In[1]:= NotebookDirectory[]

Out[1]= C:\Dropbox\Work\myweb\Courses\Math_pages\Math_430\

Equilibrium temperature distribution - 2D problem

The problem

The objective is to solve the PDE

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \text{ on } \{(x, y) \in \mid 0 \le x \le K, \ 0 \le y \le L\},$$

subject to the conditions

$u(x, 0) = f_1(x), \ u(x, L) = f_2(x)$ (call these **BCx**)

$u(0, y) = g_1(y), \ u(K, y) = g_2(y)$ (call these **BCy**)

The trick is to split this problem into **two problems**

- **Problem 1**

The objective is to solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } 0 \le x \le K, \ 0 \le y \le L,$$

subject to the conditions

$u(x, 0) = f_1(x), \ u(x, L) = f_2(x)$ (call these **BCx**)

$u(0, y) = 0, \ u(K, y) = 0$ (call these **DBCy**)

Step 1. First ignore **BCx** and solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } 0 \le x \le K, \ 0 \le y \le L,$$

subject to the conditions

$u(0, y) = 0, \; u(K, y) = 0$ (call these **DBCy**)

Using SofV method we find few solutions of this problem:

$$\text{Sin}\left[\frac{n\,\text{Pi}}{K}\,x\right]\frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,y\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,L\right]}$$

$$\text{Sin}\left[\frac{n\,\text{Pi}}{K}\,x\right]\frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,(L-y)\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,L\right]}$$

Test these solutions:

In[2]:=
```
(D[#, {x, 2}] + D[#, {y, 2}]) &[Sin[n Pi/K x] Sinh[n Pi/K y]/Sinh[n Pi/K L]]
```

Out[2]= 0

In[3]:=
```
FullSimplify[(Sin[n Pi/K x] Sinh[n Pi/K y]/Sinh[n Pi/K L]) /. {x -> {0, K}}]
```

Out[3]= $\left\{0, \text{Csch}\left[\frac{L\,n\,\pi}{K}\right]\text{Sin}[n\,\pi]\,\text{Sinh}\left[\frac{n\,\pi\,y}{K}\right]\right\}$

We need to tell *Mathematica* that n is an integer.

In[4]:=
```
FullSimplify[(Sin[n Pi/K x] Sinh[n Pi/K y]/Sinh[n Pi/K L]) /. {x -> {0, K}},
  n ∈ Integers]
```

Out[4]= {0, 0}

In[5]:= (D[#, {x, 2}] + D[#, {y, 2}]) &[

$$\text{Sin}\left[\frac{n\,\text{Pi}}{K}\,x\right]\frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,(L-y)\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,L\right]}]$$

Out[5]= 0

In[6]:= FullSimplify[

$$\left(\text{Sin}\left[\frac{n\,\text{Pi}}{K}\,x\right]\frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,(L-y)\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,L\right]}\right) \text{ /. } \{x \to \{0, K\}\}]$$

Out[6]=

$$\left\{0, \text{Csch}\left[\frac{L\,n\,\pi}{K}\right]\text{Sin}[n\,\pi]\,\text{Sinh}\left[\frac{n\,\pi\,(L-y)}{K}\right]\right\}$$

In[7]:= FullSimplify[

$$\left(\text{Sin}\left[\frac{n\,\text{Pi}}{K}\,x\right]\frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,(L-y)\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,L\right]}\right) \text{ /. } \{x \to \{0, K\}\}, n \in \text{Integers}]$$

Out[7]= {0, 0}

Step 2. Now that we have few solutions we form many solutions. This is the FFM idea, which is commonly known as the superposition principle:

$$\sum_{n=1}^{nn} a_n\,\text{Sin}\left[\frac{n\,\text{Pi}}{K}\,x\right]\frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,y\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,L\right]} +$$

$$\sum_{n=1}^{nn} b_n\,\text{Sin}\left[\frac{n\,\text{Pi}}{K}\,x\right]\frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,(L-y)\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{K}\,L\right]}$$

Next we choose a_n and b_n such that the above function satisfies **BCx** conditions. First substitute $y = 0$. This leads to the formula for b_n.

$$f_1(x) = \sum_{n=1}^{nn} b_n \,\text{Sin}\left[\frac{n\,\text{Pi}}{K}\,x\right]$$

$$f_1(x)\,\text{Sin}\left[\frac{j\,\text{Pi}}{K}\,x\right] = \sum_{n=1}^{nn} b_n \,\text{Sin}\left[\frac{n\,\text{Pi}}{K}\,x\right]\text{Sin}\left[\frac{j\,\text{Pi}}{K}\,x\right]$$

In[8]:=
```
Clear[K];
FullSimplify[Integrate[Sin[n Pi/K x] Sin[j Pi/K x], {x, 0, K}],
 And[n ∈ Integers, j ∈ Integers, Or[j > n, j < n]]]
```

Out[8]= 0

In[9]:=
```
Clear[K];
FullSimplify[Integrate[Sin[n Pi/K x] Sin[n Pi/K x], {x, 0, K}],
 And[n ∈ Integers]]
```

Out[9]= $\frac{K}{2}$

$$\int_0^K f_1(x)\,\text{Sin}\left[\frac{j\,\text{Pi}}{K}\,x\right]dx = \sum_{n=1}^{nn} b_n \int_0^K \text{Sin}\left[\frac{n\,\text{Pi}}{K}\,x\right]\text{Sin}\left[\frac{j\,\text{Pi}}{K}\,x\right]dx$$

$$\int_0^K f_1(x)\,\text{Sin}\left[\frac{j\,\text{Pi}}{K}\,x\right]dx = b_n \int_0^K \text{Sin}\left[\frac{j\,\text{Pi}}{K}\,x\right]\text{Sin}\left[\frac{j\,\text{Pi}}{K}\,x\right]dx$$

$$\int_0^K f_1(x)\,\text{Sin}\left[\frac{j\,\text{Pi}}{K}\,x\right]dx = b_j * \frac{K}{2}$$

Then substitute $y = L$. This leads to the formula for a_n.

- **Problem 2**

The objective is to solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \;\text{ on }\; 0 \le x \le K,\; 0 \le y \le L,$$

subject to the conditions

$u(x, 0) = 0, \; u(x, L) = 0$ (call these **DBCx**)

$u(0, y) = g_1(y), \; u(K, y) = g_2(y)$ (call these **BCy**)

Step 1. First ignore **BCy** and solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } 0 \le x \le K, \; 0 \le y \le L,$$

subject to the conditions

$u(x, 0) = 0, \; u(x, L) = 0$ (call these **DBCx**)

Using SofV method we find few solutions of this problem:

$$\text{Sin}\left[\frac{n\,\text{Pi}}{L}\,y\right] \frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{L}\,x\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{L}\,K\right]}$$

$$\text{Sin}\left[\frac{n\,\text{Pi}}{L}\,y\right] \frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{L}\,(K - x)\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{L}\,K\right]}$$

Test these solutions:

```
In[10]:= (D[#, {x, 2}] + D[#, {y, 2}]) &[Sin[(n Pi)/L y] Sinh[(n Pi)/L x]/Sinh[(n Pi)/L K]]

Out[10]= 0

In[11]:= FullSimplify[(Sin[(n Pi)/L y] Sinh[(n Pi)/L x]/Sinh[(n Pi)/L K]) /. {y → {0, L}},
          n ∈ Integers]

Out[11]= {0, 0}
```

```
In[12]:= (D[#, {x, 2}] + D[#, {y, 2}]) &[
           Sin[(n Pi)/L y] Sinh[(n Pi)/L (K - x)]/Sinh[(n Pi)/L K]]

Out[12]= 0

In[13]:= FullSimplify[(Sin[(n Pi)/L y] Sinh[(n Pi)/L (K - x)]/Sinh[(n Pi)/L K]) /.
           {y → {0, L}}, n ∈ Integers]

Out[13]= {0, 0}
```

Step 2. Now that we have few solutions we form many solutions. This is the FFM idea, which is commonly known as the superposition principle:

$$\sum_{n=1}^{nn} c_n \operatorname{Sin}\left[\frac{n\,Pi}{L}\,y\right] \frac{\operatorname{Sinh}\left[\frac{n\,Pi}{L}\,x\right]}{\operatorname{Sinh}\left[\frac{n\,Pi}{L}\,K\right]} +$$

$$\sum_{n=1}^{nn} d_n \operatorname{Sin}\left[\frac{n\,Pi}{L}\,y\right] \frac{\operatorname{Sinh}\left[\frac{n\,Pi}{L}\,(K-x)\right]}{\operatorname{Sinh}\left[\frac{n\,Pi}{L}\,K\right]}$$

Now we choose c_n and d_n such that the above function satisfies **BCy** conditions. First substitute $x = 0$. This leads to the formula for d_n. Then substitute $x = K$. This leads to the formula for c_n.

■ A symbolic implementation

Here are the given quantities

```
In[14]:= Clear[lK, lL, f1, f2, g1, g2, nn];
         nn = 15;
         lK = 1; lL = 1;
         f1[x_] = 4 x^2 (1 - x);
         g2[y_] = 4 y (1 - y)^2;
         f2[x_] = 0;
         g1[y_] = 0;

In[21]:= Clear[aa];
```

$$aa[n\_] = \frac{2}{lK}\,\text{Integrate}\left[f2[x]\,\text{Sin}\left[\frac{n\,Pi}{lK}\,x\right],\,\{x,\,0,\,lK\}\right]$$

Out[22]= 0

```
In[23]:= Clear[bb];
```

$$bb[n\_] = \frac{2}{lK}\,\text{Integrate}\left[f1[x]\,\text{Sin}\left[\frac{n\,Pi}{lK}\,x\right],\,\{x,\,0,\,lK\}\right]$$

Out[24]= $-\frac{1}{n^4\,\pi^4}8\left(2\,n\,\pi + 4\,n\,\pi\,\text{Cos}[n\,\pi] + \left(-6 + n^2\,\pi^2\right)\,\text{Sin}[n\,\pi]\right)$

```
In[25]:= Clear[cc];
```

$$cc[n\_] = \frac{2}{lL}\,\text{Integrate}\left[g2[y]\,\text{Sin}\left[\frac{n\,Pi}{lL}\,y\right],\,\{y,\,0,\,lL\}\right]$$

Out[26]= $\frac{1}{n^4\,\pi^4}16\,(2\,n\,\pi + n\,\pi\,\text{Cos}[n\,\pi] - 3\,\text{Sin}[n\,\pi])$

In[27]:= Clear[dd];

$$dd[n\_] = \frac{2}{lL} \text{Integrate}\left[g1[y] \,\text{Sin}\left[\frac{n\,Pi}{lL}\,y\right], \{y, 0, lL\}\right]$$

Out[28]= 0

The solution is

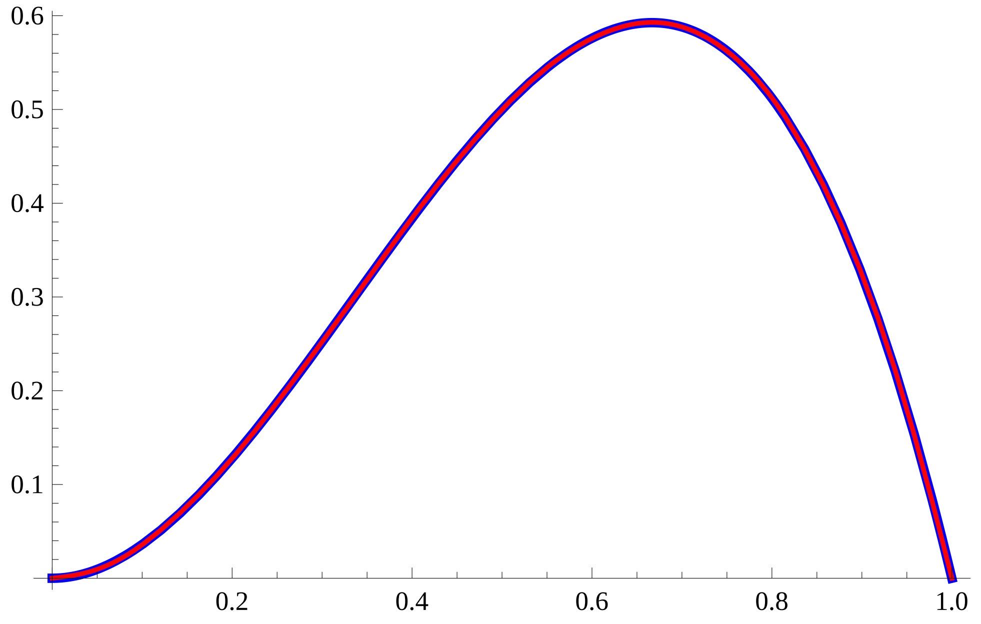
In[29]:= Clear[uu];

$$uu[x\_, y\_] = \sum_{n=1}^{nn} aa[n]\,\text{Sin}\left[\frac{n\,Pi}{lK}\,x\right] \frac{\text{Sinh}\left[\frac{n\,Pi}{lK}\,y\right]}{\text{Sinh}\left[\frac{n\,Pi}{lK}\,lL\right]} + \sum_{n=1}^{nn} bb[n]\,\text{Sin}\left[\frac{n\,Pi}{lK}\,x\right] \frac{\text{Sinh}\left[\frac{n\,Pi}{lK}\,(lL - y)\right]}{\text{Sinh}\left[\frac{n\,Pi}{lK}\,lL\right]} + \sum_{n=1}^{nn} cc[n]\,\text{Sin}\left[\frac{n\,Pi}{lL}\,y\right] \frac{\text{Sinh}\left[\frac{n\,Pi}{lL}\,x\right]}{\text{Sinh}\left[\frac{n\,Pi}{lL}\,lK\right]} + \sum_{n=1}^{nn} dd[n]\,\text{Sin}\left[\frac{n\,Pi}{lL}\,y\right] \frac{\text{Sinh}\left[\frac{n\,Pi}{lL}\,(lK - x)\right]}{\text{Sinh}\left[\frac{n\,Pi}{lL}\,lK\right]};$$

How good is our approximation for the function f1[x] in the boundary conditions? Here is a visual answer.

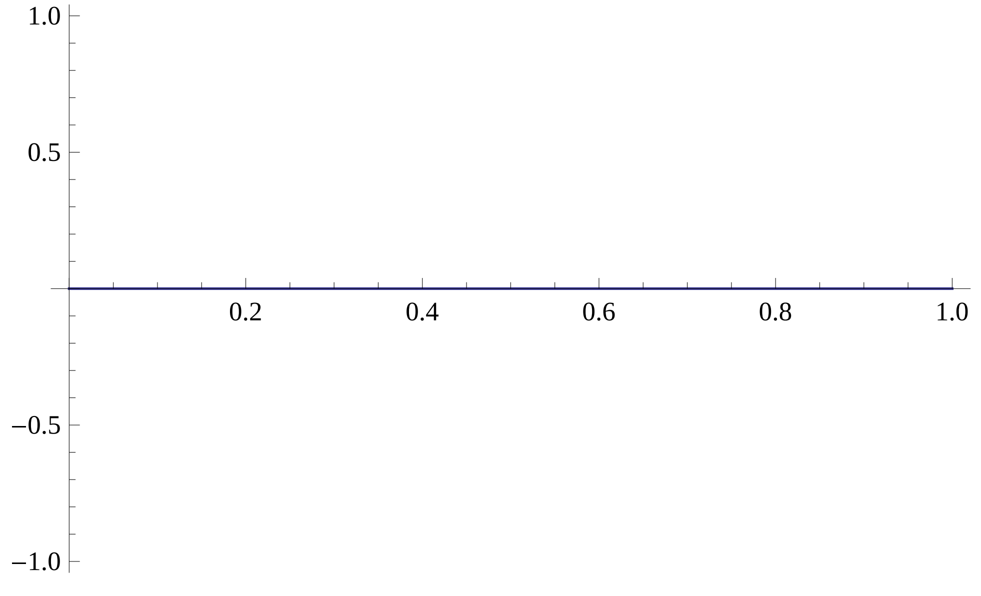
In[31]:=
```
Plot[{f1[x], uu[x, 0]}, {x, 0, 1},
 PlotStyle → {{Blue, Thickness[0.01]},
   {Red, Thickness[0.005]}}]
```

Out[31]=



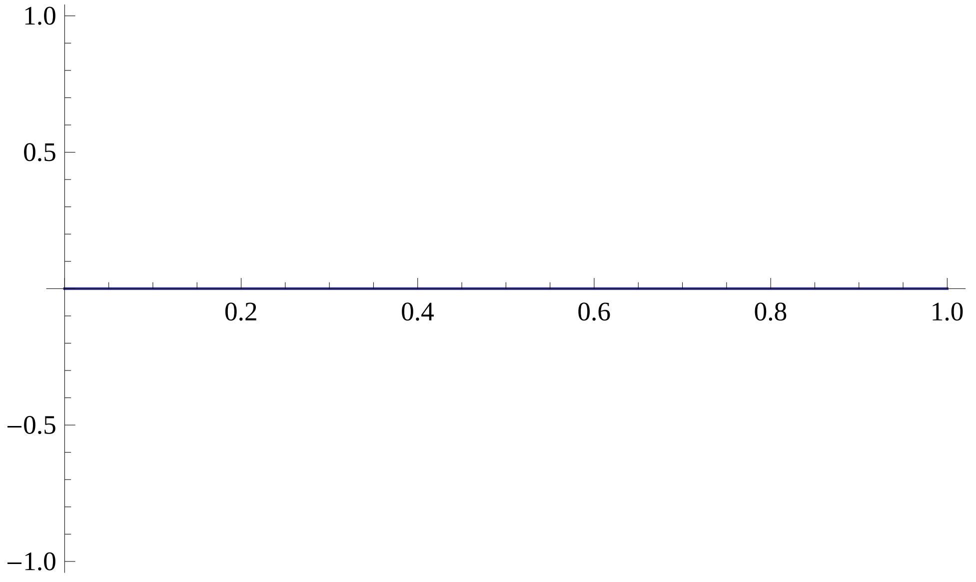
In[32]:=
```
Plot[uu[x, lL], {x, 0, 1}]
```

Out[32]=



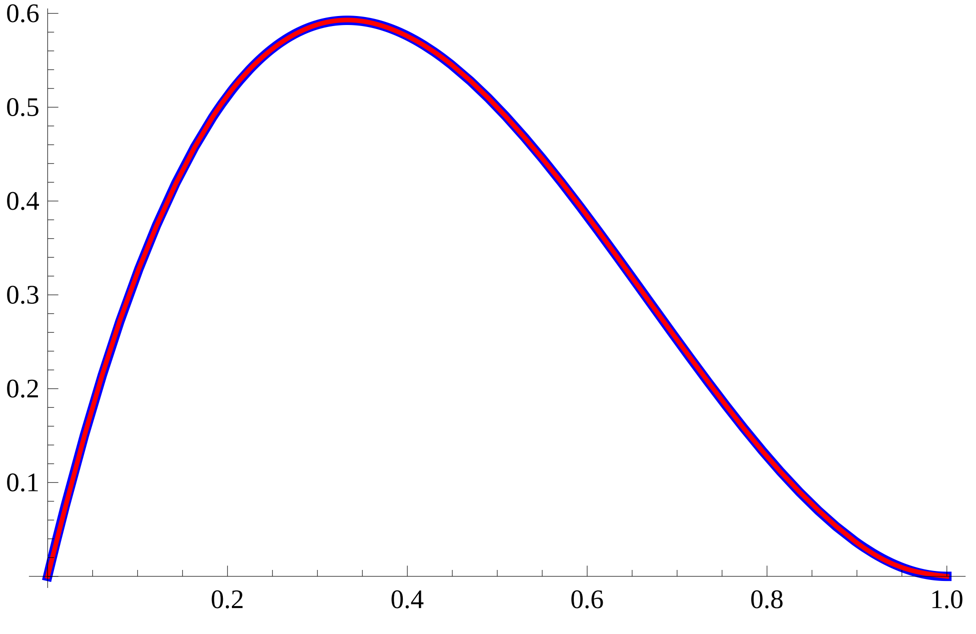
In[33]:=
```
Plot[uu[0, x], {x, 0, 1}]
```

Out[33]=



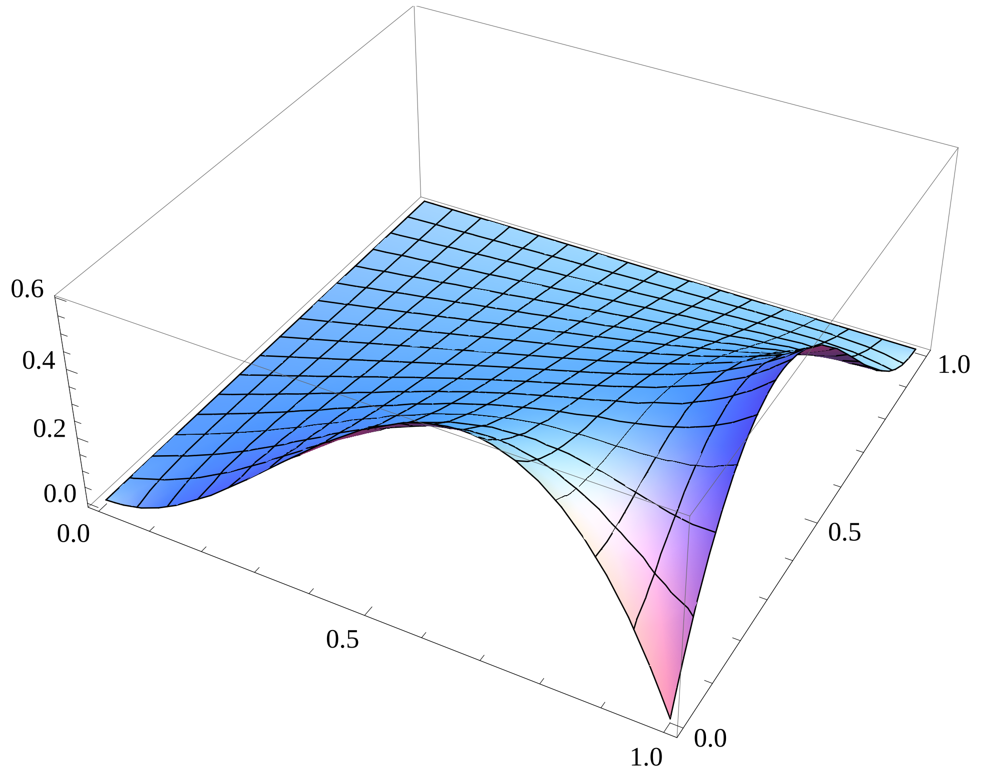
In[34]:=
```
Plot[{g2[y], uu[lK, y]}, {y, 0, 1},
 PlotStyle → {{Blue, Thickness[0.01]},
   {Red, Thickness[0.005]}}]
```

Out[34]=



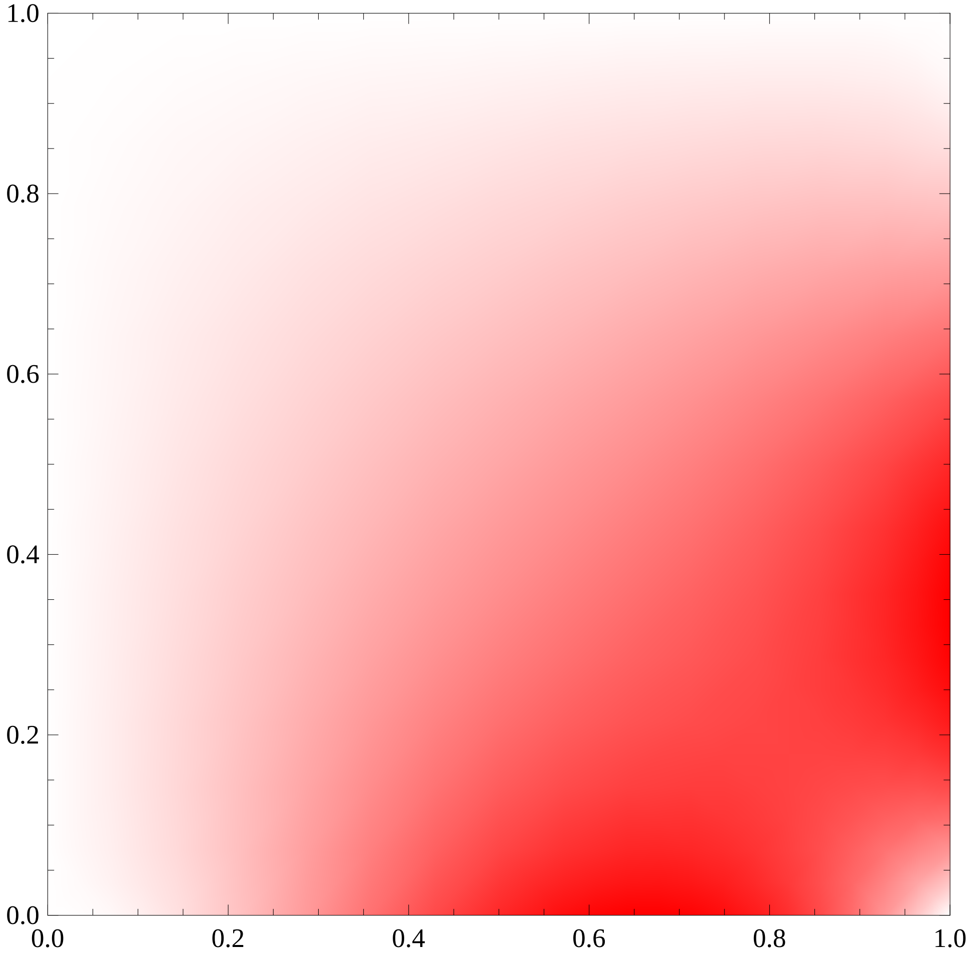
In[35]:= `Plot3D[N[uu[x, y]], {x, 0, 1}, {y, 0, 1}, Mesh → Automatic]`

Out[35]=



In[36]:=
```
DensityPlot[N[uu[x, y]], {x, 0, 1}, {y, 0, 1},
 Frame → True, PlotRange → {{0, 1}, {0, 1}},
 ColorFunction → (RGBColor[1, 1 - #, 1 - #] &)]
```

Out[36]=



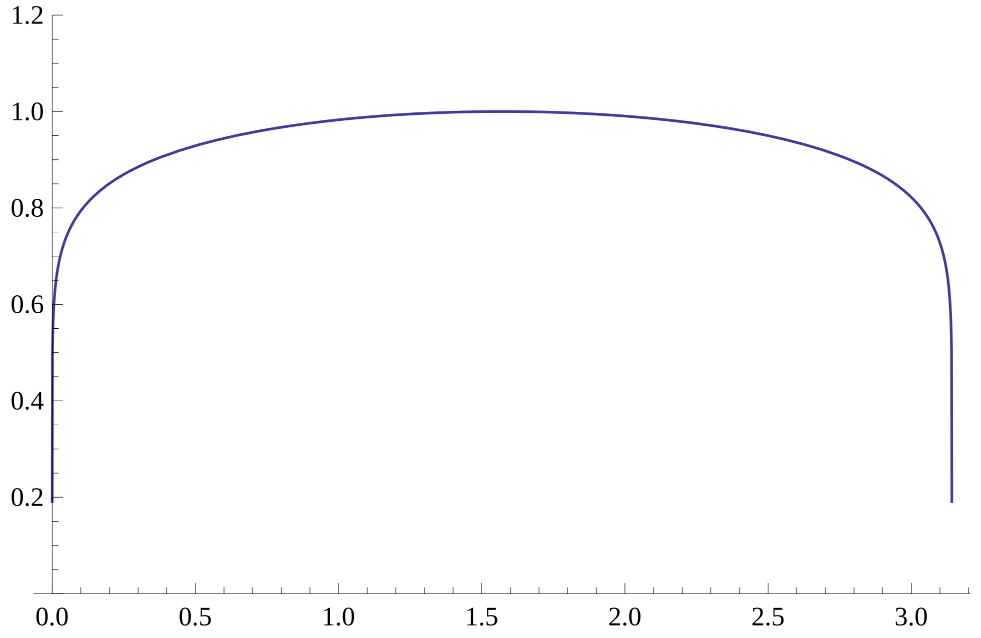
■ A numerical implementation

Here are the given quantities

In[37]:=
```
Clear[lK, lL, f1, f2, g1, g2, nn];

nn = 45;

lK = Pi; lL = Pi;

f1[x_] = (Sin[x])^(1/10); g2[y_] = 0; f2[x_] = 0;
g1[y_] = 0;
```

In[41]:=
```
Plot[f1[x], {x, 0, lK}, PlotRange → {0, 1.2}]
```

Out[41]=



In[42]:=
```
Clear[aa];

aa[n_] = 2/lK Integrate[f2[x] Sin[(n Pi)/lK x], {x, 0, lK}]
```

Out[43]= 0

In[44]:=
```
Clear[bbl];

bbl =
 Chop[Table[2/lK NIntegrate[f1[x] Sin[n Pi/lK x], {x, 0, lK},
     Method → {Automatic}, MaxRecursion → 200,
     AccuracyGoal → 12, PrecisionGoal → 16], {n, 1, nn}]]
```

Out[45]=
```
{1.23582, 0, 0.358787, 0, 0.204016, 0, 0.1408, 0,
 0.10676, 0, 0.0856006, 0, 0.0712249, 0, 0.0608478, 0,
 0.0530194, 0, 0.0469124, 0, 0.0420211, 0, 0.0380191, 0,
 0.0346867, 0, 0.0318708, 0, 0.0294614, 0, 0.0273773,
 0, 0.0255576, 0, 0.0239557, 0, 0.0225352, 0,
 0.0212672, 0, 0.0201288, 0, 0.0191014, 0, 0.0181696}
```

In[46]:=
```
Clear[cc];

cc[n_] = 2/(lL Sinh[n Pi/lL lK])
  Integrate[g2[y] Sin[n Pi/lL y], {y, 0, lL}]
```

Out[47]= 0

In[48]:=
```
Clear[dd];

dd[n_] = 2/lL Integrate[g1[y] Sin[n Pi/lL y], {y, 0, lL}]
```

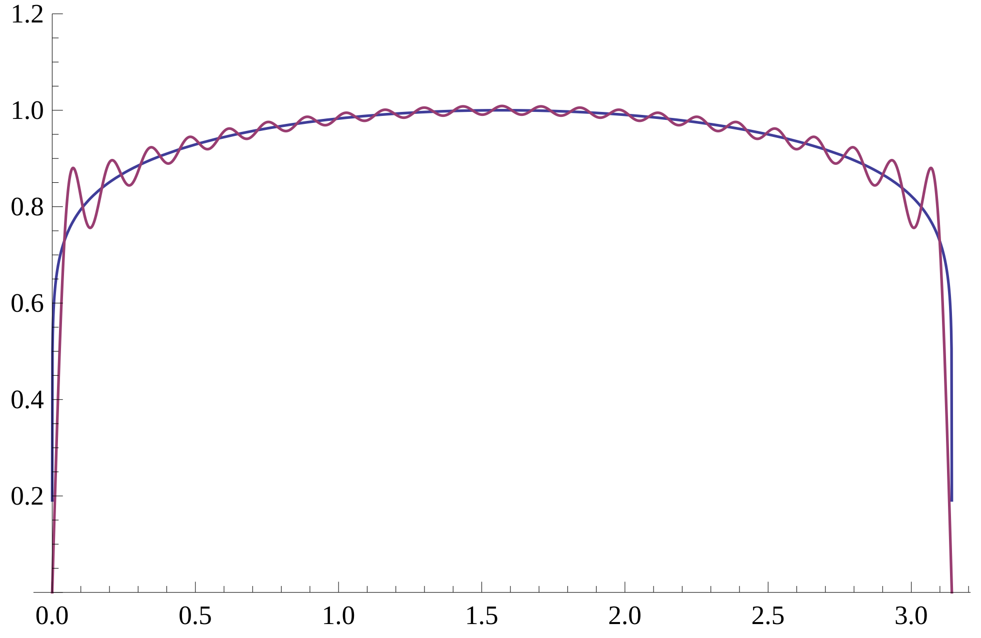
Out[49]= 0

The solution is

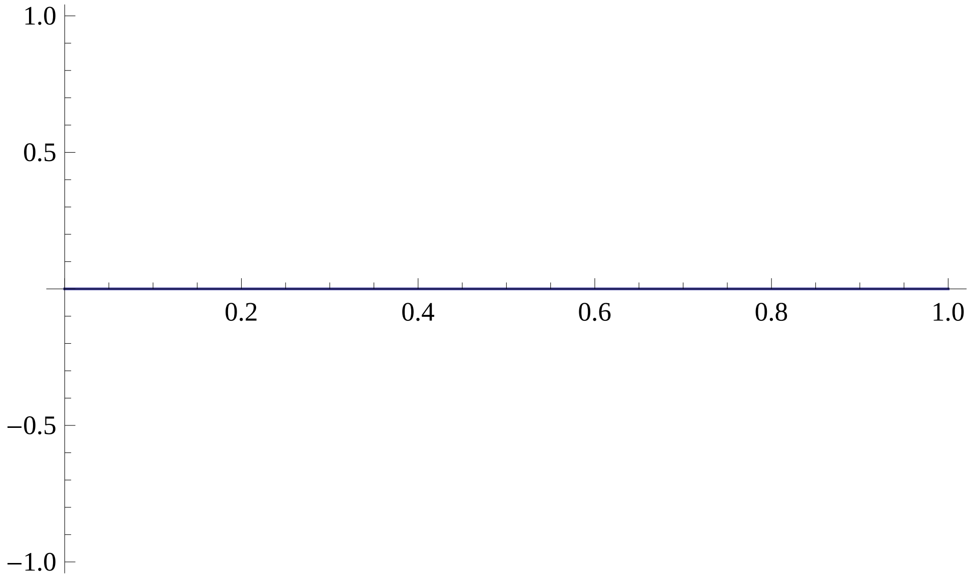
In[50]:= `Clear[uu];`

$$uu[x\_, y\_] = \sum_{n=1}^{nn} aa[n]\ Sin\left[\frac{n\ Pi}{lK}\ x\right] \frac{Sinh\left[\frac{n\ Pi}{lK}\ y\right]}{Sinh\left[\frac{n\ Pi}{lK}\ lL\right]} + \sum_{n=1}^{nn} bbl[[n]]\ Sin\left[\frac{n\ Pi}{lK}\ x\right] \frac{Sinh\left[\frac{n\ Pi}{lK}\ (lL - y)\right]}{Sinh\left[\frac{n\ Pi}{lK}\ lL\right]} + \sum_{n=1}^{nn} cc[n]\ Sin\left[\frac{n\ Pi}{lL}\ y\right] \frac{Sinh\left[\frac{n\ Pi}{lL}\ x\right]}{Sinh\left[\frac{n\ Pi}{lL}\ lK\right]} + \sum_{n=1}^{nn} dd[n]\ Sin\left[\frac{n\ Pi}{lL}\ y\right] \frac{Sinh\left[\frac{n\ Pi}{lL}\ (lK - x)\right]}{Sinh\left[\frac{n\ Pi}{lL}\ lK\right]};$$

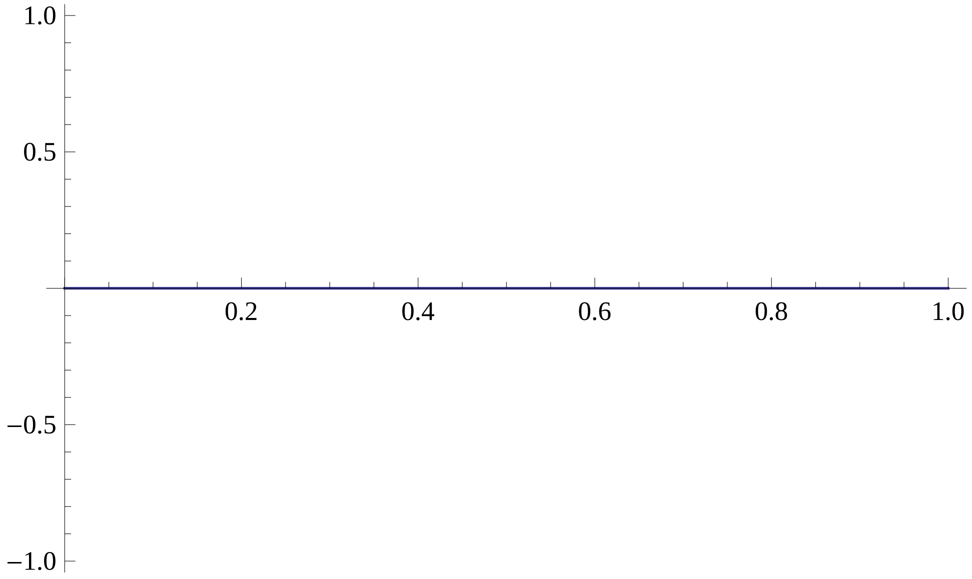
In[52]:= `Plot[{f1[x], uu[x, 0]}, {x, 0, lK}, PlotRange → {0, 1.2}]`

Out[52]=


In[53]:= `Plot[uu[x, lL], {x, 0, 1}]`

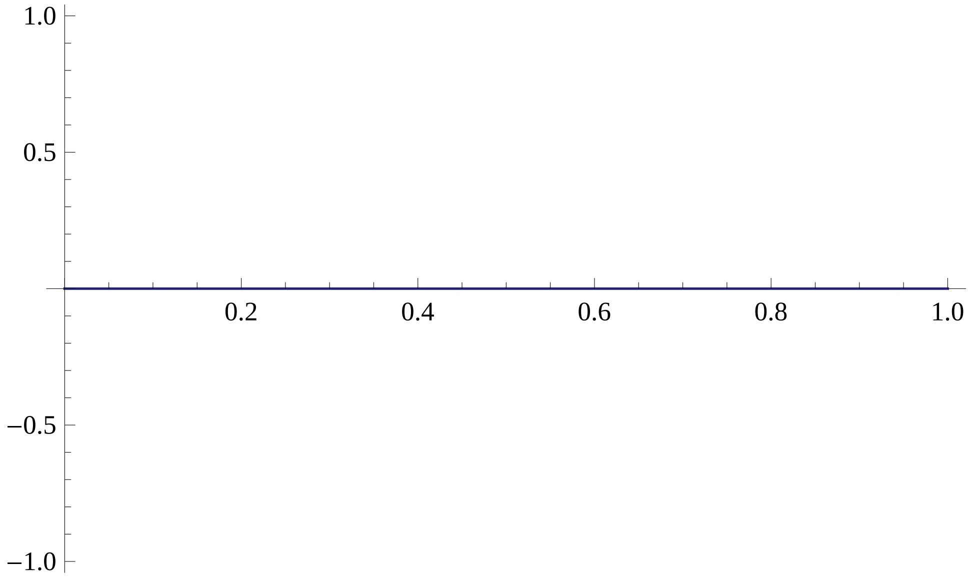
Out[53]=


In[54]:= `Plot[uu[0, x], {x, 0, 1}]`

Out[54]=


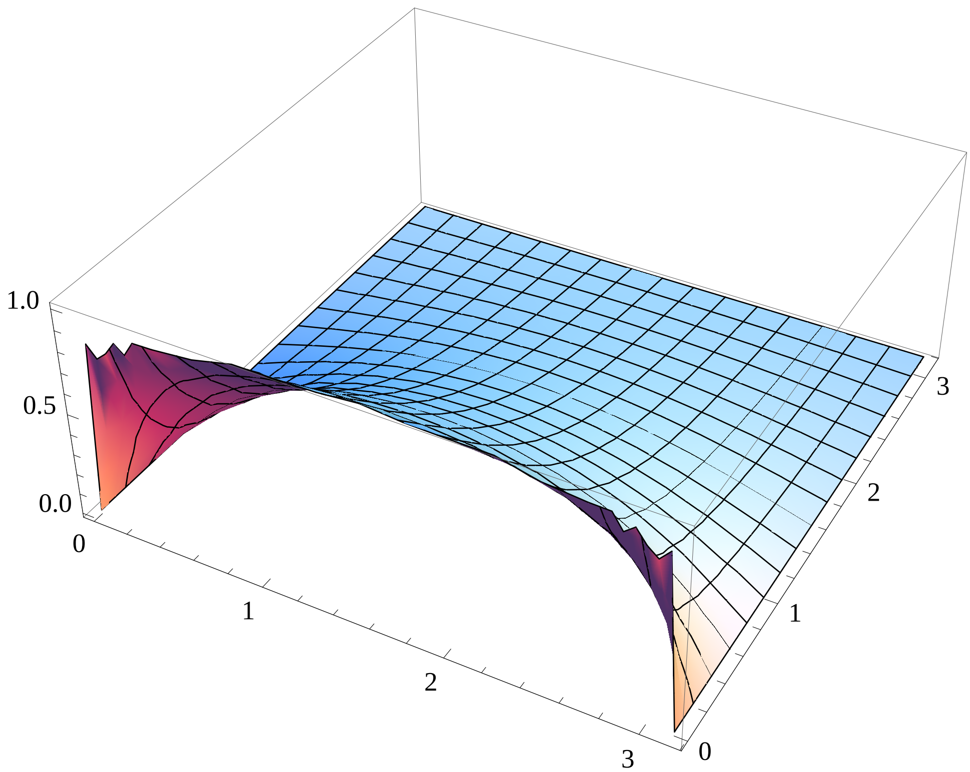
In[55]:=
```
Plot[uu[lK, x], {x, 0, 1}]
```

Out[55]=



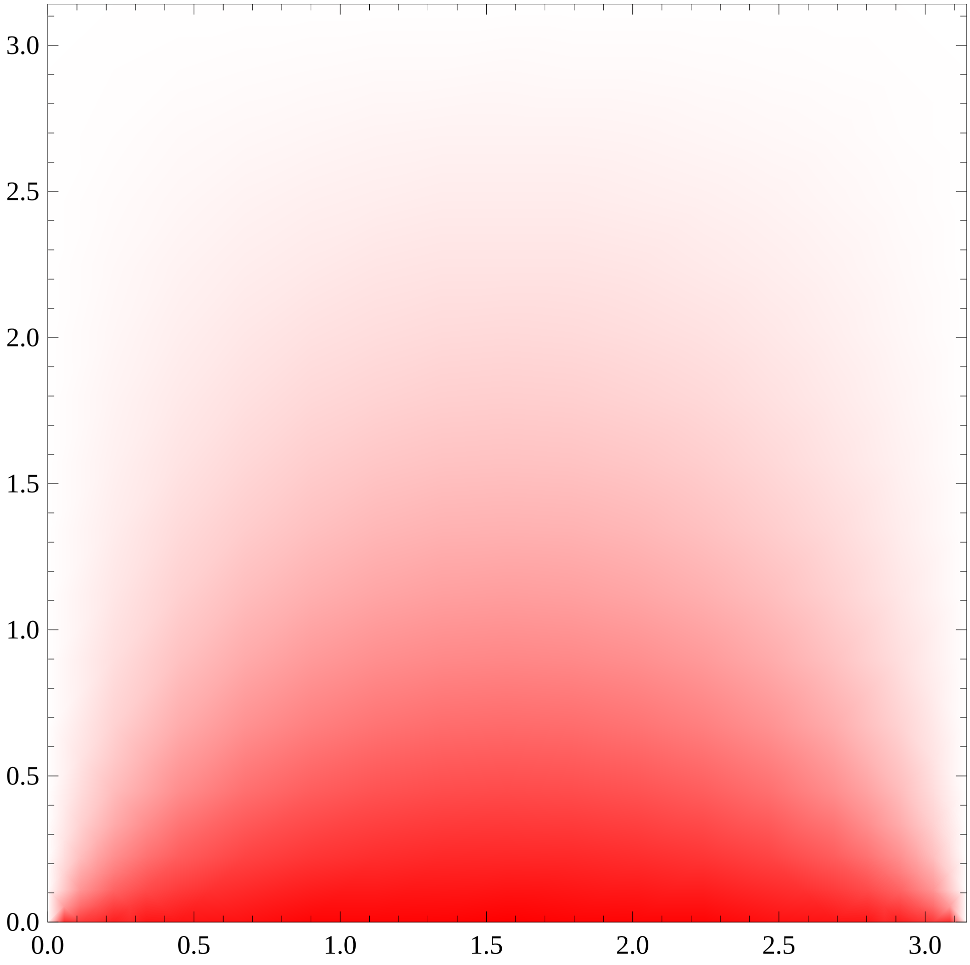
In[56]:=
```
Plot3D[N[uu[x, y]], {x, 0, lK}, {y, 0, lL}]
```

Out[56]=



In[57]:=
```
DensityPlot[N[uu[x, y]], {x, 0, lK}, {y, 0, lL},
 Frame → True, PlotRange → {{0, lK}, {0, lL}},
 ColorFunction → (RGBColor[1, 1 - #, 1 - #] &)]
```

Out[57]=



A symbolic implementation with a problem

Here are the given quantities

In[58]:=
```
Clear[lK, lL, f1, f2, g1, g2, nn];
nn = 20;
lK = 1; lL = 1;
f1[x_] = 3 x^2 + 1;
g2[y_] = 4 - 8 y (y - 1);
f2[x_] = 4 + 8 x (x - 1);
g1[y_] = 1 + 3 y^2;
```

In[65]:=
```
Clear[aa];
aa[n_] = 2/lK Integrate[f2[x] Sin[n Pi/lK x], {x, 0, lK}]
```

Out[66]= $\frac{1}{n^3 \pi^3} 2 \left(-4 \left(-4 + n^2 \pi^2\right) (-1 + \mathrm{Cos}[n \pi]) + 8 n \pi \mathrm{Sin}[n \pi]\right)$

In[67]:=
```
Clear[bb];
bb[n_] = 2/lK Integrate[f1[x] Sin[n Pi/lK x], {x, 0, lK}]
```

Out[68]= $\frac{1}{n^3 \pi^3} 2 \left(-6 + n^2 \pi^2 + \left(6 - 4 n^2 \pi^2\right) \mathrm{Cos}[n \pi] + 6 n \pi \mathrm{Sin}[n \pi]\right)$

In[69]:=
```
Clear[cc];
cc[n_] = 2/lL Integrate[g2[y] Sin[n Pi/lL y], {y, 0, lL}]
```

Out[70]= $\frac{1}{n^3 \pi^3} 2 \left(-4 \left(4 + n^2 \pi^2\right) (-1 + \mathrm{Cos}[n \pi]) - 8 n \pi \mathrm{Sin}[n \pi]\right)$

In[71]:= `Clear[dd];`

$$dd[n\_] = \frac{2}{lL} \text{Integrate}\left[g1[y] \,\text{Sin}\left[\frac{n\,Pi}{lL}\,y\right], \{y, 0, lL\}\right]$$

Out[72]= $$\frac{1}{n^3 \pi^3} 2 \left(-6 + n^2 \pi^2 + \left(6 - 4 n^2 \pi^2\right) \text{Cos}[n \pi] + 6 n \pi \,\text{Sin}[n \pi]\right)$$

The solution is

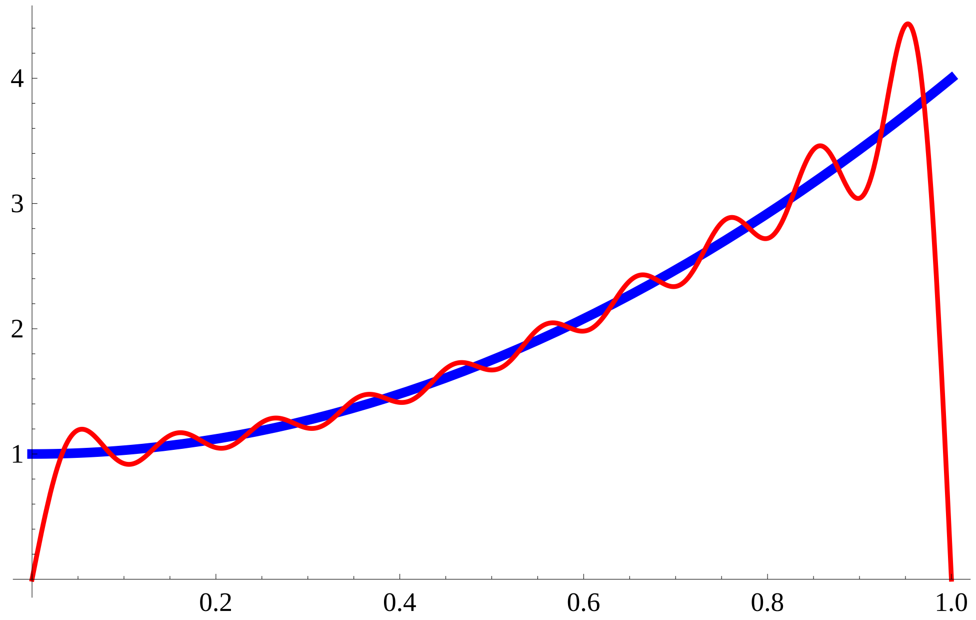
In[73]:= `Clear[uu];`

$$uu[x\_, y\_] = \sum_{n=1}^{nn} aa[n] \,\text{Sin}\left[\frac{n\,Pi}{lK}\,x\right] \frac{\text{Sinh}\left[\frac{n\,Pi}{lK}\,y\right]}{\text{Sinh}\left[\frac{n\,Pi}{lK}\,lL\right]} +$$
$$\sum_{n=1}^{nn} bb[n] \,\text{Sin}\left[\frac{n\,Pi}{lK}\,x\right] \frac{\text{Sinh}\left[\frac{n\,Pi}{lK}\,(lL - y)\right]}{\text{Sinh}\left[\frac{n\,Pi}{lK}\,lL\right]} +$$
$$\sum_{n=1}^{nn} cc[n] \,\text{Sin}\left[\frac{n\,Pi}{lL}\,y\right] \frac{\text{Sinh}\left[\frac{n\,Pi}{lL}\,x\right]}{\text{Sinh}\left[\frac{n\,Pi}{lL}\,lK\right]} +$$
$$\sum_{n=1}^{nn} dd[n] \,\text{Sin}\left[\frac{n\,Pi}{lL}\,y\right] \frac{\text{Sinh}\left[\frac{n\,Pi}{lL}\,(lK - x)\right]}{\text{Sinh}\left[\frac{n\,Pi}{lL}\,lK\right]};$$

How good are the approximations? Here are visual answers:

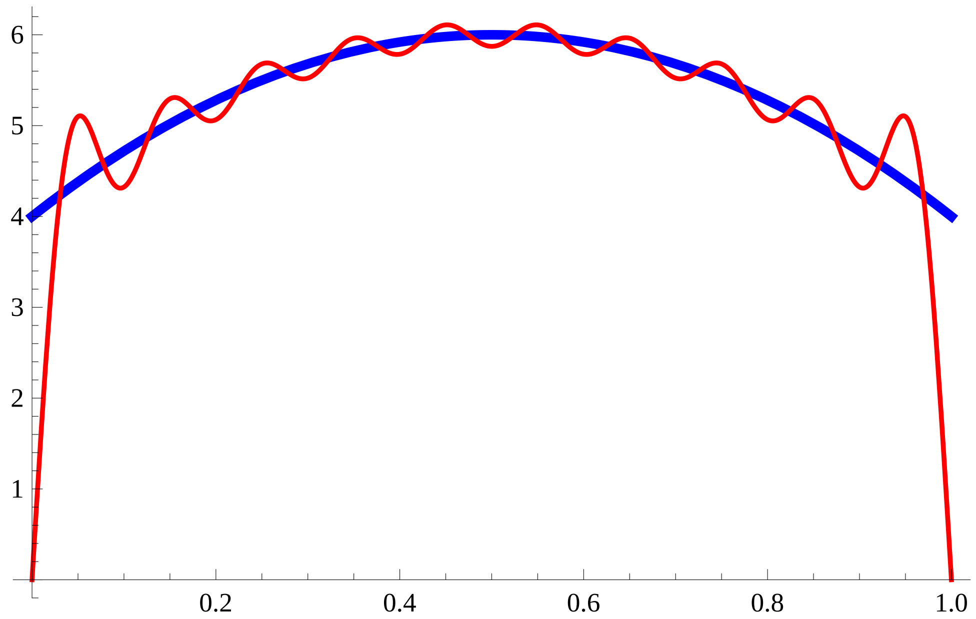
In[75]:=
```
Plot[{f1[x], uu[x, 0]}, {x, 0, 1},
 PlotStyle → {{Blue, Thickness[0.01]},
   {Red, Thickness[0.005]}}, PlotRange → All]
```

Out[75]=



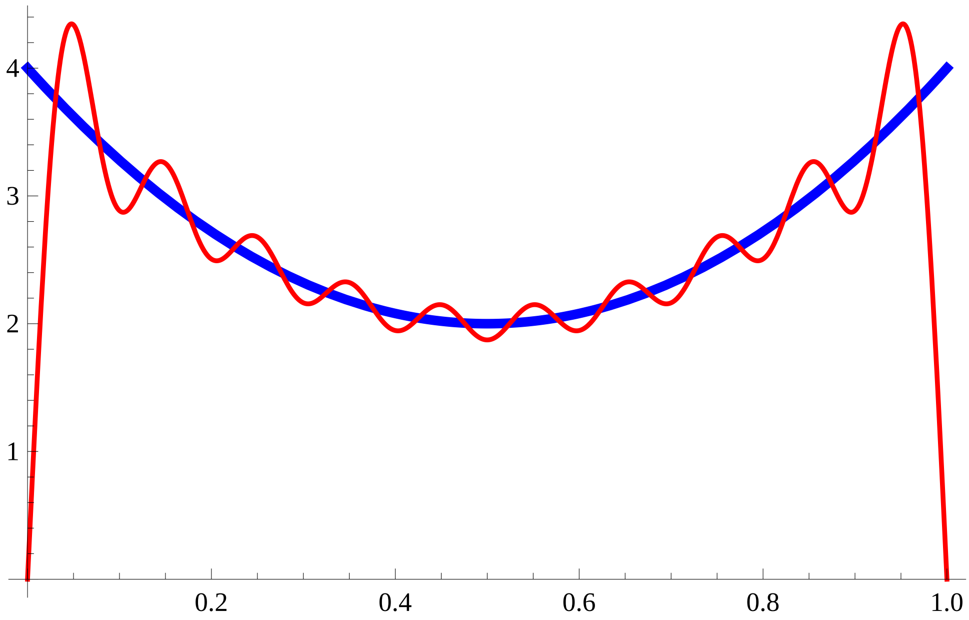
In[76]:=
```
Plot[{g2[y], uu[lK, y]}, {y, 0, 1},
 PlotStyle → {{Blue, Thickness[0.01]},
   {Red, Thickness[0.005]}}, PlotRange → All]
```

Out[76]=



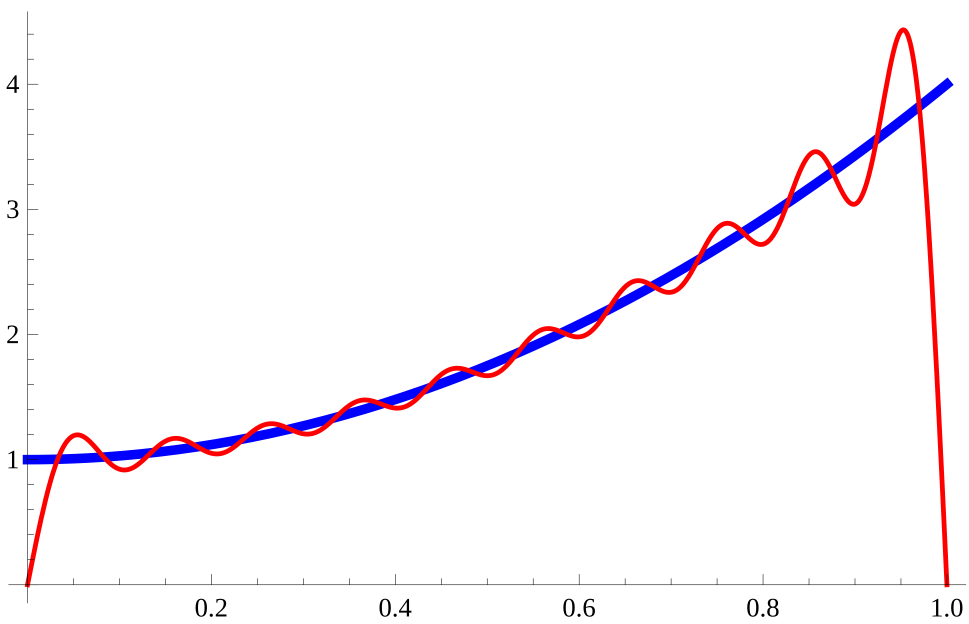
In[77]:=
```
Plot[{f2[x], uu[x, lL]}, {x, 0, 1},
 PlotStyle → {{Blue, Thickness[0.01]},
   {Red, Thickness[0.005]}}, PlotRange → All]
```

Out[77]=



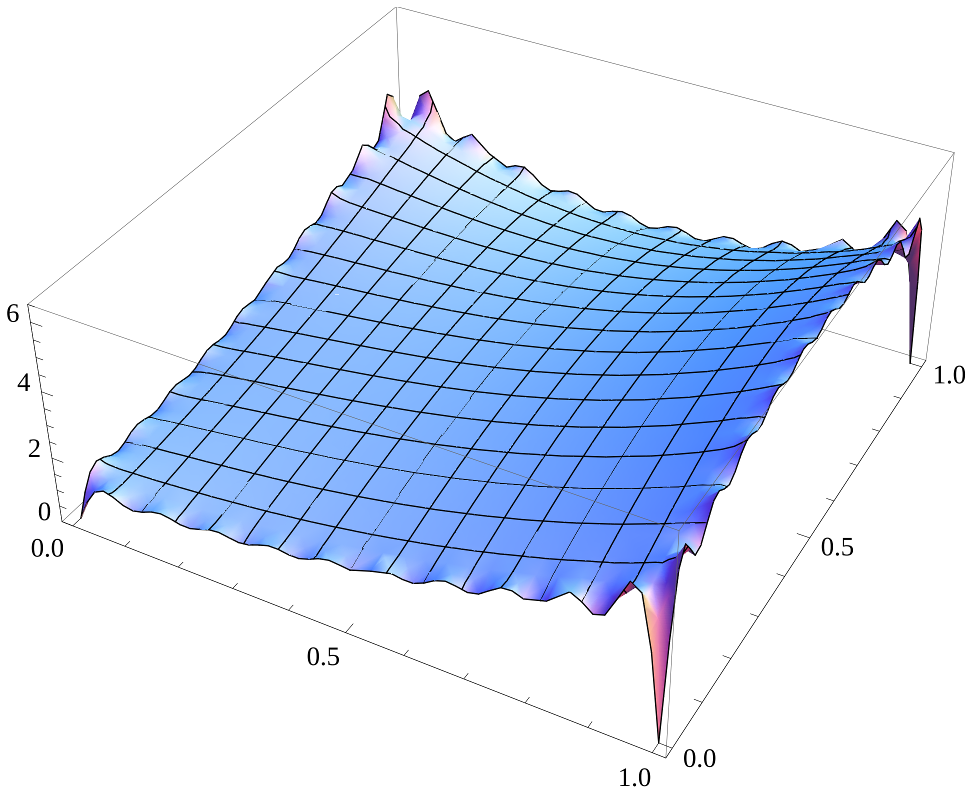
In[78]:=
```
Plot[{g1[y], uu[0, y]}, {y, 0, 1},
 PlotStyle → {{Blue, Thickness[0.01]},
   {Red, Thickness[0.005]}}, PlotRange → All]
```

Out[78]=



In[79]:=
```
Plot3D[N[uu[x, y]], {x, 0, 1}, {y, 0, 1}, Mesh → Automatic,
  PlotRange → {0, 6.5}]
```

Out[79]=



In[80]:=
```
DensityPlot[N[uu[x, y]], {x, 0, 1}, {y, 0, 1},
 Frame → True, PlotRange → {{0, 1}, {0, 1}},
 ColorFunction → (RGBColor[1, 1 - #, 1 - #] &)]
```

Out[80]=