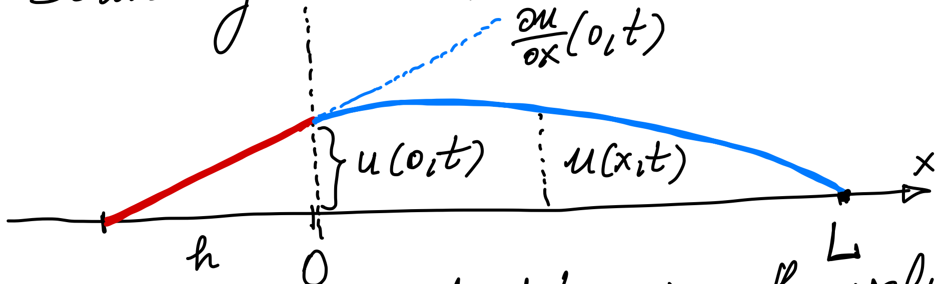
Vibrating String whose Left-end is soaked in super-glue

Robin Boundary Conditions

Consider a vibrating string whose left-end of length h is soaked in super-glue, so it is not flexible. This physical situation leads to Robin Boundry conditions at 0.



The red super-glued part determines the value of the slope of the string at 0, that is $\frac{\partial u}{\partial x}(0,t)$:

$$\frac{\partial u}{\partial x}(0,t) = \frac{u(0,t)}{h}$$

This is a Robin Boundary Condition:

$$u(0,t) - h\frac{\partial u}{\partial x}(0,t) = 0.$$

We consider a uniform string with tension force T_0, mass density ρ_0 and we set $c = \sqrt{T_0/\rho_0}$. In this case, the vibrating string problem is: With $L > 0$, $h \in \mathbb{R}\setminus\{0\}$

PDE: $\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t)$, $x \in [0,L]$, $t \geq 0$

BCs: $u(0,t) - h\frac{\partial u}{\partial x}(0,t) = 0$, $u(L,t) = 0$ $\forall t \geq 0$

ICs: $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$ $\forall x \in [0,L]$

Here we excluded $h = 0$ since in this case we have a Dirichlet Boundary condition which we considered earlier. (Although this case will be implicitly included in our considerations.)

Here f and g are piecewise smooth functions such that

$$f(0) - hf'(0) = 0 \text{ and } f(L) = 0$$

(no boundary conditions on g)

Separation of variables $u(x,t) = A(x)B(t)$ leads to the second order equation for B:

$$B''(t) = -\lambda c^2 B(t)$$

and the boundary-eigenvalue problem for A:

$$A''(x) = -\lambda A(x)$$

$$A(0) - hA'(0) = 0$$

$$A(L) = 0$$

It will be proved for an arbitrary Sturm-Liouville problem that such a problem does not have non-real eigenvalues.

So we consider three possibilities for λ:

Case 1. $\lambda < 0$ Case 2. $\lambda = 0$ Case 3. $\lambda > 0$

Case 1. $\lambda < 0$. We set $\lambda = -\mu^2$ with $\mu > 0$.

We need to find those values for μ for which there exist a non-zero function $A(x)$ such that:

$A''(x) = \mu^2 A(x)$ and $A(0) - h A'(0) = 0$ and $A(L) = 0$.

The reasoning in problems like this is based on the fact that we know the fundamental set of solutions of

$$A''(x) = \mu^2 A(x).$$

In this case we chose to work with the fundamental set of solutions $\cosh(\mu x)$ and $\sinh(\mu x)$, which we abbreviate as $ch(\mu x)$ and $sh(\mu x)$: We need to find (nonzero)

$$A(x) = C_1\, ch(\mu x) + C_2\, sh(\mu x)$$

which satisfies the boundary conditions

a big step in right direction and function A is converted to red c_1, c_2

BCs: $A(0) - h\,A'(0) = 0$ and $A(L) = 0$.

So, the question is: For which $\mu > 0$ there exist nonzero $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ such that $c_1 \operatorname{ch}(\mu x) + c_2 \operatorname{sh}(\mu x)$ satisfies the above BCs.

To answer this question we substitute into BCs and consider the linear system that we obtain:

$$A(0) = c_1, \quad A'(0) = \mu c_2, \quad A(L) = c_1 \operatorname{ch}(\mu L) + c_2 \operatorname{sh}(\mu L)$$

The system is:

$$c_1 - h\mu c_2 = 0$$

$$c_1 \operatorname{ch}(\mu L) + c_2 \operatorname{sh}(\mu L) = 0$$

Written as a matrix equation this system reads

$$\begin{bmatrix} 1 & -h\mu \\ \operatorname{ch}(\mu L) & \operatorname{sh}(\mu L) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The logic here is similar to finding eigenvalues of a 2×2 matrix. The difficulty here is that μ is involved in ch and sh.

The above matrix equation has a nontrivial solution $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ if and only if

$$\begin{vmatrix} 1 & -h\mu \\ \operatorname{ch}(\mu L) & \operatorname{sh}(\mu L) \end{vmatrix} = 0,$$

that is

$$\operatorname{sh}(\mu L) + h\mu \operatorname{ch}(\mu L) = 0.$$

It is not likely that we can find a symbolic solution of this equation. So, let us solve it "graphically":

$ch(\mu L) > 0$, so
$$\underbrace{\frac{sh(\mu L)}{ch(\mu L)}}_{th(\mu L)} = -h\mu$$

So we need to understand the solutions of the equation
$$th(\mu L) = -h\mu.$$

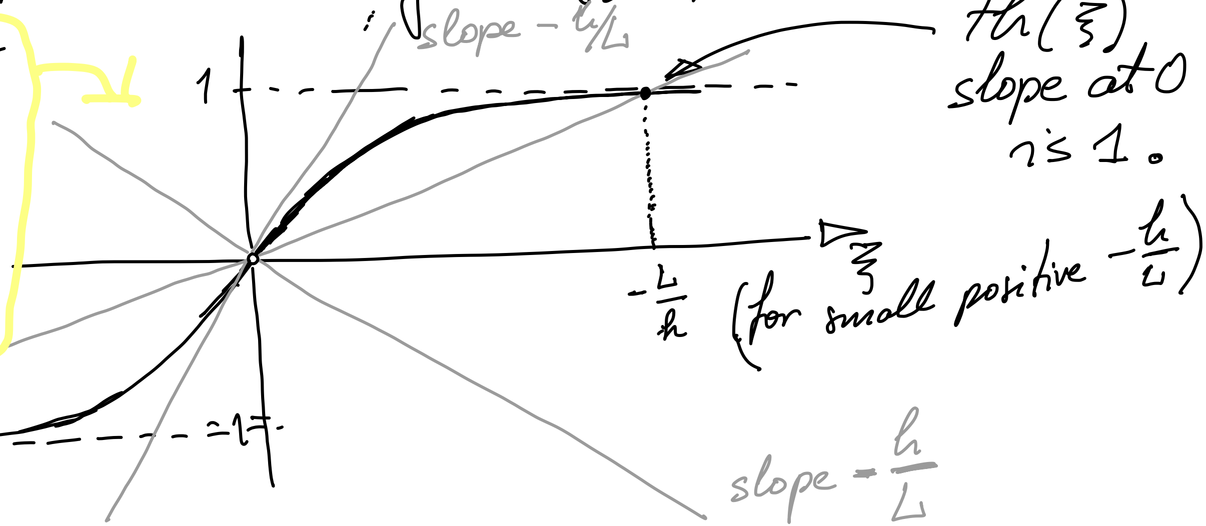
It is easier to picture the solutions of this equation if we introduce a new variable $\xi = \mu L$ and study

$\xi > 0$ since $\mu > 0$

$$th(\xi) = -\frac{h}{L}\xi.$$

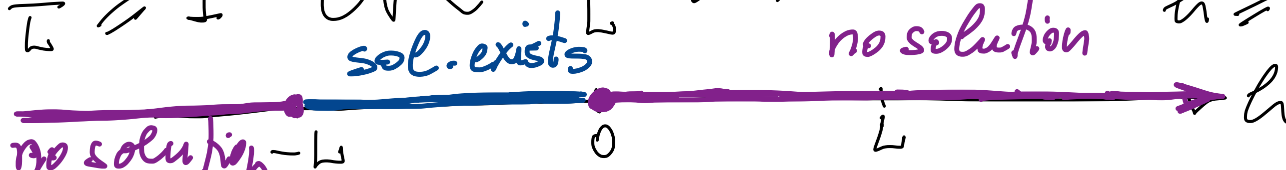
Now we can graph $th(\xi)$ and the line through the origin and analyze the existence of a solution:

we are not interested in $\xi \le 0$ since $\mu > 0$.



What we see from the above graph is that the equation $th(\xi) = -\frac{h}{L}\xi$ has no solution if

$-\frac{h}{L} \geq 1$ OR $-\frac{h}{L} \leq 0$, that is $-h \geq L$ or $h \geq 0$



Thus the solution exists if $-L < h < 0$.
If $-L < h < 0$ is satisfied, then Mathematica can solve the equation

$$\operatorname{th}(\mu L) = -h\mu .$$

in Mathematica

We use the command FindRoot[,].

Notice that if $-\frac{h}{L}$ is a small positive number, then a good approximation for the solution for μ is $-\frac{1}{h}$.

comment.

Denote the solution found by Mathematica by μ_{-1}. Then $\lambda_{-1} = -(\mu_{-1})^2$.

Now we go back to the matrix equation that we need to solve:

$$\begin{bmatrix} 1 & -h\mu_{-1} \\ ch(\mu_{-1}L) & sh(\mu_{-1}L) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

With the μ_{-1} that we found the 2x2 matrix is singular and since we need only one pair $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ we can choose $c_1 = h\mu_{-1}$ and $c_2 = 1$

Hence the eigenvalue (negative one) is $\lambda_{-1} = -(\mu_{-1})^2$ and a corresponding eigenfunction is:

$$h\mu_{-1}\, ch(\mu_{-1}x) + sh(\mu_{-1}x)$$

Now we go back and solve the equation for $B(t)$:

$$B''(t) = (\mu_{-1})^2 c^2 B(t).$$

The general solution of this equation is:

$$a_{-1}\, ch(\mu_{-1}ct) + b_{-1}\, sh(\mu_{-1}ct)$$

where a_{-1} and b_{-1} are arbitrary constants.

Finally we write two separated solutions

$$ch(\mu_{-1}ct)\left(h\mu_{-1}\, ch(\mu_{-1}x) + sh(\mu_{-1}x)\right)$$

and $$sh(\mu_{-1}ct)\left(h\mu_{-1}\, ch(\mu_{-1}t) + sh(\mu_{-1}t)\right)$$

These two solutions correspond to "natural modes" of vibrations; the only difference being that these solutions do not vibrate; practically they mean the string will break.

Case 2 $\lambda = 0$. We need to answer the question: Is there a function $A(x)$ such that

$$A''(x) = 0$$

and $A(0) - h A'(0) = 0$ and $A(L) = 0$. (BCs)

A fundamental set of solutions of $A''(x) = 0$ is $\{1, x\}$.

The general solution is $A(x) = c_1 + c_2 x$.

Substitute into BCs:

$$c_1 - h c_2 = 0$$
$$c_1 + L c_2 = 0$$

Written as a matrix equation:

$$\begin{bmatrix} 1 & -h \\ 1 & L \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The preceding matrix equation has a nontrivial solution if and only if $\begin{vmatrix} 1 & -h \\ 1 & L \end{vmatrix} = 0$, that is $L + h = 0$.

Hence $\lambda = 0$ is an eigenvalue if and only if $h = -L$.

In this case a corresponding eigenfunction is

$L - x$ (here $c_1 = L$, $c_2 = -1$)

Now solve the time equation $B''(t) = 0$.
The general solution is $a_0 \cdot 1 + b_0 t$.
Thus, two special "separated" solutions of the PDE and the BCs are

$$1(L-x) \quad \text{and} \quad t(L-x)$$

these solution are relevant only in the case $h=-L$.
Whenever $h \neq -L$, 0 is <u>NOT</u> an eigenvalue.

<u>Case 3</u> $\lambda > 0$. Set $\lambda = \mu^2$ with $\mu \geq 0$.
We need to find all positive values of μ for which there exists a nonzero function $A(x)$ such

that $$A''(x) = -\mu^2 A(x)$$

and BCs $A(0) - h A'(0) = 0 \quad A(L) = 0.$

The fundamental set of solutions is $\{\cos(\mu x), \sin(\mu x)\}$.

the general solution is $$A(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)$$

This expression is a big step in right direction.
The unknown function $A(x)$ is replaced by two unknow numbers C_1 and C_2; certainly an easier task to find.

Calculate: $A(0) = C_1$, $A'(0) = \mu C_2$

$$A(L) = C_1 c(\mu L) + C_2 s(\mu L)$$

and substitute into boundry conditions:

$$C_1 - h\mu C_2 = 0$$

$$c(\mu L) C_1 + s(\mu L) C_2 = 0$$

Write it as a matrix equation:

$$\begin{bmatrix} 1 & -h\mu \\ c(\mu L) & s(\mu L) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

this matrix equation has a nontrivial solution if and only if

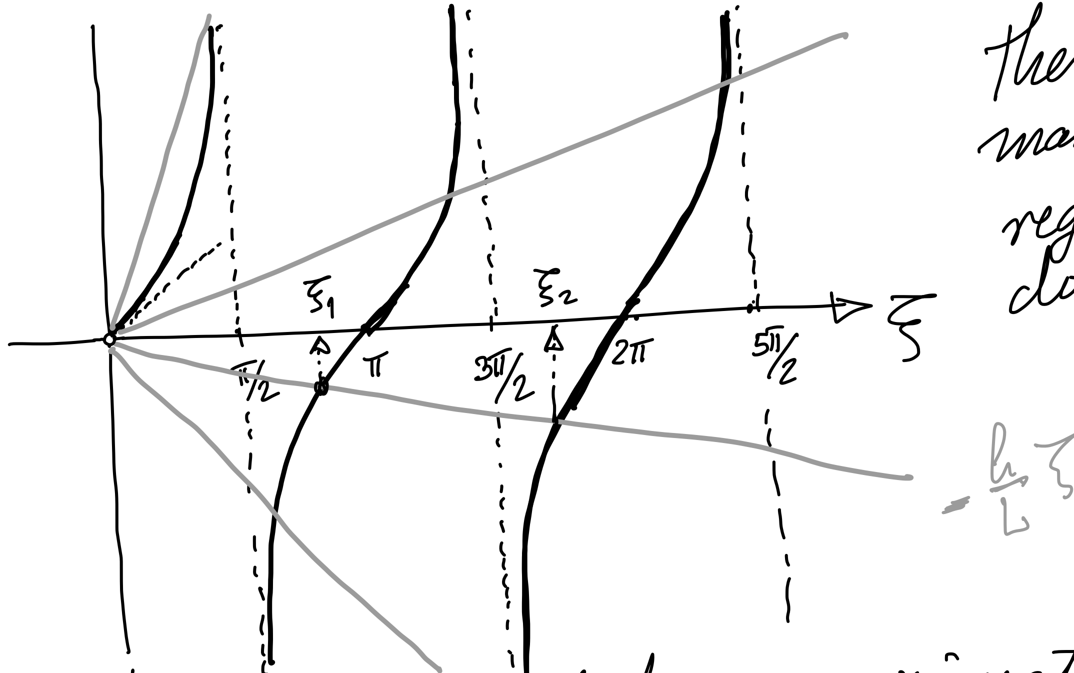
$$s(\mu L) + h\mu c(\mu L) = 0. \quad (*)$$

It is desirable to reduce burning two two places. We can achieve that by dividing by $c(\mu L)$. For that we need $c(\mu L) \neq 0$. So, we first consider the case $\mu L = (2k-1)\frac{\pi}{2}$ with $k \in \mathbb{N}$. In this case $\cos(\mu L) = 0$ and $\sin(\mu L) = (-1)^{k+1}$. In this case (*) becomes $(-1)^{k+1} + 0 = 0$ which is never true. Therefore we will lose no solutions if we assume $c(\mu L) \neq 0$. Then (*) is equivalent to

$$\tan(\mu L) = -h\mu.$$

Introducing $\xi = \mu L$ we look for the positive solutions of

$$\underset{\text{black graph}}{\tan(\xi)} = \underbrace{-\frac{h}{L}\xi}_{\text{gray line}}.$$



There are countably many solutions regardless of the choice of h and L.

(In this case $h > 0$. In fact $h \approx \frac{L}{2}$.)

Mathematica can calculate approximations for $\mu_1, \mu_2, \mu_3, \dots, \mu_n, \dots$. In fact, when calculating these approximations I use the equation $(*)$

$$\sin(\mu L) + \mu L \cos(\mu L) = 0.$$

Now we can consider $\mu_1, \mu_2, \mu_3, \ldots, \mu_n, \ldots$ green. We know them (to some extend). What are the corresponding eigenfunctions? The eigenfunction corresponding to the eigenvalue $\lambda_n = (\mu_n)^2$ is

$$\mu_n L \cdot \cos(\mu_n x) + \sin(\mu_n x)$$

To find the natural modes of vibrations we need to solve $B''(t) = -(\mu_n)^2 c^2 B(t)$

the fundamental set of solutions is

$$\cos(\mu_n c t),\ \sin(\mu_n c t)$$

A typical natural mode of vibration is

$$\cos(\mu_n c t)\left(\mu_n h \cdot \cos(\mu_n x) + \sin(\mu_n x)\right)$$

We can ignore other forms (of nat-modes) since they are just shifts (time shifts) and scales of this one.

the End!