We want to prove that the functions {1, $(\text{Cos}[t])^2$, $(\text{Cos}[t])^4$, $(\text{Cos}[t])^6$} are linearly independent. We need to prove that when the linear combination

a1 * 1 + a2 * (Cos[t])^2 + a3 * (Cos[t])^4 + a4 * (Cos[t])^6

equals to 0 for all $t \in \mathbb{R}$, then all the coefficients a1, a2, a3, a4 must equal to 0.

Since the equality a1∗1 + a2∗$(\text{Cos}[t])^2$ + a3∗$(\text{Cos}[t])^4$ + a4∗$(\text{Cos}[t])^6$ = 0 for all $t \in \mathbb{R}$, we can substitute t with for carefully selected special values:

```
In[∘]:= (a1 * 1 + a2 * (Cos[t])^2 + a3 * (Cos[t])^4 + a4 * (Cos[t])^6) /. {t → Pi / 2}
Out[∘]= a1
```

```
In[∘]:= (a1 * 1 + a2 * (Cos[t])^2 + a3 * (Cos[t])^4 + a4 * (Cos[t])^6) /. {t → 0}
Out[∘]= a1 + a2 + a3 + a4
```

```
In[∘]:= (a1 * 1 + a2 * (Cos[t])^2 + a3 * (Cos[t])^4 + a4 * (Cos[t])^6) /. {t → Pi / 4}
```

Out[∘]= $a1 + \frac{a2}{2} + \frac{a3}{4} + \frac{a4}{8}$

```
In[∘]:= (a1 * 1 + a2 * (Cos[t])^2 + a3 * (Cos[t])^4 + a4 * (Cos[t])^6) /. {t → Pi / 6}
```

Out[∘]= $a1 + \frac{3\,a2}{4} + \frac{9\,a3}{16} + \frac{27\,a4}{64}$

Now we have four equations with four unknowns. We collect these four equations in a list of equations and ask Mathematica to solve them, as follows:

In[∘]:= **Solve**$\left[\left\{a1 == 0,\ a1 + a2 + a3 + a4 == 0,\ a1 + \frac{a2}{2} + \frac{a3}{4} + \frac{a4}{8} == 0,\ a1 + \frac{3\,a2}{4} + \frac{9\,a3}{16} + \frac{27\,a4}{64} == 0\right\},\ \{a1, a2, a3, a4\}\right]$

```
Out[∘]= {{a1 → 0, a2 → 0, a3 → 0, a4 → 0}}
```

Mathematica gives us the only solution.

Using linear algebra methods we would form the matrix of the homogeneous system and row reduce that matrix:

In[∘]:= **MatrixForm**$\left[\text{RowReduce}\left[\left\{\{1, 0, 0, 0\}, \{1, 1, 1, 1\}, \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right\}, \left\{1, \frac{3}{4}, \frac{9}{16}, \frac{27}{64}\right\}\right\}\right]\right]$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus the only solution of the above homogenous system is the trivial solution.

Next we explore some linear combinations of the given four functions:

{1, (Cos[t])^2, (Cos[t])^4, (Cos[t])^6}

The above is the list of the given functions.

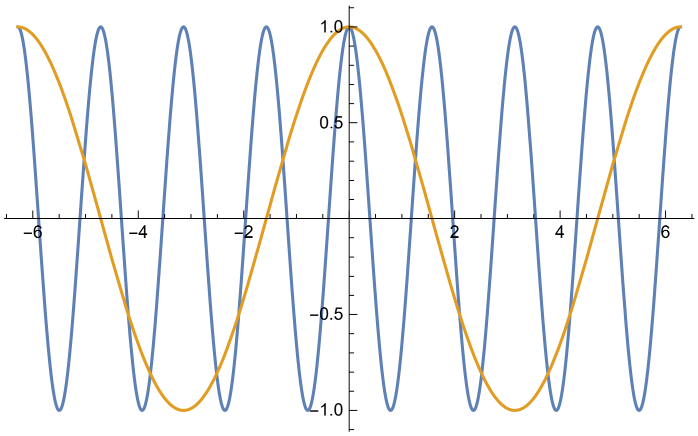
The next command makes a specific linear combination of the given functions:

```
In[ ]:= {1, -8, 8, 0}.{1, (Cos[t])^2, (Cos[t])^4, (Cos[t])^6}
Out[ ]= 1 - 8 Cos[t]^2 + 8 Cos[t]^4
```

Next plot this linear combination:

```
In[ ]:= Plot[{1 - 8 Cos[t]^2 + 8 Cos[t]^4, Cos[t]}, {t, -2 Pi, 2 Pi}]
```

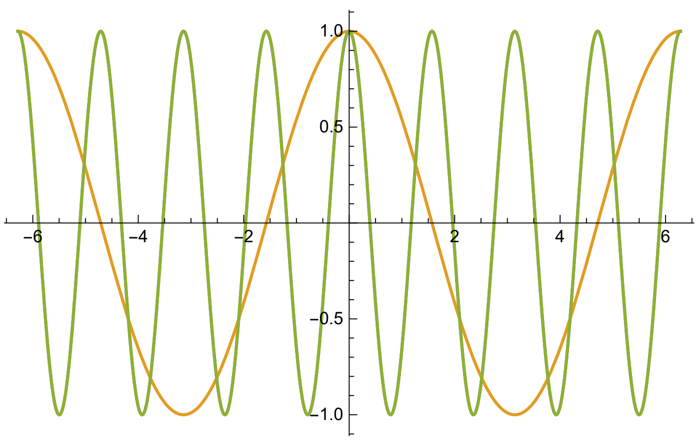
Out[ ]=



We ques that the linear combination equals Cos[4 t]. We verify it with a plot

```
In[ ]:= Plot[{1 - 8 Cos[t]^2 + 8 Cos[t]^4, Cos[t], Cos[4 t]}, {t, -2 Pi, 2 Pi}]
```

Out[ ]=



Does Mathematica know this identity?

```
In[ ]:= Simplify[1 - 8 Cos[t]^2 + 8 Cos[t]^4 - Cos[4 t]]
Out[ ]= 0
```

Yes!

Let us check the next linear combination:

```
In[ ]:= {-1, 18, -48, 32}.{1, (Cos[t])^2, (Cos[t])^4, (Cos[t])^6}
Out[ ]= -1 + 18 Cos[t]^2 - 48 Cos[t]^4 + 32 Cos[t]^6
```

```
In[•]:= FullSimplify[{-1, 18, -48, 32}.{1, (Cos[t])^2, (Cos[t])^4, (Cos[t])^6} - Cos[6 t]]
Out[•]= 0
```

Now we explore the four functions

```
In[•]:= {1, Cos[2 t], Cos[4 t], Cos[6 t]}
Out[•]= {1, Cos[2 t], Cos[4 t], Cos[6 t]}
```

The coordinate vectors of these functions are verified below:

```
In[•]:= FullSimplify[{1, 0, 0, 0}.{1, (Cos[t])^2, (Cos[t])^4, (Cos[t])^6} - 1]
Out[•]= 0

In[•]:= FullSimplify[{-1, 2, 0, 0}.{1, (Cos[t])^2, (Cos[t])^4, (Cos[t])^6} - Cos[2 t]]
Out[•]= 0

In[•]:= FullSimplify[{1, -8, 8, 0}.{1, (Cos[t])^2, (Cos[t])^4, (Cos[t])^6} - Cos[4 t]]
Out[•]= 0

In[•]:= FullSimplify[{-1, 18, -48, 32}.{1, (Cos[t])^2, (Cos[t])^4, (Cos[t])^6} - Cos[6 t]]
Out[•]= 0
```

Next we use the coordinate vectors to prove that the functions

```
{1, Cos[2 t], Cos[4 t], Cos[6 t]}
```

are linearly independent:

```
In[•]:= RowReduce[{{1, 0, 0, 0}, {-1, 2, 0, 0}, {1, -8, 8, 0}, {-1, 18, -48, 32}}]
Out[•]= {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

Thus, the coordinate vectors are linearly independent. Therefore, (citation needed) the functions

```
{1, Cos[2 t], Cos[4 t], Cos[6 t]}
```

are linearly independent.