Complete Proof of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Branko Ćurgus

March 9, 2022 19:09

Abstract

In this paper, I present my variation on the proof of the famous equality in the title. The key idea in this proof is attributed to the French 19th-century mathematician Augustin-Louis Cauchy. My contribution is that I prove all the significant mathematical facts that are used in the paper. I do not use any big theorems. If I use something, I prove it. It is my goal to build the proof from "first principles" as much as possible.

1 Background knowledge

In this section we present two main tools for the proof of the equality in the title. Notice that the trigonometric inequality at the beginning of the proof of the next proposition is also used in the proof of the limit $\lim_{x\to 0}(\sin x)/x = 1$.

Proposition 1.1. *For all* $\theta \in \big(0, \pi/2\big)$ *we have*

$$\frac{1}{(\sin\theta)^2} - 1 < \frac{1}{\theta^2} < \frac{1}{(\sin\theta)^2}. \tag{1.1}$$

Proof. Let $\theta \in (0, \pi/2)$ and let the angle $\angle DOC$ in Figure 1 be equal to θ. In Figure 1 the length of $\overline{OC}$ is 1, the length of $\overline{AB}$ is $\sin\theta$, and the length of $\overline{CD}$ is $\tan\theta$. Therefore the area of the triangle $\triangle OCB$ is $(\sin\theta)/2$ and the area of the triangle $\triangle OCD$ is $(\tan\theta)/2$. The area of the sector $\heartsuit OCB$ of the unit disk bounded by the line segments $\overline{OC}$ and $\overline{OB}$ and the circular arc joining points C and B is $\theta/2$.

Since clearly

$$\triangle OCB \subset \heartsuit OCB \subset \triangle OCD,$$

for the corresponding areas we have

$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}.$$

Multiplying by 2, squaring and taking the reciprocals yield

$$\frac{(\cos\theta)^2}{(\sin\theta)^2} < \frac{1}{\theta^2} < \frac{1}{(\sin\theta)^2},$$

which is equivalent to the inequality in the proposition. □

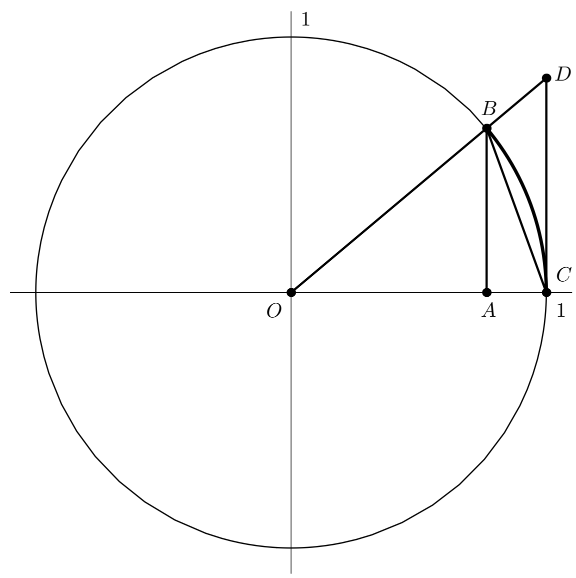

Figure 1: The unit circle

The proof of the next theorem is a little longer and will be presented in Section 4. In this note $\mathbb{N}$ denotes the set of positive integers. At few places we will work with the set of all nonnegative integers. That set we represent as the union $\mathbb{N} \cup \{0\}$.

Theorem 1.2. *For all* $n \in \mathbb{N}$ *we have*

$$\sum_{k=1}^{n} \frac{1}{\Big(\sin\big(\frac{k\pi}{2n+1}\big)\Big)^2} = \frac{2}{3}n(n+1).$$

2 Proof of the equality in the title

We first establish a squeeze for the partial sums of the series in the title.

Theorem 2.1. *For all* $n \in \mathbb{N}$ *we have*

$$\frac{\pi^2}{6}\left(1 - \frac{3}{2n+1}\right) \;\le\; \sum_{k=1}^{n}\frac{1}{k^2} \;\le\; \frac{\pi^2}{6}\left(1 - \frac{1}{(2n+1)^2}\right). \tag{2.1}$$

Proof. Let $n \in \mathbb{N}$ be arbitrary. For all $k \in \{1, \ldots, n\}$ we have that

$$\frac{k\,\pi}{2n+1} \in \left(0, \frac{\pi}{2}\right).$$

Substituting $\theta = k\pi/(2n+1)$ in (1.1) we obtain that for all $k \in \{1, \ldots, n\}$ we have

$$\frac{1}{\left(\sin\left(\frac{k\,\pi}{2n+1}\right)\right)^2} - 1 \le \frac{(2n+1)^2}{\pi^2 k^2} \le \frac{1}{\left(\sin\left(\frac{k\,\pi}{2n+1}\right)\right)^2}. \tag{2.2}$$

Summing the expressions in (2.2) we get

$$-n + \sum_{k=1}^{n} \frac{1}{\left(\sin\left(\frac{k\,\pi}{2n+1}\right)\right)^2} \le \frac{(2n+1)^2}{\pi^2} \sum_{k=1}^{n} \frac{1}{k^2} \le \sum_{k=1}^{n} \frac{1}{\left(\sin\left(\frac{k\,\pi}{2n+1}\right)\right)^2}. \tag{2.3}$$

The identity from Theorem 1.2 yields

$$-n + \frac{2}{3}n(n+1) \le \frac{(2n+1)^2}{\pi^2} \sum_{k=1}^{n} \frac{1}{k^2} \le \frac{2}{3}n(n+1).$$

A simplification of the expressions on the left-hand side and on the right-hand side of the preceding inequalities gives

$$\frac{1}{6}\big((2n+1)^2 - 3(2n+1) + 2\big) \le \frac{(2n+1)^2}{\pi^2} \sum_{k=1}^{n} \frac{1}{k^2} \le \frac{1}{6}\big((2n+1)^2 - 1\big).$$

Since clearly

$$(2n+1)^2 - 3(2n+1) \le (2n+1)^2 - 3(2n+1) + 2,$$

from the last two displayed inequalities we deduce

$$\frac{2n+1}{6}\big((2n+1) - 3\big) \le \frac{(2n+1)^2}{\pi^2} \sum_{k=1}^{n} \frac{1}{k^2} \le \frac{1}{6}\big((2n+1)^2 - 1\big).$$

Multiplying the last expression by $\pi^2/(2n+1)^2$ leads to the inequality in the theorem. □

Corollary 2.2. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$

Proof. It follows from the squeeze in Theorem 2.1 that

$$0 < \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \le \frac{\pi^2}{2(2n+1)} < \frac{\pi^2}{4n}.$$

Let $\epsilon > 0$ be arbitrary. The preceding inequality, yields the following implication

$$\text{for all} \quad n \in \mathbb{N} \quad n > \frac{\pi^2}{4\epsilon} \quad \text{implies} \quad 0 < \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} < \epsilon,$$

which, by the definition of convergence, proves that the sequence of partial sums converges to $\pi^2/6$, that is

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

The corollary is proved. □

3 Background knowledge for proof of Theorem 1.2

3.1 Trigonometric identities

We need the angle addition formulas for the sine function

$$\sin(\alpha+\beta) = (\sin\alpha)(\cos\beta) + (\cos\alpha)(\sin\beta), \quad \sin(\alpha-\beta) = (\sin\alpha)(\cos\beta) - (\cos\alpha)(\sin\beta)$$

and the double angle formulas

$$\sin(2\alpha) = 2(\sin\alpha)(\cos\alpha), \quad \cos(2\alpha) = 1 - 2(\sin\alpha)^2.$$

We also need the triple angle formula

$$\begin{aligned}
\sin(3\alpha) &= \sin(2\alpha)(\cos\alpha) + \cos(2\alpha)(\sin\alpha) \\
&= (\sin\alpha)\big(2(\cos\alpha)^2 + 1 - 2(\sin\alpha)^2\big) \\
&= (\sin\alpha)\big(2 - 2(\sin\alpha)^2 + 1 - 2(\sin\alpha)^2\big) \\
&= (\sin\alpha)\big(3 - 4(\sin\alpha)^2\big) \\
&= 3(\sin\alpha) - 4(\sin\alpha)^3.
\end{aligned}$$

3.2 Addition formulas

For all $n \in \mathbb{N}$ the following summation formulas hold

$$1 + 3 + 5 + \cdots + (2n-1) = n^2 \tag{3.1}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1). \tag{3.2}$$

(If you know mathematical induction you can prove these identities by induction. However, it might be good to see another way of proving these formulas.) There is a convenient way of proving these summation formulas using telescoping sums. To prove the first formula we recall that for all $k \in \mathbb{N}$ we have

$$k^2 - (k-1)^2 = 2k - 1.$$

Now we use the preceding identity to represent the sum of positive odd integers as a telescoping sum of the differences of squares:

$$\begin{aligned}
\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n}\big(k^2 - (k-1)^2\big) \\
&= (1^2 - 0^2) + (2^2 - 1^2) + (3^2 - 2^2) + \cdots + \big((n-1)^2 - (n-2)^2\big) + \big(n^2 - (n-1)^2\big) \\
&= n^2.
\end{aligned}$$

This proves the first summation formula (3.1).

We use the same method to prove the second summation formula (3.2). For convenience we set

$$q_k = \frac{1}{6}k(k+1)(2k+1) \quad \text{for all} \quad k \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

Then we verify that for all $k \in \mathbb{N}$ we have

$$q_k - q_{k-1} = \frac{k}{6}\big((k+1)(2k+1) - (k-1)(2k-1)\big) = \frac{k}{6}\big(2k^2 + 3k + 1 - (2k^2 - 3k + 1)\big) = k^2.$$

Now we use the preceding identity to represent the sum of squares as a telescoping sum of differences of q_k-s: For all $n \in \mathbb{N}$ we have:

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} \bigl(q_k - q_{k-1}\bigr) \\ &= \bigl(q_1 - q_0\bigr) + \bigl(q_2 - q_1\bigr) + \cdots + \bigl(q_{n-1} - q_{n-2}\bigr) + \bigl(q_n - q_{n-1}\bigr) \\ &= q_n.\end{aligned}$$

Hence, (3.2) is proved.

In the following proposition we characterize the sequence $\{q_n\}$ defined in (3.3).

Proposition 3.1. *Let $\{b_n\}$ be a sequence of real numbers. The following two statements are equivalent:*

(I) *For all $n \in \mathbb{N} \cup \{0\}$ we have*

$$b_n = \frac{1}{6} n(n+1)(2n+1) \tag{3.4}$$

(II) *We have $b_0 = 0$, $b_1 = 1$, and for all $n \in \mathbb{N}$ we have*

$$\bigl(b_{n+1} - b_n\bigr) - \bigl(b_n - b_{n-1}\bigr) = 2n + 1. \tag{3.5}$$

Proof. First we prove (I)⇒(II). Assume (I). That is assume that for all $k \in \mathbb{N} \cup \{0\}$ we have $b_k = q_k$. Then, we calculate $b_0 = 0$ and $b_1 = 1$. Since we already established that for all $k \in \mathbb{N}$ we have $q_k - q_{k-1} = k^2$, then we also have $b_k - b_{k-1} = k^2$ for all $k \in \mathbb{N}$. Further, for all $k \in \mathbb{N}$ we have

$$\bigl(b_{k+1} - b_k\bigr) - \bigl(b_k - b_{k-1}\bigr) = (k+1)^2 - k^2 = 2k + 1.$$

That is, we proved (II).

Now we prove the converse. That is we prove (II)⇒(I). Assume (II). Let $n \in \mathbb{N}$ be arbitrary. Notice that the following sum is a telescopic sum (and work out how the cancaletion works in it), so we have the following equality

$$\begin{aligned}\sum_{k=1}^{n-1} \bigl((b_{k+1} - b_k) - (b_k - b_{k-1})\bigr) &= \bigl((b_2 - b_1) - (b_1 - b_0)\bigr) + \cdots + \bigl((b_n - b_{n-1}) - (b_{n-1} - b_{n-2})\bigr) \\ &= -1 + (b_n - b_{n-1}).\end{aligned}$$

Since we assume (II) we have

$$\sum_{k=1}^{n-1} \bigl((b_{k+1} - b_k) - (b_k - b_{k-1})\bigr) = \sum_{k=1}^{n-1} (2k+1).$$

By (3.1) we have

$$\sum_{k=1}^{n-1} (2k+1) = 3 + 5 + \cdots + 2n - 1 = n^2 - 1.$$

From the last three displayed equalities we have

$$-1 + (b_n - b_{n-1}) = n^2 - 1.$$

Since $n \in \mathbb{N}$ was arbitrary, the preceding equality proves that for all $n \in \mathbb{N}$ we have $b_n - b_{n-1} = n^2$. Consequently, using a telescoping sum and (3.2), we obtain that

$$b_n = \sum_{k=1}^{n} \bigl(b_k - b_{k-1}\bigr) = \sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

is true for all $n \in \mathbb{N}$. This proves (I). □

3.3 Coefficients and roots of polynomials

It is familiar from our experience with quadratic polynomials that there are formulas relating the coefficients of a polynomial to the roots of a polynomial. Let us review some of those formulas. Assume that $a_0, a_1, a_2 \in \mathbb{R}$ and assume that $a_0 \neq 0$ and $a_2 \neq 0$. Let r_1 and r_2 be distinct real roots of the quadratic

$$a_0 + a_1 x + a_2 x^2.$$

Then

$$a_0 + a_1 x + a_2 x^2 = a_2 (x - r_1)(x - r_2).$$

Multiplying out the product on the right-hand side we get

$$a_0 + a_1 x + a_2 x^2 = a_2 r_1 r_2 - a_2 (r_1 + r_2) x + a_2 x^2.$$

Therefore,

$$a_0 = a_2 r_1 r_2, \quad a_1 = -a_2 (r_1 + r_2).$$

Since we assume that $a_0 \neq 0$ and $a_2 \neq 0$, we have that $r_1 \neq 0$ and $r_2 \neq 0$ and

$$-\frac{a_1}{a_0} = \frac{a_2(r_1 + r_2)}{a_2 r_1 r_2} = \frac{1}{r_1} + \frac{1}{r_2}.$$

That is, the ratio $-a_1/a_0$ of coefficients equals to the sum of the reciprocals of the roots. In the next proposition we prove that the preceding relationship between the coefficients a_0 and a_1 and the roots is universal, that is, it holds for an arbitrary polynomial with a nonzero constant coefficient a_0.

Proposition 3.2. *Let $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Let*

$$P(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

be a polynomial of degree n with $a_0 \neq 0$ and $a_n \neq 0$. Assume that $P(x)$ has distinct real roots $r_1, \ldots, r_n$. Then $r_k \neq 0$ for all $k \in \{1, \ldots, n\}$ and for all $x \in \mathbb{R}$ we have

$$P(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n = a_0 \prod_{k=1}^{n} \left(1 - \frac{x}{r_k}\right). \tag{3.6}$$

In particular,

$$\sum_{k=1}^{n} \frac{1}{r_k} = -\frac{a_1}{a_0}.$$

Proof. By repeated factoring of linear terms we obtain the identity

$$P(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n = a_n \prod_{k=1}^{n} \big(x - r_k\big). \tag{3.7}$$

By multiplying out the linear factors we calculate the constant coefficient on the right-hand side to be

$$a_0 = (-1)^n a_n \rho,$$

where we set $\rho = r_1 \cdots r_n$. Since $a_0 \neq 0$ it follows that $\rho \neq 0$ and therefore $r_k \neq 0$ for each $k \in \{1, \ldots, n\}$. Next we factor out $-r_k$ in each of the linear factors in (3.7), that is $\big(x - r_k\big) = (-r_k)\big(1 - x/r_k\big)$, to get

$$P(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n = (-1)^n a_n \rho \prod_{k=1}^{n} \left(1 - \frac{x}{r_k}\right).$$

Since $a_0 = (-1)^n a_n \rho$, the preceding equality proves (3.6).

To calculate the coefficient of x in the product in (3.6), for each $k \in \{1, \ldots, n\}$ we multiply $-\dfrac{x}{r_k}$ by all 1s in the remaining factors and take into account the distributivity of multiplication to conclude that the coefficient of x in the product in (3.6) is

$$-a_0 \left(\frac{1}{r_1} + \cdots + \frac{1}{r_n}\right).$$

Consequently, equating the coefficients of x on the left and right side of the last equality in (3.6) we obtain

$$a_1 = -a_0 \left(\frac{1}{r_1} + \cdots + \frac{1}{r_n}\right).$$

The last claim in the proposition follows by dividing by $-a_0 \neq 0$. □

4 Proof of Theorem 1.2

In the trigonometry background knowledge we proved the triple angle formula for the sine function

$$\sin(3\theta) = (\sin\theta)\big(3 - 4(\sin\theta)^2\big).$$

Our next goal is to establish the analogous formulas for $\sin(5\theta)$, $\sin(7\theta)$, in fact for all odd multiple angles:

$$\sin\big((2n+1)\theta\big) \quad \text{where} \quad n \in \mathbb{N} \cup \{0\}.$$

To accomplish this task we will use a remarkable idea due to the Russian mathematician Pafnuty Chebyshev who lived in 19th century. To expand $\sin(5\theta)$ as a linear combination of sines we will use the sine addition formulas. We think of $\sin(5\theta)$ as $\sin(3\theta + 2\theta)$, but at the same time we think of $\sin(\theta)$ as $\sin(3\theta - 2\theta)$ to obtain the following two formulas:

$$\begin{aligned} \sin(5\theta) &= \sin(3\theta)\cos(2\theta) + \cos(3\theta)\sin(2\theta), \\ \sin(\theta) &= \sin(3\theta)\cos(2\theta) - \cos(3\theta)\sin(2\theta). \end{aligned}$$

Adding the preceding two formulas and moving $\sin(\theta)$ to the right-hand side gives us

$$\sin(5\theta) = 2\sin(3\theta)\cos(2\theta) - \sin(\theta).$$

Next we substitute the triple angle formula and the double angle formula for the cosine and we get

$$\sin(5\theta) = (\sin\theta)\Big(2\big(3 - 4(\sin\theta)^2\big)\big(1 - 2(\sin\theta)^2\big) - 1\Big). \tag{4.1}$$

Now it remains to simplify the last expression to get

$$\sin(5\theta) = (\sin\theta)\big(5 - 20(\sin\theta)^2 + 16(\sin\theta)^4\big).$$

Our reader can imagine that these formulas are becoming more complicated. To facilitate their easier writing we introduce the following polynomials:

$$P_0(x) = 1, \quad P_1(x) = 3 - 4x, \quad P_2(x) = 5 - 20x + 16x^2. \tag{4.2}$$

With this new notation we can write the odd multiple angle formulas that we know so far as

$$\sin\theta = (\sin\theta)P_0\big((\sin\theta)^2\big), \quad \sin(3\theta) = (\sin\theta)P_1\big((\sin\theta)^2\big), \quad \sin(5\theta) = (\sin\theta)P_2\big((\sin\theta)^2\big).$$

Moreover, formula (4.1) can now be rewritten as

$$(\sin\theta)P_2\big((\sin\theta)^2\big) = (\sin\theta)\Big(2P_1\big((\sin\theta)^2\big)\big(1 - 2(\sin\theta)^2\big) - P_0\big((\sin\theta)^2\big)\Big).$$

Dividing by $\sin\theta \neq 0$ and setting $x = (\sin\theta)^2$ the last formula tells us the following recursive relationship between $P_0(x)$, $P_1(x)$, and $P_2(x)$.

$$P_2(x) = 2P_1(x)(1 - 2x) - P_0(x).$$

This pattern continues. To calculate $\sin(7\theta)$ we would expand $\sin(5\theta + 2\theta)$ and $\sin(3\theta) = \sin(5\theta - 2\theta)$ using the angle addition formulas, add the resulting expressions, substitute $\cos(2\theta)$ by $1 - 2(\sin\theta)^2$ and simplify. Or, what is equivalent, we can calculate

$$P_3(x) = 2P_2(x)(1 - 2x) - P_1(x) = 7 - 56x + 112x^2 - 64x^3,$$

and then we have

$$\sin(7\theta) = (\sin\theta)P_3\big((\sin\theta)^2\big).$$

What we did for $n = 2$ (for $\sin(5\theta)$) and $n = 3$ (for $\sin(7\theta)$) we can do for every $n \in \mathbb{N}$. Assuming that we already have

$$\sin\big((2n-1)\theta\big) = (\sin\theta)P_{n-1}\big((\sin\theta)^2\big) \quad \text{and} \quad \sin\big((2n+1)\theta\big) = (\sin\theta)P_n\big((\sin\theta)^2\big),$$

we use Chebyshev's idea to calculate $\sin\big((2n+3)\theta\big)$. As before,

$$\begin{aligned} \sin\big((2n+3)\theta\big) &= \sin\big((2n+1)\theta + 2\theta\big) = \sin\big((2n+1)\theta\big)\cos(2\theta) + \cos\big((2n+1)\theta\big)\sin(2\theta), \\ \sin\big((2n-1)\theta\big) &= \sin\big((2n+1)\theta - 2\theta\big) = \sin\big((2n+1)\theta\big)\cos(2\theta) - \cos\big((2n+1)\theta\big)\sin(2\theta). \end{aligned}$$

Adding the last two identities, moving $\sin\big((2n-1)\theta\big)$ to the right-hand side and substituting $\cos(2\theta) = 1 - 2(\sin\theta)^2$ we get

$$\sin\big((2n+3)\theta\big) = 2\sin\big((2n+1)\theta\big)\big(1 - 2(\sin\theta)^2\big) - \sin\big((2n-1)\theta\big). \tag{4.3}$$

As we assume that we already know the identities

$$\sin\big((2n-1)\theta\big) = (\sin\theta)P_{n-1}\big((\sin\theta)^2\big) \quad \text{and} \quad \sin\big((2n+1)\theta\big) = (\sin\theta)P_n\big((\sin\theta)^2\big),$$

substituting the preceding two identities in (4.3) yields

$$\sin\big((2n+3)\theta\big) = (\sin\theta)\Big(2P_n\big((\sin\theta)^2\big)\big(1-2(\sin\theta)^2\big) - P_{n-1}\big((\sin\theta)^2\big)\Big). \tag{4.4}$$

Consequently, defining the new polynomial $P_{n+1}(x)$ by

$$P_{n+1}(x) = 2P_n(x)(1-2x) - P_{n-1}(x),$$

enables us to write (4.4) as

$$\sin\big((2n+3)\theta\big) = (\sin\theta)P_{n+1}\big((\sin\theta)^2\big).$$

In the above reasoning we constructed a special sequence of polynomials $P_n(x)$ of degree n where n is a nonnegative integer. In the next proposition we use the following notation for these polynomials

$$P_n(x) = a_{0,n} + a_{1,n}x + \cdots + a_{n,n}x^n,$$

that is, for $k \in \{0,1,\ldots,n\}$ the coefficient with the k-th monomial x^k is denoted by $a_{k,n}$.

Proposition 4.1. *Let $P_n(x)$ be the recursively defined sequence of polynomials:*

$$P_0(x) = 1, \quad P_1(x) = 3-4x, \quad \text{and} \quad P_{n+1}(x) = 2P_n(x)(1-2x) - P_{n-1}(x) \tag{4.5}$$

for all $n \in \mathbb{N}$. Then each polynomial $P_n(x)$ is of degree n and for all $n \in \mathbb{N}\cup\{0\}$ we have

$$a_{0,n} = 2n+1, \quad a_{1,n} = -\frac{2}{3}n(n+1)(2n+1), \quad a_{n,n} = (-4)^n. \tag{4.6}$$

For every $n \in \mathbb{N}\cup\{0\}$ and for all $\theta \in \mathbb{R}$ we have

$$\sin\big((2n+1)\theta\big) = (\sin\theta)P_n\big((\sin\theta)^2\big). \tag{4.7}$$

Proof. **Step 1.** By the recursive definition in (4.5) the polynomial $P_0(x) = 1$ is of degree 0 and $P_1(x)$ is of degree 1. The recursive formula produces $P_2(x)$ of degree 2, and in general, each next polynomial will have the degree one higher than the preceding one. Hence, $P_n(x)$ will be of degree n. This is also confirmed by calculating each $a_{n,n}$. Clearly, $a_{0,0} = 1 = (-4)^0$ and $a_{1,1} = -4$. To calculate the leading coefficient $a_{n+1,n+1}$ of $P_{n+1}(x)$ it is convenient to write

$$P_{n+1}(x) = (-4)xP_n(x) + 2P_n(x) - P_{n-1}(x).$$

Hence, $a_{n+1,n+1} = (-4)a_{n,n}$. Consequently, $a_{2,2} = (-4)^2$, $a_{3,3} = (-4)^3$, and, so on, $a_{n,n} = (-4)^n$. This proves the third formula in (4.6).

Step 2. Now we prove the formula for $a_{0,n}$. It follows from (4.5) that

$$a_{0,0} = 1, \quad a_{0,1} = 3, \quad a_{0,n+1} = 2a_{0,n} - a_{0,n-1} \tag{4.8}$$

for all $n \in \mathbb{N}$. Rewriting the preceding recursion we get the recursion for the differences

$$a_{0,1} - a_{0,0} = 2, \quad a_{0,n+1} - a_{0,n} = a_{0,n} - a_{0,n-1}$$

for all $n \in \mathbb{N}$. Consequently, $a_{0,n} - a_{0,n-1} = 2$ for all $n \in \mathbb{N}$. Now we represent $a_{0,n}$ as the sum of differences and deduce

$$\begin{aligned} a_{0,n} &= (a_{0,n} - a_{0,n-1}) + (a_{0,n-1} - a_{0,n-2}) + \cdots + (a_{0,1} - a_{0,0}) + a_{0,0} \\ &= 2 + 2 + \cdots + 2 + 1 \\ &= 2n+1. \end{aligned}$$

This proves the first formula in (4.6).

Step 3. To prove the formula for $a_{1,n}$ we set $a_{1,0} = 0$, pretending that $P_0(x) = 1 + 0x$. Then it follows from (4.5) that

$$a_{1,0} = 0, \quad a_{1,1} = -4, \quad a_{1,n+1} = 2a_{1,n} - 4a_{0,n} - a_{1,n-1}$$

for all $n \in \mathbb{N}$. Since we already know that $a_{0,n} = 2n + 1$ for all $n \in \mathbb{N} \cup \{0\}$ we can rewrite the preceding recursion as

$$a_{1,0} = 0, \quad a_{1,1} = -4, \quad a_{1,n+1} = 2a_{1,n} - 4(2n + 1) - a_{1,n-1} \tag{4.9}$$

Set $b_n = -a_{1,n}/4$ for all $n \in \mathbb{N} \cup \{0\}$. Now we divide all equalities in (4.9) by -4 and rewrite (4.9) in terms of b_n to get

$$b_0 = 0, \quad b_1 = 1, \quad b_{n+1} - 2b_n + b_{n-1} = 2n + 1 \tag{4.10}$$

for all $n \in \mathbb{N}$. The recursion in (4.10) is identical to the recursion in Proposition 3.1(II). By the implication (II)$\Rightarrow$(I) proved in Proposition 3.1 we conclude that $b_n = n(n + 1)(2n + 1)/6$ for all $n \in \mathbb{N} \cup \{0\}$. Since $b_n = -a_{1,n}/4$, we have

$$a_{1,n} = -4\frac{1}{6}n(n + 1)(2n + 1) = -\frac{2}{3}n(n + 1)(2n + 1)$$

for all $n \in \mathbb{N} \cup \{0\}$. This proves the second formula in (4.6).

Step 4. The identity (4.7) was proved by the recursive reasoning preceding this proposition. $\square$

Finally we can prove Theorem 1.2 which is the key tool in the proof of the equality in the title. First we will restate the theorem.

Theorem 1.2. *For all $n \in \mathbb{N}$ we have*

$$\sum_{k=1}^{n} \frac{1}{\Bigl(\sin\bigl(\frac{k\pi}{2n+1}\bigr)\Bigr)^2} = \frac{2}{3}n(n + 1).$$

Proof. Let $n \in \mathbb{N}$ be arbitrary. Consider the polynomial $P_n(x)$ of degree n from Proposition 4.1. Then, as stated in (4.7), for all $\theta \in \mathbb{R}$ we have

$$\sin\bigl((2n + 1)\theta\bigr) = (\sin\theta)P_n\bigl((\sin\theta)^2\bigr). \tag{4.11}$$

Notice that

$$0 < \frac{\pi}{2n + 1} < \frac{2\pi}{2n + 1} < \cdots < \frac{(n - 1)\pi}{2n + 1} < \frac{n\pi}{2n + 1} < \frac{\pi}{2}.$$

Further notice that the function $\theta \mapsto (\sin\theta)^2$ defined for $\theta \in (0, \pi/2)$, is strictly increasing from 0 to 1 and consequently

$$0 < \Bigl(\sin\frac{\pi}{2n + 1}\Bigr)^2 < \Bigl(\sin\frac{2\pi}{2n + 1}\Bigr)^2 < \cdots < \Bigl(\sin\frac{(n - 1)\pi}{2n + 1}\Bigr)^2 < \Bigl(\sin\frac{n\pi}{2n + 1}\Bigr)^2 < 1. \tag{4.12}$$

Substituting

$$\theta = \frac{k\pi}{2n + 1} \in \Bigl(0, \frac{\pi}{2}\Bigr), \quad \text{with} \quad k \in \{1, \dots, n\},$$

in (4.7) we get

$$\Big(\sin\frac{k\pi}{2n+1}\Big)P_n\bigg(\Big(\sin\frac{k\pi}{2n+1}\Big)^2\bigg)=\sin\Big((2n+1)\frac{k\pi}{2n+1}\Big)=\sin(k\pi)=0.$$

Since $\sin\big(\frac{k\pi}{2n+1}\big)\neq 0$, we conclude that

$$P_n\bigg(\Big(\sin\frac{k\pi}{2n+1}\Big)^2\bigg)=0.$$

Thus,

$$r_k=\Big(\sin\frac{k\pi}{2n+1}\Big)^2 \quad \text{with} \quad k\in\{1,\dots,n\}$$

are the roots of $P_n(x)$. The roots r_k are n distinct numbers in $(0,1)$ which are listed in (4.12). Since $P_n(x)$ is of degree n it has at most n roots. Hence, we have all the roots of $P_n(x)$.

From Proposition 4.1 we know two coefficients of $P_n(x)$ to be

$$a_{0,n}=2n+1, \quad a_{1,n}=-\frac{2}{3}n(n+1)(2n+1).$$

Since r_k with $k\in\{1,\dots,n\}$ are the roots of $P_n(x)$, using Proposition 3.2 we deduce

$$\sum_{k=1}^{n}\frac{1}{\Big(\sin\big(\frac{k\pi}{2n+1}\big)\Big)^2}=\sum_{k=1}^{n}\frac{1}{r_k}=-\frac{a_1}{a_0}=\frac{2}{3}\frac{n(n+1)(2n+1)}{2n+1}=\frac{2}{3}n(n+1).$$

The identity in the theorem is proved. □

A Acknowledgments

I thank my student Zachery Wall and my colleague and friend Aad Dijksma for the constructive comments that contributed to the improvement of this note.