Homogeneous Second Order Linear Differential Equations

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The symbol $\mathbb{R}$ denotes the set of all real numbers. Let $a, b \in \mathbb{R}$ be such that $a < b$. Then we use the following terminology and notation:

set	name	notation
$\{x \in \mathbb{R} : a < x < b\}$	*open finite interval*	(a, b)
$\{x \in \mathbb{R} : a \leq x \leq b\}$	*closed finite interval*	$[a, b]$
$\{x \in \mathbb{R} : a \leq x < b\}$	*half-open finite interval*	$[a, b)$
$\{x \in \mathbb{R} : a < x \leq b\}$	*half-open finite interval*	$(a, b]$
$\{x \in \mathbb{R} : a < x\}$	*open infinite interval*	$(a, +\infty)$
$\{x \in \mathbb{R} : a \leq x\}$	*closed infinite interval*	$[a, +\infty)$
$\{x \in \mathbb{R} : x < a\}$	*open infinite interval*	$(-\infty, a)$
$\{x \in \mathbb{R} : x \leq a\}$	*closed infinite interval*	$(-\infty, a]$
$\mathbb{R}$	*infinite interval*	$\mathbb{R}$

The term *interval* relates to any of the above sets.

Let I be an interval. The general (non-homogeneous) second order linear differential equation is

$$y''(x) + P(x)y'(x) + Q(x)y(x) = R(x), \quad x \in I, \tag{1}$$

Note that this equation is "normalized", that is the coefficient with the second derivative is 1. This is very important since the theorems below do not apply directly to equations that are not normalized.

Theorem 1. *Let P, Q and R be continuous real valued functions defined on an interval I. Let x_0 be any number in I and let y_0 and v_0 be any real numbers. Then the initial value problem*

$$\begin{gathered} y''(x) + P(x)y'(x) + Q(x)y(x) = R(x), \quad x \in I, \\ y(x_0) = y_0, \quad y'(x_0) = v_0, \end{gathered} \tag{2}$$

has a unique solution defined on the interval I.

This theorem states that there exists one and only one function $y : I \to \mathbb{R}$ which satisfies the equation (1) and such that $y(x_0) = y_0$ and $y'(x_0) = v_0$. Clearly the function $y : I \to \mathbb{R}$ is a continuous function and it has continuous first and second derivative on the interval I. If the interval I is not specifically given than you need to determine the maximum interval on which all the functions P, Q and R are continuous and which contains the point x_0 and work with that interval.

We will mostly be interested in homogeneous second order linear differential equations

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \quad x \in I.$$

The natural problem is to find all solutions of this equation. The following theorem helps with this task. I nicknamed it "from two all" (FTA or more mathematically "F2∀").

Theorem 2. *Let P and Q be continuous real valued functions defined on an interval I. Let $y_1 : I \to \mathbb{R}$ and $y_2 : I \to \mathbb{R}$ be linearly independent solutions of the homogeneous equation*

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \; x \in I. \tag{3}$$

Then all solutions of the homogeneous equation (3) *are given by the general solution formula*

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad x \in I,$$

where c_1 and c_2 are arbitrary constants.

Recall that two functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are *linearly dependent* on I if one is a constant multiple of the other. Otherwise-that is, if neither is a constant multiple of the other-they are called *linearly independent on I*. Also recall that if $f(x) = 0$ for all $x \in I$, then f and g are linearly dependent for every function g, since $f = 0 \cdot g$. The functions f and g are *linearly independent on I* if $C_1 f(x) + C_2 g(x) = 0$ for all $x \in I$ implies that $C_1 = C_2 = 0$.

One way to verify whether two solutions of (3) are linearly independent is to check their *Wronskian*: For two differentiable functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ the Wronskian $W(f, g) : I \to \mathbb{R}$ is defined by

$$W(f,g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x), \; x \in I.$$

Theorem 3. *Let P and Q be continuous real valued functions defined on an interval I. Let $y_1 : I \to \mathbb{R}$ and $y_2 : I \to \mathbb{R}$ be solutions of the homogeneous equation* (3)*. Then, either $W(y_1, y_2)(x) = 0$ for all $x \in I$ (in this case y_1 and y_2 are linearly dependent), or else $W(y_1, y_2)(x) \neq 0$ for all $x \in I$ (in this case y_1 and y_2 are linearly independent).*

A pair of linearly independent solutions of the homogeneous equation (3) is called a *fundamental set of solutions* of (3).

Example 4. Let m, c and k be real numbers and $m \neq 0$. Consider the homogeneous equation with constant coefficients

$$my''(x) + cy'(x) + ky(x) = 0, \quad x \in I. \tag{4}$$

Since $m \neq 0$, dividing by m we get a normalized equation with coefficients $P(x) = \frac{c}{m}$ and $Q(x) = \frac{k}{m}$. Since these functions are continuous on entire real line it is natural to take $I = \mathbb{R}$. Substituting $y(x) = e^{rx}$ in (4) you can find that the function e^{rx} is a solution of (4) if and only if r is a root of the quadratic equation

$$mr^2 + cr + k = 0. \tag{5}$$

If (5) has two distinct roots r_1 and r_2 we get two linearly independent solutions of (4):

$$y_1(x) = e^{r_1 x} \quad \text{and} \quad y_2(x) = e^{r_2 x}. \tag{6}$$

If r_1 and r_2 are real numbers, then the solutions in (6) are real and we are done. If r_1 and r_2 are non-real numbers, then the solutions in (6) are not real valued functions and cannot be used, since we seek real valued solutions only.

If r_1 and r_2 are non-real numbers, then there exist real numbers α and $\beta \neq 0$ such that $r_1 = \alpha + i\beta$ and $r_1 = \alpha - i\beta$. Here i is the imaginary unit: $i^2 = -1$. Using the Euler's formula

$$e^{ix} = \cos(x) + i\sin(x), \quad x \in \mathbb{R},$$

you can show that the real valued functions

$$y_1(x) = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2(x) = e^{\alpha x}\sin(\beta x) \tag{7}$$

are linearly independent solutions of (4).

If (5) has only one real root r_1 then two linearly independent solutions of (4) are the functions

$$y_1(x) = e^{r_1 x} \quad \text{and} \quad y_2(x) = xe^{r_1 x}. \tag{8}$$

Example 5. Let p and q be real numbers. Consider the homogeneous equation

$$x^2 y''(x) + pxy'(x) + qy(x) = 0, \quad x \in I. \tag{9}$$

To get a normalized form of the equation I have to divide by x^2. I get a normalized equation with coefficients $P(x) = \frac{p}{x}$ and $Q(x) = \frac{q}{x^2}$. Since these functions are continuous on the interval $(0, +\infty)$, it is natural to take $I = (0, +\infty)$.

Substituting $y(x) = x^r$ in (9) you can find that the function x^r is a solution of (9) if and only if r is a root of the quadratic equation

$$r(r-1) + pr + q = 0. \tag{10}$$

If (10) has two distinct roots r_1 and r_2 we get two linearly independent solutions of (9):

$$y_1(x) = x^{r_1} \quad \text{and} \quad y_2(x) = x^{r_2}, \quad x > 0. \tag{11}$$

If r_1 and r_2 are real numbers, then the solutions in (11) are real and we are done. If r_1 and r_2 are non-real numbers, then the solutions in (11) are not real valued functions and cannot be used, since we seek real valued solutions only.

If r_1 and r_2 are non-real numbers, then there exist real numbers α and $\beta \neq 0$ such that $r_1 = \alpha + i\beta$ and $r_1 = \alpha - i\beta$. Using the Euler's formula again you can show that the real valued functions

$$y_1(x) = x^\alpha \cos(\beta \ln(x)) \quad \text{and} \quad y_2(x) = x^\alpha \sin(\beta \ln(x)), \quad x > 0, \tag{12}$$

are linearly independent solutions of (4).

If (10) has only one real root r_1 then two linearly independent solutions of (9) are the functions

$$y_1(x) = x^{r_1} \quad \text{and} \quad y_2(x) = x^{r_1} \ln(x). \tag{13}$$

Remark 6. Rather then doing all the calculations required in Examples 4 and 5 by hand, it is convenient to use the computer algebra system *Mathematica* to do calculations for us. Another reason to use *Mathematica* is that we will encounter second order homogeneous linear equations that we can not solve using simple tricks as in Examples 4 and 5. The *Mathematica*'s library of functions is huge and it can find solutions in terms of functions that you have not encountered before. *Mathematica* can find the general solutions for a huge class of second order homogeneous linear equations. Very often the solution will be in terms of the functions that you are not familiar with. In that case you can use *Mathematica* to plot unfamiliar functions and get some idea about behaviour of these functions. The only problem is that by default *Mathematica* works with complex valued functions and it sometimes takes a special effort to get *Mathematica* to write a solution in terms of real valued functions. *Mathematica* commands that are helpful to get real valued solutions are `ComplexExpand`, `Re`, `Im`, and `Simplify` with specific restrictions on the variable.

Problem 7. Consider the equation

$$y''(x) - y(x) = 0. \tag{14}$$

(a) Find the general solution of the equation (14).

(b) What is the natural interval I on which the equation (14) is to be considered?

(c) Find the solution of the equation (14) which satisfies $y(0) = 0$ and $y'(0) = 1$.

(d) Find the solution of the equation (14) which satisfies $y(0) = 1$ and $y'(0) = 0$.

(e) Find all solutions of the equation (14) which are bounded on the interval $[0, +\infty)$. Are there any solutions of the equation (14) which are bounded on $\mathbb{R}$?

Problem 8. Consider the equation

$$y''(x) + y(x) = 0 \ . \tag{15}$$

(a) Find the general solution of the equation (15).

(b) What is the natural interval I on which the equation (15) is to be considered?

(c) Find the solution of the equation (15) which satisfies $y(0) = 0$ and $y'(0) = 1$.

(d) Find the solution of the equation (15) which satisfies $y(0) = 1$ and $y'(0) = 0$.

Remark 9. Because of the similarity of Problems 7 and 8, the solution obtained in Problem 7 (c) is called *hyperbolic sine* : $\sinh(x) = \frac{e^x - e^{-x}}{2}$, and the solution obtained in Problem 7 (d) is called *hyperbolic cosine* : $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Show that sinh and cosh are linearly independent on $\mathbb{R}$. Therefore sinh and cosh form a fundamental set of solutions of (14).

Problem 10. Consider the equation

$$x^2 y''(x) + x y'(x) - y(x) = 0 \ . \tag{16}$$

(a) Find the general solution of the equation (16). (Use the method of Example 5. Do not use *Mathematica*.)

(b) What is the natural interval I on which the equation (16) is to be considered?

(c) Find the solution of the equation (16) which satisfies $y(0) = 0$ and $y'(0) = 1$.

(d) Does Theorem 1 guarantees the existence of the solution of the equation (16) which satisfies $y(0) = 0$ and $y'(0) = 1$?

(e) Does there exist a solution of the equation (16) which satisfies $y(0) = 1$ and $y'(0) = 0$?

(f) Find all solutions of the equation (16) which are bounded on the interval $(0, 1)$. Are there any solutions of the equation (16) which are bounded on $(0, +\infty)$?

(g) Use *Mathematica* to find a general solution of the equation (16). Have in mind that complex valued functions are not acceptable as solutions. Which fundamental set of solutions of (16) is suggested by *Mathematica*'s solution? Is there an easy modification of the general solution offered by *Mathematica* to make it acceptable? Explain.

Problem 11. Consider the equation

$$x^2y''(x) + xy'(x) + y(x) = 0 \; . \tag{17}$$

(a) Find the general solution of the equation (17).

(b) What is the natural interval I on which the equation (17) is to be considered?

(c) Find all solutions of the equation (17) which are bounded on $(0, +\infty)$.

Problem 12. Consider the equation

$$y''(x) - xy(x) = 0 \; . \tag{18}$$

(a) Find the general solution of the equation (18). (Use *Mathematica*.)

(b) What is the natural interval I on which the equation (18) is to be considered?

(c) Use *Mathematica* to plot the functions in the fundamental set of solutions used in (a). Based on these graphs, can you informally identify all solutions of the equation (18) which are bounded on $\mathbb{R}$?

(d) Find the solution of the equation (18) which satisfies $y(0) = 0$ and $y'(0) = 1$. Find the solution of the equation (18) which satisfies $y(0) = 1$ and $y'(0) = 0$.

(e) Compare the behaviour of the solutions in the fundamental set of solutions used in (a) and the solutions found in (d) to the solutions of Problems 7 and 8. (Use Remark 9.)