Complex Numbers

February 7, 2024

1 Complex Numbers

Quadratic equations arise often in mathematical calculations and applications of mathematics. However, the basic properties of real numbers yield that the square of an arbitrary real number is nonnegative. Therefore, the simple quadratic equation $x^2 = -1$ does not have a real number as a solution. What to do? The answer is: Let us introduce a new kind of number, called the **imaginary unit** whose square is -1. Commonly, the imaginary unit is denoted by i. Thus, **imaginary unit** i is the complex number for which

$$\boxed{i^2 = -1}.$$

A **complex number** is a number of the form $a+bi$ where a and b are real numbers and i is the imaginary unit. For example $1+2i$ and $2-3i$ are complex numbers. The notations $a+ib$ and $a+bi$ are used interchangeably.

The set of all complex numbers is denoted by $\mathbb{C}$. All real numbers are complex numbers since we can write $a = a+0i$. Thus, $\mathbb{R} \subset \mathbb{C}$. The numbers of the form $0+bi$, with b being a real number, are called **imaginary numbers**. The set of all imaginary numbers is denoted by $\mathbb{R}i$. The only number which is both real and imaginary is the complex number zero $0 = 0+0i$. The most notable nonzero real number is $1 = 1+0i$ and the most notable nonzero imaginary number is the imaginary unit $i = 0+1i$.

Complex numbers are often denoted by one symbol only: $z = a+bi$. The real number a in $z = a+bi$ is called the **real part** of z; it is denoted by $\mathrm{Re}(z)$. The real number b in $z = a+bi$ is called the **imaginary part** of z; it is denoted by $\mathrm{Im}(z)$. For example

$$\mathrm{Re}(2-3i) = 2, \qquad \mathrm{Im}(2-3i) = -3.$$

In calculations with complex numbers the objective is to express the result as a sum of the real part and the imaginary part, where the later is multiplied by i. This approach mirrors the calculations with rational numbers, where the objective is to consolidate the result into a single fraction.

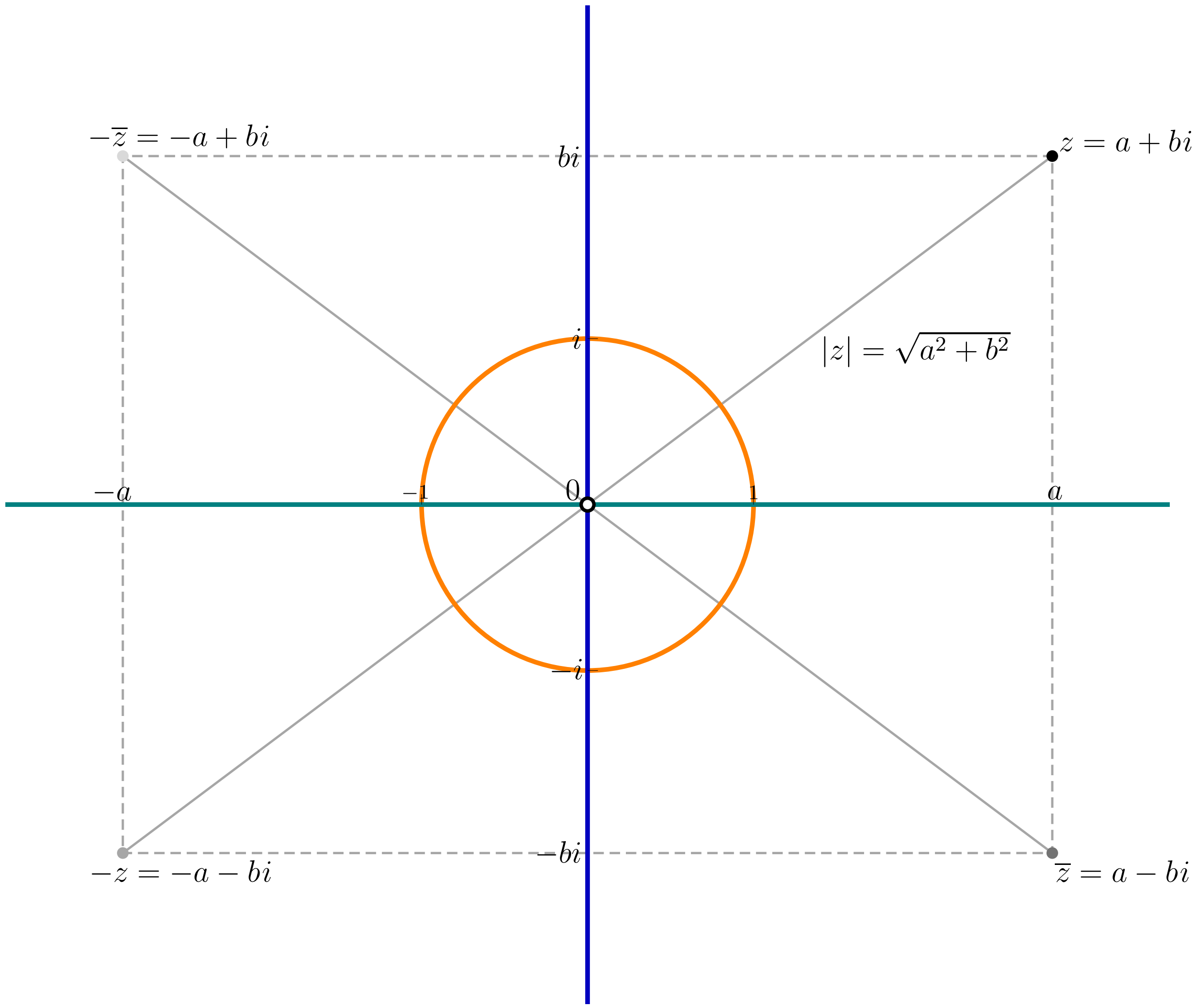

Fig. 1: The Complex Plane

2 The Complex Plane

Complex numbers are visualized in the **Complex Plane**.

In Figure1 of the complex plane, I emphasize the five important complex numbers

$$0, \quad 1, \quad i, \quad -1, \quad -i.$$

In Figure1 the complex plane, I also emphasize two axes in the complex plane: the real axis and the imaginary axis. The real axis is in teal, the imaginary axes is in blue.

Figure1 of the complex plane, I also emphasize two important symmetries in the complex plane. The first symmetry is the symmetry with respect to the origin, that is to the complex number 0. The complex numbers

$$z = a + bi \quad \text{and} \quad -z = -a - bi$$

are symmetric with respect to the complex number zero. The complex number $-z = -a - bi$ is called the **opposite** of z.

The second important symmetry is in the reflection across the real axis. The complex numbers

$$z = a + bi \quad \text{and} \quad \overline{z} = a - bi$$

are reflections of each other across the real axis.

To complete a rectangle, I added the complex number $-\overline{z} = -a - bi$ which is the opposite of $\overline{z}$ and the conjugate of $-z$.

The importance of the conjugate $\overline{z} = a - bi$ of $z = a + bi$ is in the following calculation:

$$z\,\overline{z} = (a + bi)(a - bi) = a^2 + b^2.$$

The quantity $a^2 + b^2$ is familiar from the Pythagorean Theorem. The number

$$\sqrt{a^2 + b^2} = \sqrt{z\,\overline{z}}$$

is the length of the hypothenuse of the right triangle whose sides have lengths a and b. In Figure1 that is the distance between the complex number 0 and the complex number z. This number provides a generalization of the absolute value function to complex numbers. It is called the **modulus** of a complex number and it is denoted by $|z|$:

$$|z| = \sqrt{z\,\overline{z}} = \sqrt{a^2 + b^2}.$$

The only complex numbers which coincide with their conjugates are real numbers. That is, a complex number z is real if and only if $z = \overline{z}$.

The only complex numbers whose opposites coincide with their conjugates are imaginary numbers. That is, a complex number z is imaginary if and only if $-z = \overline{z}$.

3 Arithmetic with Complex Numbers

Complex numbers can be added, subtracted, multiplied and divided. Whenever we do operations with complex numbers it is important to clearly identify the resulting complex number with its real and imaginary part.

To add (subtract) two complex numbers, simply add (subtract) the corresponding real and imaginary parts. If $z = a + bi$ and $w = c + di$ are complex numbers, then

$$z + w = (a + c) + (b + d)i, \qquad z - w = (a - c) + i(b - d).$$

Notice that here we show how to calculate the real and the imaginary part of the sum and the difference of two complex numbers in terms of the real and the imaginary part of the given complex numbers.

To multiply two complex numbers, use distributive law and the fact that $i^2 = -1$. With z and w as above,

$$\begin{aligned} zw &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

For example,

$$(2 - 5i)(3 + 4i) = 6 + 8i - 15i - 20i^2 = 6 - 7i - 20(-1) = 26 - 7i$$

Notice that here we show how to calculate the real and the imaginary part of the product of two complex numbers in terms of the real and the imaginary part of the given complex numbers.

Next we show how to calculate the real and the imaginary part of the result of the division of two complex numbers in terms of the real and the imaginary part of the given complex numbers. The goal is to replace question marks with real numbers:

$$\frac{2 - 5i}{3 + 4i} = ? + ?\, i$$

To accomplish this task it is important to note the following multiplication:

$$(a + bi)(a - bi) = (a - b(-b)) + (a(-b) + ba)i = a^2 + b^2.$$

For example,

$$(3 + 4i)(3 - 4i) = 9 - 12i + 12i - 16i^2 = 9 + 16 = 25.$$

We see that the result of this multiplication is a real number. This is important, since we can use this to find the real and imaginary part in the fraction of two complex numbers above:

$$\frac{2 - 5i}{3 + 4i} = \frac{(2 - 5i)(3 - 4i)}{(3 + 4i)(3 - 4i)} = \frac{6 - 8i - 15i + 20i^2}{25} = \frac{-14 - 23i}{25} = -\frac{14}{25} - \frac{23}{25}\, i.$$

The key here is to multiply $3 + 4i$ by $3 - 4i$ to get a real number in denominator. This can be done for any complex number. But first introduce the following terminology.

For a complex number $z = a + bi$, then the number $a - bi$ is called the **complex conjugate** of z; it is denoted by z^* or $\overline{z}$. The notation z^* is used in electrical engineering. In mathematics $\overline{z}$ is more common. Thus

$$\overline{z} = z^* = \mathrm{Re}(z) - i\,\mathrm{Im}(z).$$

Complex conjugates are often very useful. For example, if $z = a + bi$ and $w = c + di$ are complex numbers, then

$$\begin{aligned}
\frac{w}{z} &= \frac{w\,\overline{z}}{z\,\overline{z}} \\
&= \frac{(c+di)(a-bi)}{(a+bi)(a-bi)} \\
&= \frac{ca - cbi + dai + db}{a^2+b^2} \\
&= \frac{ca+db}{a^2+b^2} + \frac{da-cb}{a^2+b^2}\, i.
\end{aligned}$$

The complex conjugation has the following properties:

$$\overline{(z+w)} = \overline{z} + \overline{w}, \quad \overline{(z-w)} = \overline{z} - \overline{w}, \quad \overline{(z\,w)} = \overline{z}\,\overline{w} \quad \overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}, \quad \overline{(\overline{z})} = z.$$

We also have

$$z + \overline{z} = 2\,\mathrm{Re}(z) \qquad \text{and} \qquad z - \overline{z} = 2i\,\mathrm{Im}(z)$$

Consequently

$$\mathrm{Re}(z) = \frac{z+\overline{z}}{2} \qquad \text{and} \qquad \mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$$

4 Euler's Identity and the Complex Exponential Function

From calculus you are familiar with the exponential function e^t, here t is any real number. This function is remarkable since it equals its own derivative and $e^0 = 1$:

$$\frac{d}{dt}\big(e^t\big) = e^t \qquad \text{and} \qquad e^0 = 1.$$

Other remarkable properties of the exponential function are

$$e^{a+b} = e^a\, e^b, \quad e^{-a} = \frac{1}{e^a}, \quad \frac{d}{dt}\big(e^{at}\big) = ae^{at}, \tag{4.1}$$

where a, b and t are real numbers.

In the previous section we learned the arithmetic with complex numbers. The next step is to learn the complex exponential function:

How to calculate e^z where $z = x + yi$ with $x, y \in \mathbb{R}$?

To answer the displayed question we need to find real functions $f(x,y)$ and $g(x,y)$ of the real variables $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ such that

$$\boxed{e^z = f(x,y) + g(x,y)i.}$$

In other words, we have to find

$$\mathrm{Re}\bigl(e^z\bigr) \quad \text{and} \quad \mathrm{Im}\bigl(e^z\bigr).$$

We expect that the complex exponential function will have the properties listed in (4.1). With complex numbers z and w and real t we have

$$e^0 = 1, \quad e^{z+w} = e^z\, e^w, \quad e^{-z} = \frac{1}{e^z}, \quad \frac{d}{dt}\bigl(e^{zt}\bigr) = ze^{zt}. \tag{4.2}$$

The first step is to get the formula for the real and imaginary part of e^{it} where t is a real number. That is the famous Euler's identity

$$\boxed{e^{it} = \cos t + i \sin t \quad \text{for all} \quad t \in \mathbb{R}}$$

You might be asking: Why is this formula valid? A rigorous *derivation* of Euler's identity involves infinite series representation of the exponential function.

However, we can *justify* Euler's identity just by using the properties in (4.2). Here is a *justification*: Consider the function

$$r(t) = \bigl(\cos t + i \sin t\bigr)\, e^{-it}, \quad \text{where} \quad t \quad \text{is a real number.}$$

Clearly, by (4.2), $r(0) = 1$. Next we calculate the derivative $f'(t)$. We first use the product rule and (4.2), and after that algebra with complex numbers: and the usual properties of differentiation:

$$\begin{aligned} r'(t) &= \bigl(-\sin t + i \cos t\bigr)\, e^{-it} + \bigl(\cos t + i \sin t\bigr)\,(-i)e^{-it} \\ &= \bigl(-\sin t + i \cos t\bigr)\, e^{-it} - \bigl(i \cos t + i^2 \sin t\bigr)\, e^{-it} \\ &= \bigl(-\sin t + i \cos t\bigr)\, e^{-it} - \bigl(-\sin t + i \cos t\bigr)\, e^{-it} \\ &= 0. \end{aligned}$$

Since $r'(t) = 0$ for all real numbers t, the function r is constant. Since $r(0) = 1$, we conclude that $r(t) = 1$ for all real t. That is

$$1 = \bigl(\cos t + i \sin t\bigr)\, e^{-it} = \bigl(\cos t + i \sin t\bigr)\, \frac{1}{e^{it}} \quad \text{for all real} \quad t.$$

Therefore,

$$e^{it} = \cos t + i \sin t \quad \text{for all real} \quad t.$$

In other words, Euler's formula is a natural consequence of the rules (4.2) and the rules for the derivative.

With Euler's identity and the rules (4.2) we can calculate the real and the imaginary part of e^z for $z = x + iy$:

$$e^z = e^{x+iy} = e^x \, e^{iy} = e^x \big(\cos y + i \sin y\big) = e^x \cos y + i e^x \sin y.$$

Euler's identity is an endless source of interesting formulas. For example

$$\boxed{e^{i\frac{\pi}{2}} = i}, \qquad \boxed{e^{i\pi} = -1}, \qquad \boxed{e^{i\frac{3\pi}{2}} = -i}, \qquad \boxed{e^{2i\pi} = 1}.$$

In fact, for any integer $k \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ we have

$$\boxed{e^{2k\pi i} = 1}.$$

5 The Modulus and the Argument of a Complex Number

The complex numbers are visualized as points in the complex plane. In this plane the real numbers are on the horizontal axis and purely imaginary numbers are on the vertical axes. A complex number $z = x + iy$ is represented by the point with the coordinates (x, y).

As you might have seen in calculus that the points in the xy-plane can be represented in polar coordinates as well. The polar coordinates of the point (x, y) are (r, θ) where $r \geq 0$ and $\theta \in (-\pi, \pi]$ and

$$x = r \cos \theta \qquad \text{and} \qquad y = r \sin \theta.$$

For given x and y the polar coordinates are calculated by

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \begin{cases} \arccos\left(\dfrac{x}{r}\right) & \text{for} \quad y \geq 0 \\ -\arccos\left(\dfrac{x}{r}\right) & \text{for} \quad y < 0 \end{cases}$$

In the terminology of complex numbers r is called the absolute value or the modulus of $z = x + iy$, it is denoted by $|z|$, and θ is called the argument of z, it is denoted by $\arg(z)$. That is,

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = \sqrt{z\,\overline{z}} \qquad \text{and} \qquad \arg(z) = \begin{cases} \arccos\left(\dfrac{\operatorname{Re} z}{|z|}\right) & \text{for} \quad \operatorname{Im} z \geq 0 \\ -\arccos\left(\dfrac{\operatorname{Re} z}{|z|}\right) & \text{for} \quad \operatorname{Im} z < 0 \end{cases}$$

With $|z|$ and $\theta = \arg(z)$ the complex number can be written in the following form

$$\boxed{z = |z|\,\big(\cos\theta + i\sin\theta\big) = |z|\,e^{i\theta}}.$$

This formula together with the rules for the exponential function provides a geometric explantation how complex numbers are multiplied. If z_1 and z_2 are complex numbers and $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$, then

$$z_1 z_2 = |z_1|\,e^{i\theta_1}\,|z_2|\,e^{i\theta_2} = |z_1|\,|z_2|\,e^{i\theta_1}\,e^{i\theta_2} = |z_1|\,|z_2|\,e^{i\theta_1+i\theta_2} = |z_1|\,|z_2|\,e^{i(\theta_1+\theta_2)}.$$

The rule is that the modulus of the product $z_1 z_2$ is product of the moduli $|z_1|$ and $|z_2|$ and the argument of the product $z_1 z_2$ is the **sum** of the arguments of z_1 and z_2.

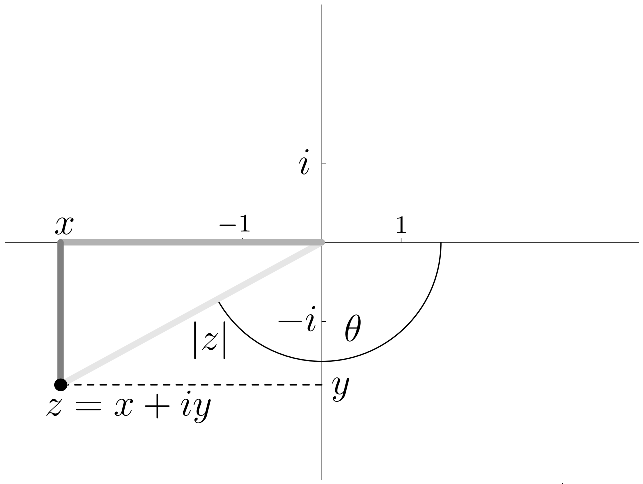

Fig. 2: $x < 0,\ y < 0,\ -\pi < \theta < -\pi/2$

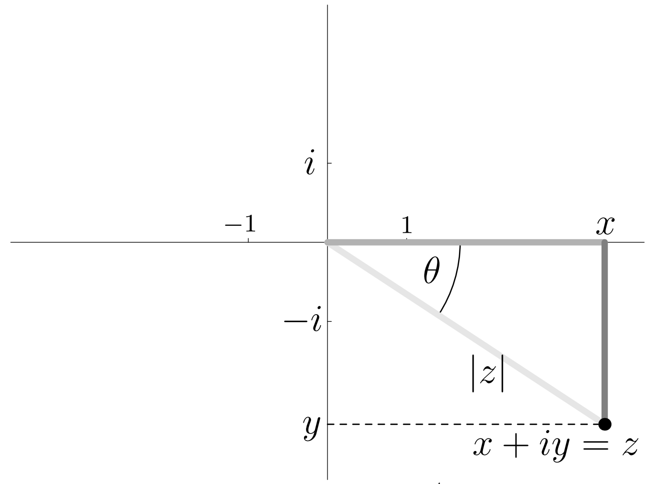

Fig. 3: $x > 0,\ y < 0,\ -\pi/2 < \theta < 0$

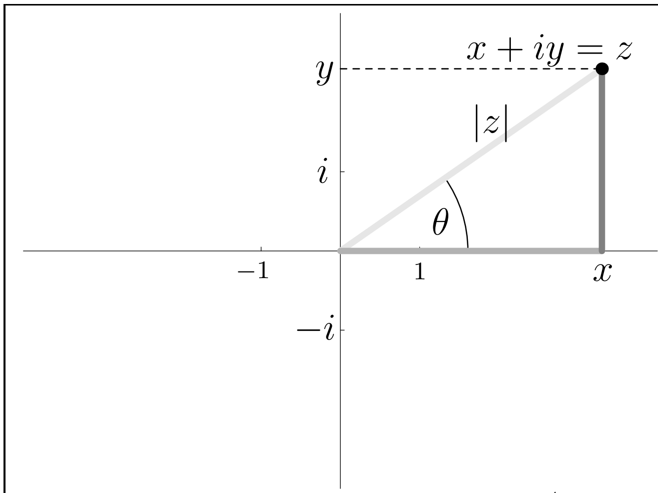

Fig. 4: $x > 0,\ y > 0,\ 0 < \theta < \pi/2$

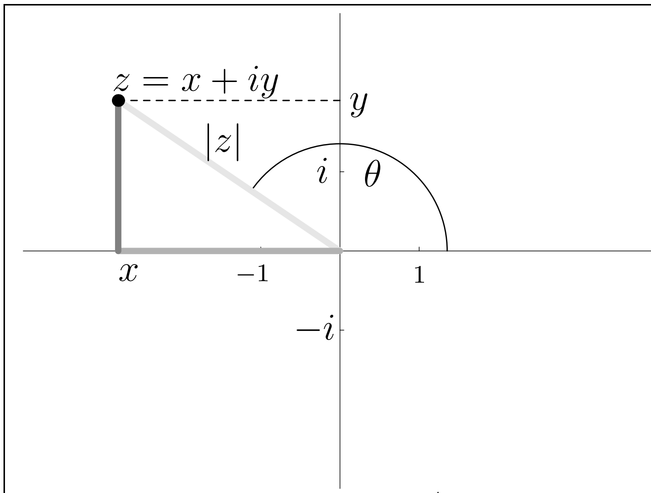

Fig. 5: $x < 0,\ y > 0,\ \pi/2 < \theta < \pi$

In each of the examples pictured above we have

$$\mathrm{Re}(z) = x = |z|\cos\theta, \qquad \mathrm{Im}(z) = y = |z|\sin\theta.$$

The angle θ can be calculated as

$$\begin{aligned} \theta &= \arccos\left(\frac{\mathrm{Re}(z)}{|z|}\right), \quad \text{if} \quad \mathrm{Im}(z) \geq 0, \quad \text{(see Figures 4 and 5)} \\ \theta &= -\arccos\left(\frac{\mathrm{Re}(z)}{|z|}\right), \quad \text{if} \quad \mathrm{Im}(z) < 0, \quad \text{(see Figures 2 and 3)} \end{aligned}$$

6 Exercises

1. Express the following complex numbers in the form $x + iy$.

$$(5-6i)+(3+2i), \quad \left(4-\frac{1}{2}i\right)-\left(9+\frac{5}{2}i\right), \quad (2+5i)(4-i), \quad (1-2i)(8-3i)$$

2. Find two distinct complex numbers z and w such that $z^2 = w^2 = -1$.

3. Find four distinct complex numbers z, u, v and w such that $z^4 = u^4 = v^4 = w^4 = 1$.

4. Evaluate the real and the imaginary part of the product $\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\,i\right)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right)$.

5. Evaluate the complex conjugates of the numbers

$$6+7i, \quad -2-3i, \quad 2i\left(\frac{1}{2}-i\right), \quad e^{i\pi/3}.$$

   Express your answers in the form $x + iy$.

6. Express the following complex numbers in the form $x + iy$:

$$\frac{1+4i}{3+2i}, \quad \frac{3+2i}{1-4i}, \quad \frac{1}{1+i}, \quad \frac{3}{4-3i}, \quad \frac{1}{i}.$$

7. Express the reciprocals of the following numbers in the form $x + iy$:

$$1-i, \quad 3-2i, \quad i, \quad 2\sqrt{3}-2i, \quad -\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\,i.$$

8. Write the following three complex numbers in the form $x + iy$:

$$e^{-\frac{2\pi}{3}i}, \quad 4\,e^{\frac{\pi}{6}i}, \quad e^{\frac{2\pi}{3}i}+e^{\frac{4\pi}{3}i}+e^{2\pi i}.$$

9. Write the following four numbers in the form $x + iy$:

$$3\,e^{\frac{3\pi}{4}i}, \quad 6e^{-\frac{22\pi}{3}i}, \quad 11\,e^{\frac{14\pi}{2}i}, \quad 3\,e^{\frac{3\pi}{2}i},$$

10. Write the following numbers in polar form ($|z|\,e^{i\arg(z)}$):

$$4-4i, \quad -2i, \quad 7\sqrt{3}-7i, \quad -2\sqrt{3}+2i, \quad 2-2\sqrt{3}i$$

11. Write the complex number $\sqrt{3}-i$ in polar form. Use the polar form to calculate $\left(\sqrt{3}-i\right)^8$. Express the result in the form $x + iy$.