$\mathbf{p} . \nabla (\mathbf{F} . \mathbf{q}) - \mathbf{q} . \nabla (\mathbf{F} . \mathbf{p})$$

By the identity

$$(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) = (\mathbf{a}.\mathbf{c})\,(\mathbf{d}.\mathbf{b}) - (\mathbf{b}.\mathbf{c})\,(\mathbf{d}.\mathbf{a})$$

that is the identity

$$(\mathbf{a}.\mathbf{c})\,(\mathbf{d}.\mathbf{b}) - (\mathbf{b}.\mathbf{c})\,(\mathbf{d}.\mathbf{a}) = (\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d})$$

formally applied to

$$\mathbf{p} . \nabla (\mathbf{F} . \mathbf{q}) - \mathbf{q} . \nabla (\mathbf{F} . \mathbf{p})$$

we get that

$$\mathbf{p} . \nabla (\mathbf{F} . \mathbf{q}) - \mathbf{q} . \nabla (\mathbf{F} . \mathbf{p}) = (\mathbf{p} \times \mathbf{q}) . (\nabla \times \mathbf{F})$$

Now it is easy to answer the following question: In which direction is the magnitude of the circulation density the largest? Answer: The magnitude is the largest when the vector $\mathbf{p} \times \mathbf{q}$ is collinear with the vector $\nabla \times \mathbf{F}$. This explains the importance of the Curl of $\mathbf{F}$ given by $\nabla \times \mathbf{F}$.