In[19]:= `NotebookFileName[]`

Out[19]= `C:\Dropbox\Work\myweb\Courses\Math_pages\Math_430\20221104_EqTempDisk.nb`

I want this to be the boundary temperature distribution along the unit circle (which is the boundary of the unit disk):

In[20]:= `Clear[ff]; ff[θ_] = Pi^2 - θ^2`

Out[20]= $\pi^2 - \theta^2$

The average temperature along the unit circle is:

In[21]:= $\frac{1}{2\,\text{Pi}}$ `Integrate[ff[θ], {θ, -Pi, Pi}]`

Out[21]= $\frac{2\pi^2}{3}$

For our calculations we will need the inner squares of the functions 1 (the constant function) and the positive integer multiples of cosine

In[22]:= `Integrate[1 * 1, {θ, -Pi, Pi}]`

Out[22]= 2π

In[23]:= `Table[Integrate[ Cos[n θ] Cos[n θ], {θ, -Pi, Pi}], {n, 1, 20}]`

Out[23]= $\{\pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi\}$

These are the Fourier coefficients of the upside down parabola that we have chosen for the temperature along the boundary (unit circle):

In[24]:= `Table[` $\frac{1}{\text{Pi}}$ `Integrate[ff[θ] Cos[n θ], {θ, -Pi, Pi}], {n, 1, 20}]`

Out[24]= $\left\{4, -1, \frac{4}{9}, -\frac{1}{4}, \frac{4}{25}, -\frac{1}{9}, \frac{4}{49}, -\frac{1}{16}, \frac{4}{81}, -\frac{1}{25}, \frac{4}{121}, -\frac{1}{36}, \frac{4}{169}, -\frac{1}{49}, \frac{4}{225}, -\frac{1}{64}, \frac{4}{289}, -\frac{1}{81}, \frac{4}{361}, -\frac{1}{100}\right\}$

It is an amazing fact that I can write a partial sum of the Fourier series for the boundary function as the dot product of the list of the coefficients and the list of cosines.

In[25]:= `Table[Cos[n θ], {n, 1, 20}]`

Out[25]= `{Cos[θ], Cos[2 θ], Cos[3 θ], Cos[4 θ], Cos[5 θ], Cos[6 θ], Cos[7 θ], Cos[8 θ], Cos[9 θ], Cos[10 θ], Cos[11 θ], Cos[12 θ], Cos[13 θ], Cos[14 θ], Cos[15 θ], Cos[16 θ], Cos[17 θ], Cos[18 θ], Cos[19 θ], Cos[20 θ]}`

This is the inner product

In[26]:=

```
{4, -1, 4/9, -1/4, 4/25, -1/9, 4/49, -1/16, 4/81, -1/25, 4/121, -1/36, 4/169, -1/49, 4/225,
  -1/64, 4/289, -1/81, 4/361, -1/100}.{Cos[θ], Cos[2 θ], Cos[3 θ], Cos[4 θ], Cos[5 θ],
  Cos[6 θ], Cos[7 θ], Cos[8 θ], Cos[9 θ], Cos[10 θ], Cos[11 θ], Cos[12 θ], Cos[13 θ],
  Cos[14 θ], Cos[15 θ], Cos[16 θ], Cos[17 θ], Cos[18 θ], Cos[19 θ], Cos[20 θ]}
```

Out[26]=

$$4\,\mathrm{Cos}[\theta] - \mathrm{Cos}[2\,\theta] + \frac{4}{9}\mathrm{Cos}[3\,\theta] - \frac{1}{4}\mathrm{Cos}[4\,\theta] + \frac{4}{25}\mathrm{Cos}[5\,\theta] - \frac{1}{9}\mathrm{Cos}[6\,\theta] + \frac{4}{49}\mathrm{Cos}[7\,\theta] - \frac{1}{16}\mathrm{Cos}[8\,\theta] + \frac{4}{81}\mathrm{Cos}[9\,\theta] - \frac{1}{25}\mathrm{Cos}[10\,\theta] + \frac{4}{121}\mathrm{Cos}[11\,\theta] - \frac{1}{36}\mathrm{Cos}[12\,\theta] + \frac{4}{169}\mathrm{Cos}[13\,\theta] - \frac{1}{49}\mathrm{Cos}[14\,\theta] + \frac{4}{225}\mathrm{Cos}[15\,\theta] - \frac{1}{64}\mathrm{Cos}[16\,\theta] + \frac{4}{289}\mathrm{Cos}[17\,\theta] - \frac{1}{81}\mathrm{Cos}[18\,\theta] + \frac{4}{361}\mathrm{Cos}[19\,\theta] - \frac{1}{100}\mathrm{Cos}[20\,\theta]$$

Now define a Fourier approximation for out boundary function

In[27]:=

```
faff[θ_] = 2 π^2/3 + 4 Cos[θ] - Cos[2 θ] + 4/9 Cos[3 θ] - 1/4 Cos[4 θ] +
  4/25 Cos[5 θ] - 1/9 Cos[6 θ] + 4/49 Cos[7 θ] - 1/16 Cos[8 θ] + 4/81 Cos[9 θ] - 1/25 Cos[10 θ] +
  4/121 Cos[11 θ] - 1/36 Cos[12 θ] + 4/169 Cos[13 θ] - 1/49 Cos[14 θ] + 4/225 Cos[15 θ] -
  1/64 Cos[16 θ] + 4/289 Cos[17 θ] - 1/81 Cos[18 θ] + 4/361 Cos[19 θ] - 1/100 Cos[20 θ]
```

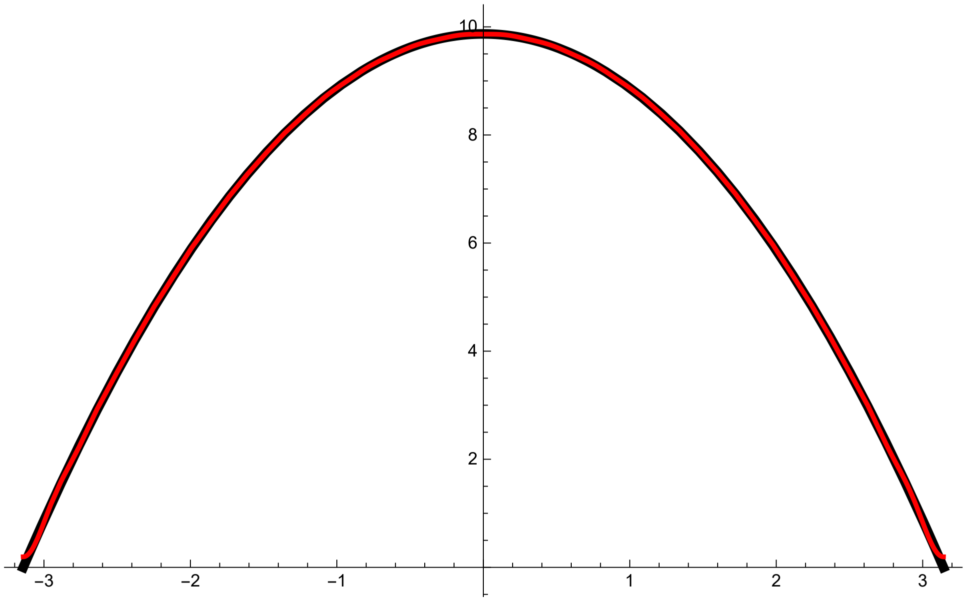
Out[27]=

$$\frac{2\,\pi^2}{3} + 4\,\mathrm{Cos}[\theta] - \mathrm{Cos}[2\,\theta] + \frac{4}{9}\mathrm{Cos}[3\,\theta] - \frac{1}{4}\mathrm{Cos}[4\,\theta] + \frac{4}{25}\mathrm{Cos}[5\,\theta] - \frac{1}{9}\mathrm{Cos}[6\,\theta] + \frac{4}{49}\mathrm{Cos}[7\,\theta] - \frac{1}{16}\mathrm{Cos}[8\,\theta] + \frac{4}{81}\mathrm{Cos}[9\,\theta] - \frac{1}{25}\mathrm{Cos}[10\,\theta] + \frac{4}{121}\mathrm{Cos}[11\,\theta] - \frac{1}{36}\mathrm{Cos}[12\,\theta] + \frac{4}{169}\mathrm{Cos}[13\,\theta] - \frac{1}{49}\mathrm{Cos}[14\,\theta] + \frac{4}{225}\mathrm{Cos}[15\,\theta] - \frac{1}{64}\mathrm{Cos}[16\,\theta] + \frac{4}{289}\mathrm{Cos}[17\,\theta] - \frac{1}{81}\mathrm{Cos}[18\,\theta] + \frac{4}{361}\mathrm{Cos}[19\,\theta] - \frac{1}{100}\mathrm{Cos}[20\,\theta]$$

Let us see how good our approximation is:

In[28]:=
```
Plot[{ff[θ], faff[θ]}, {θ, -Pi, Pi},
 PlotStyle → {{RGBColor[0, 0, 0], Thickness[0.01]}, {RGBColor[1, 0, 0], Thickness[0.005]}},
 ImageSize → 500]
```

Out[28]=

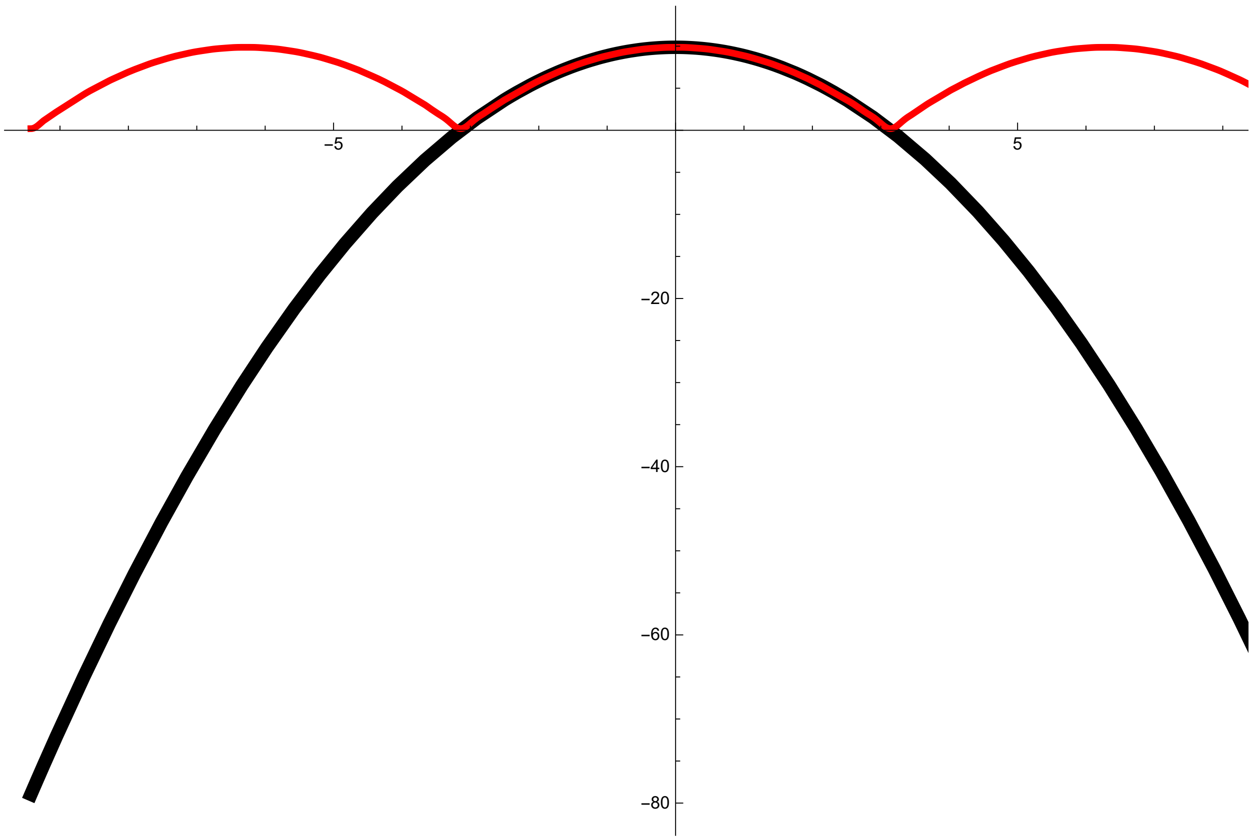


Quite good.

Notice that the Fourier approximation extends the function periodically to the entire real line

In[29]:=
```
Plot[{ff[θ], faff[θ]}, {θ, -3 Pi, 3 Pi},
 PlotStyle → {{RGBColor[0, 0, 0], Thickness[0.01]}, {RGBColor[1, 0, 0], Thickness[0.005]}},
 ImageSize → 700]
```

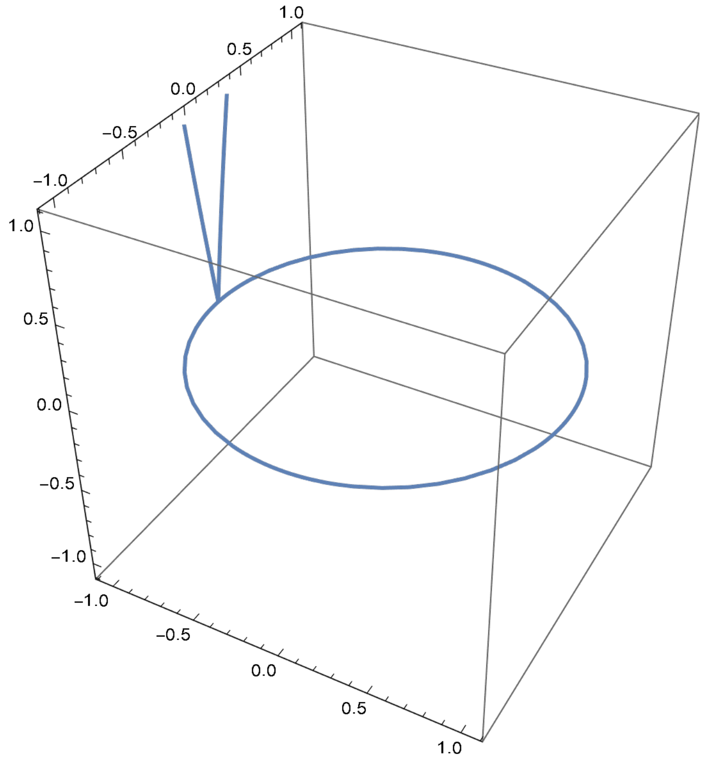
Out[29]=



Now show the unit circle and the boundary temperature in 3-space:

In[30]:=
```
Show[ParametricPlot3D[{Cos[θ], Sin[θ], 0}, {θ, -Pi, Pi}],
 ParametricPlot3D[{Cos[θ], Sin[θ], ff[θ]}, {θ, -Pi, Pi}], BoxRatios → {1, 1, 1}]
```

Out[30]=



Now we want to write an approximation for the solution. Again using the dot product

In[31]:= $\left\{4, -1, \frac{4}{9}, -\frac{1}{4}, \frac{4}{25}, -\frac{1}{9}, \frac{4}{49}, -\frac{1}{16}, \frac{4}{81}, -\frac{1}{25}, \frac{4}{121}, -\frac{1}{36}, \frac{4}{169}, -\frac{1}{49}, \frac{4}{225}, -\frac{1}{64}, \frac{4}{289}, -\frac{1}{81}, \frac{4}{361}, -\frac{1}{100}\right\}$.Table[r^n Cos[n θ], {n, 1, 20}]

Out[31]= $4\, r\, \text{Cos}[\theta] - r^2\, \text{Cos}[2\,\theta] + \frac{4}{9} r^3\, \text{Cos}[3\,\theta] - \frac{1}{4} r^4\, \text{Cos}[4\,\theta] + \frac{4}{25} r^5\, \text{Cos}[5\,\theta] - \frac{1}{9} r^6\, \text{Cos}[6\,\theta] + \frac{4}{49} r^7\, \text{Cos}[7\,\theta] - \frac{1}{16} r^8\, \text{Cos}[8\,\theta] + \frac{4}{81} r^9\, \text{Cos}[9\,\theta] - \frac{1}{25} r^{10}\, \text{Cos}[10\,\theta] + \frac{4}{121} r^{11}\, \text{Cos}[11\,\theta] - \frac{1}{36} r^{12}\, \text{Cos}[12\,\theta] + \frac{4}{169} r^{13}\, \text{Cos}[13\,\theta] - \frac{1}{49} r^{14}\, \text{Cos}[14\,\theta] + \frac{4}{225} r^{15}\, \text{Cos}[15\,\theta] - \frac{1}{64} r^{16}\, \text{Cos}[16\,\theta] + \frac{4}{289} r^{17}\, \text{Cos}[17\,\theta] - \frac{1}{81} r^{18}\, \text{Cos}[18\,\theta] + \frac{4}{361} r^{19}\, \text{Cos}[19\,\theta] - \frac{1}{100} r^{20}\, \text{Cos}[20\,\theta]$

We are ready to write an approximation for the solution

In[32]:=
```
Clear[sol];
sol[r_, θ_] = 2 π^2/3 + 4 r Cos[θ] - r^2 Cos[2 θ] + 4/9 r^3 Cos[3 θ] - 1/4 r^4 Cos[4 θ] + 4/25 r^5 Cos[5 θ] -
  1/9 r^6 Cos[6 θ] + 4/49 r^7 Cos[7 θ] - 1/16 r^8 Cos[8 θ] + 4/81 r^9 Cos[9 θ] - 1/25 r^10 Cos[10 θ] +
  4/121 r^11 Cos[11 θ] - 1/36 r^12 Cos[12 θ] + 4/169 r^13 Cos[13 θ] - 1/49 r^14 Cos[14 θ] + 4/225 r^15 Cos[15 θ] -
  1/64 r^16 Cos[16 θ] + 4/289 r^17 Cos[17 θ] - 1/81 r^18 Cos[18 θ] + 4/361 r^19 Cos[19 θ] - 1/100 r^20 Cos[20 θ]
```

Out[32]=

$$\frac{2\pi^2}{3} + 4 r \operatorname{Cos}[\theta] - r^2 \operatorname{Cos}[2\theta] + \frac{4}{9} r^3 \operatorname{Cos}[3\theta] - \frac{1}{4} r^4 \operatorname{Cos}[4\theta] + \frac{4}{25} r^5 \operatorname{Cos}[5\theta] - \frac{1}{9} r^6 \operatorname{Cos}[6\theta] + \frac{4}{49} r^7 \operatorname{Cos}[7\theta] - \frac{1}{16} r^8 \operatorname{Cos}[8\theta] + \frac{4}{81} r^9 \operatorname{Cos}[9\theta] - \frac{1}{25} r^{10} \operatorname{Cos}[10\theta] + \frac{4}{121} r^{11} \operatorname{Cos}[11\theta] - \frac{1}{36} r^{12} \operatorname{Cos}[12\theta] + \frac{4}{169} r^{13} \operatorname{Cos}[13\theta] - \frac{1}{49} r^{14} \operatorname{Cos}[14\theta] + \frac{4}{225} r^{15} \operatorname{Cos}[15\theta] - \frac{1}{64} r^{16} \operatorname{Cos}[16\theta] + \frac{4}{289} r^{17} \operatorname{Cos}[17\theta] - \frac{1}{81} r^{18} \operatorname{Cos}[18\theta] + \frac{4}{361} r^{19} \operatorname{Cos}[19\theta] - \frac{1}{100} r^{20} \operatorname{Cos}[20\theta]$$

See it in 3-space

In[33]:=
```
Show[ParametricPlot3D[{Cos[θ], Sin[θ], 0}, {θ, -Pi, Pi}],
 ParametricPlot3D[{Cos[θ], Sin[θ], ff[θ]}, {θ, -Pi, Pi}],
 ParametricPlot3D[{r Cos[θ], r Sin[θ], sol[r, θ]}, {r, 0, 1}, {θ, -Pi, Pi},
  PlotPoints → {30, 150}], BoxRatios → {1, 1, 1}, PlotRange → All, ImageSize → 500]
```

Out[33]=