

# Positive differential operators in Krein space $L^2(\mathbb{R}^n)$

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To Heinz Langer on the occasion of his 60th birthday.

We characterize a class of indefinite partial differential operators which are similar to selfadjoint operators in the Hilbert space  $L^2(\mathbb{R}^n)$ .

## 1. Introduction

In this paper we consider the weighted eigenvalue problem

$$Lu = \lambda (\operatorname{sgn} x_n)u, \tag{1.1}$$

on the whole space  $\mathbb{R}^n$  where  $L = p(D)$  is a positive symmetric partial differential operator with constant coefficients. Our goal is to characterize a class of nonnegative polynomials  $p$  for which the operator associated with the problem (1.1) in the Hilbert space  $L^2(\mathbb{R}^n)$  is similar to a selfadjoint operator. For example, our results imply that the operator  $(\operatorname{sgn} x_n)\Delta$  defined on  $H^2(\mathbb{R}^n)$  is similar to a selfadjoint operator in  $L^2(\mathbb{R}^n)$ .

The natural setting to study the problem (1.1) is the space  $L^2(\mathbb{R}^n)$  with the indefinite inner product  $[u, v] = \int u(x)\overline{v(x)}\operatorname{sgn} x_n dx$ . The space  $L^2(\mathbb{R}^n)$  with this inner product is a Krein space. The operator  $A = (\operatorname{sgn} x_n)L$  is positive in this Krein space. In order to apply H. Langer's spectral theory of definitizable operators in Krein spaces we need to prove that the resolvent set  $\rho(A)$  is not empty. In the setting of this paper, a useful tool for this is a simple result stated in Lemma 2.1. The spectral theory of definitizable operators is a generalization of the spectral theory of selfadjoint operators in Hilbert spaces. In particular, a definitizable operator in a Krein space has a spectral function. With exception of finitely many critical points this spectral function has properties analogous to the properties of the spectral function of a selfadjoint operator in a Hilbert space. Definitizable operators in this paper are of the simplest kind: positive operators in a Krein space with nonempty resolvent set. For such operators only 0 and  $\infty$  may be critical points. The projector valued spectral function  $G$  of a positive operator  $A$  with nonempty resolvent set is defined on open intervals in  $\mathbb{R}$  with the endpoints different from 0 and  $\infty$ . The ranges of projectors corresponding to intervals with positive endpoints are Hilbert subspaces and the ranges of projectors corresponding

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to intervals with negative endpoints are anti-Hilbert subspaces of the Krein space  $L^2(\mathbb{R}^n)$ . In general, for a definitizable operator  $T$  with the spectral function  $E$  in a Krein space  $\mathcal{K}$  a spectral point  $\lambda$  is of *positive type* (*negative type*) if there exists an open interval  $\iota$  such that  $\lambda \in \iota$  and the range  $E(\iota)\mathcal{K}$  is a Hilbert (anti-Hilbert) subspace of  $\mathcal{K}$ . A spectral point of  $T$  is *critical* if it is neither of positive nor of negative type. A critical point  $\lambda$  is *regular* if the spectral function is bounded near  $\lambda$ . A critical point is *singular* if it is not regular. For a positive operator  $A$  the points 0 and  $\infty$  are the only possible critical points of  $A$ .

We are primarily interested in the case when neither 0 nor  $\infty$  is a singular critical point of  $A$ . In this case  $A$  is similar to a selfadjoint operator in  $L^2(\mathbb{R}^n)$ . When  $n = 1$  and  $p(t) = t^2$  we proved in [5] that  $A$  is similar to a selfadjoint operator in  $L^2(\mathbb{R})$ . In [13] this result was extended to more general weight functions (see also Example 3.6 below) and in [6] the result was extended to more general polynomials  $p$  (see also Corollary 3.5 below). In this paper we characterize a class of polynomials  $p$  in  $n$  variables for which the corresponding operator  $A = (\text{sgn } x_n)p(D)$  is similar to a selfadjoint operator in the Hilbert space  $L^2(\mathbb{R}^n)$ . The problem with a definite discontinuous weight has recently been considered in [19].

The question of regularity of the critical point  $\infty$  of definitizable operators in Krein spaces has attracted considerable interest, see for example [2, 14, 15, 23]. Corresponding questions for the Sturm-Liouville problem and the elliptic eigenvalue problem with indefinite weight were also studied extensively, see the references in [3, 4, 10, 11, 12, 24]. One of the reasons for this is the following: if a definitizable operator  $T$  in a Krein space  $\mathcal{K}$  has a discrete spectrum, only  $\infty$  may be an accumulation point of spectral points of both positive and negative type. In this case regularity of the critical point  $\infty$  is equivalent to the existence of a Riesz basis of  $\mathcal{K}$  which consists of eigenvectors and generalized eigenvectors of  $T$  (see [4, Proposition 2.3]). The regularity of the critical point  $\infty$  of a definitizable operator was characterized in [2] in terms of the operator domain. This was used in [3] (case  $n = 1$ ) and in [4] (case  $n > 1$ ) to prove regularity of the critical point  $\infty$  for differential operators with more general weight functions and more general differential expressions  $L$ .

Our main interest in this paper is the case when the operator  $A$  is positive (not uniformly positive as in [4]) and this is why the critical point 0 may appear as a critical point. If the spectrum of  $A$  accumulates at 0 from both sides, then 0 is a critical point of  $A$ . To determine whether it is singular or regular we need to investigate the range of  $A$ . This question is harder than the investigation of the domain.

For the readers convenience in Section 2. we prove several simple lemmas that we use later on in the paper. We give a sufficient condition for  $\text{ran}(B + V) = \text{ran}(B)$  for a closed operator  $B$ . For further results related to the stability of the range under additive perturbations see [7]. From [6] we recall a necessary and sufficient condition for  $\text{ran}(B) = \text{ran}(C)$  for multiplication operators  $B, C$  in  $L^2(\mathbb{R}^n)$ .

In Section 3. we prove several stability theorems for the regularity of the critical points 0 and  $\infty$  of positive definitizable operators in a Krein space. As a conse-

quence we get a stability theorem for the similarity to a selfadjoint operator in a Hilbert space. These results are improvements of the corresponding results in [6] since they do not require a priori knowledge of nonemptiness of the resolvent sets of the resulting operators. For related results in this direction see [14].

In Section 4. we consider partial differential operators with constant coefficients. For polynomials  $p$  of the form  $p(\hat{x}, x_n) = q(\hat{x}) + r(x_n)$  we establish the formula (4.5) expressing the spectral function of  $A$  in terms of the spectral functions of the operators

$$(\operatorname{sgn} x_n) \left( r \left( \frac{1}{i} \frac{d}{dx_n} \right) + q(\hat{x})I \right) .$$

For such polynomials we give a detailed analysis of the spectrum and the critical points. We show that  $\infty$  is a regular critical point and give sufficient conditions for 0 to be a regular critical point. These results about critical points are extended to more general polynomials  $p$  using the perturbation results from Section 2. These perturbation results are used in Section 5. to treat a variable coefficient operator.

The study of spectral properties of indefinite eigenvalue problems for differential operators has been motivated by the investigation of the half-range completeness property, see [1]. It follows from the general operator theory in Krein spaces (see [3, 6]) that an operator which is positive in the Krein space  $(L^2(\mathbb{R}^n), [\cdot, \cdot])$  and similar to a selfadjoint operator in the Hilbert space  $L^2(\mathbb{R}^n)$  has the half-range completeness property. Therefore our results in Sections 4. and 5. give sufficient conditions for the half-range completeness property for the problem (1.1).

For definitions and basic results of the theory of definitizable operators see [8, 17].

## 2. Preliminaries

We start with a simple lemma that assures preservation of nonemptiness of resolvent sets under bounded additive perturbations. For a closed operator  $T$  in a Hilbert space  $\mathcal{H}$ ,  $\rho(T)$  denotes the resolvent set of  $T$ .

**Lemma 2.1.** *Let  $A$  be an operator in a Hilbert space  $\mathcal{H}$  which is similar to a selfadjoint operator and let  $B$  be a bounded operator in  $\mathcal{H}$ . There exists  $K > 0$  such that  $\lambda \in \rho(A + B)$  whenever  $|\operatorname{Im} \lambda| > K$ .*

*Proof.* Since  $A$  is similar to a selfadjoint operator there exists a constant  $C > 0$  such that  $\|(A - \lambda I)^{-1}\| < C|\operatorname{Im} \lambda|^{-1}$  for all  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . Therefore,  $B(A - \lambda I)^{-1}$  is a bounded operator with norm  $< 1$  whenever  $|\operatorname{Im} \lambda| > C\|B\|$ . Thus,  $I + B(A - \lambda I)^{-1}$  has a bounded inverse for all  $\lambda \in \mathbb{C}$  such that  $|\operatorname{Im} \lambda| > C\|B\|$ . Since  $A + B - \lambda I = (I + B(A - \lambda I)^{-1})(A - \lambda I)$ , it follows that  $\lambda \in \rho(A + B)$  whenever  $|\operatorname{Im} \lambda| > C\|B\|$ .  $\square$

**Lemma 2.2.** *Let  $A$  and  $B$  be definitizable operators in the Krein space  $(\mathcal{K}, [\cdot, \cdot])$*

such that 0 is neither an eigenvalue of  $A$  nor of  $B$ . Assume that  $\text{ran}(A) = \text{ran}(B)$ . Then 0 is not a singular critical point of  $A$  if and only if 0 is not a singular critical point of  $B$ .

Proof. Both operators  $A^{-1}$  and  $B^{-1}$  are definitizable and 0 is not a singular critical point of  $A$  if and only if  $\infty$  is not a singular critical point of  $A^{-1}$ . Since  $\text{dom}(A^{-1}) = \text{dom}(B^{-1})$ , [2, Corollary 3.3] implies that  $\infty$  is not a singular critical point of  $A^{-1}$  if and only if  $\infty$  is not a singular critical point of  $B^{-1}$ . Since  $\infty$  is not a singular critical point of  $B^{-1}$  if and only if 0 is not a singular critical point of  $B$ , the lemma is proved.  $\square$

Motivated by Lemma 2.2 we prove a result on the preservation of ranges under additive perturbations. The following is a restatement of [16, Lemma VI.2.30].

**Lemma 2.3.** *Let  $A$  and  $V$  be closed densely defined operators in the Hilbert space  $\mathcal{H}$ . Let  $A$  be injective. Assume that  $\text{dom}(A^*) \subseteq \text{dom}(V^*)$  and that there exists  $\beta \geq 0$  such that*

$$\|V^*x\| \leq \beta\|A^*x\| \quad \text{for all } x \in \text{dom}(A^*) . \quad (2.1)$$

Then  $\text{ran}(V) \subseteq \text{ran}(A)$  and  $\|A^{-1}Vy\| \leq \beta\|y\|$  for all  $y \in \text{dom}(V)$ .

**Corollary 2.4.** *In addition to the assumptions of Lemma 2.3 assume that (2.1) holds with  $\beta < 1$ . Then  $A + V$  is injective and*

$$\text{ran}(A + V) = \text{ran}(A) .$$

Proof. Lemma 2.3 implies that  $\text{ran}(A + V) \subseteq \text{ran}(A)$ . Next we prove the opposite inclusion. We have  $\text{dom}((A + V)^*) \subseteq \text{dom}(V^*)$ . Further it follows from (2.1) that

$$\|V^*u\| \leq \beta\|A^*u\| \leq \beta\|(A^* + V^*)u\| + \beta\|V^*u\|,$$

implying

$$\|V^*u\| \leq \frac{\beta}{1 - \beta}\|(A^* + V^*)u\| \quad \text{for all } u \in \text{dom}((A + V)^*) .$$

Applying Lemma 2.3 to the operators  $A + V$  and  $-V$  we conclude that  $\text{ran}(A) = \text{ran}((A + V) - V) \subseteq \text{ran}(A + V)$ .

From Lemma 2.3 it also follows that the operator  $A^{-1}V$  is defined on  $\text{dom}(V)$  and bounded and with the norm is less than or equal to  $\beta$ . If  $x \in \text{dom}(V)$  satisfies  $(A + V)x = 0$ , then  $x = -A^{-1}Vx$ . Therefore  $x = 0$ .  $\square$

**Corollary 2.5.** *Let  $A$  be selfadjoint and  $V$  a closed symmetric operator in the Hilbert space  $\mathcal{H}$  and  $\text{dom}(A) \subseteq \text{dom}(V)$ . Assume that (2.1) holds with  $\beta < 1$ . Then*

$$\text{ran}(A + V) = \text{ran}(A) \quad \text{and} \quad \text{dom}(A + V) = \text{dom}(A) .$$

Proof. It follows from (2.1) that  $\ker(A) \subseteq \ker(V)$ . Denote the closure of  $\text{ran}(A)$  by  $\mathcal{L}$ . Then  $\mathcal{L}$  is invariant under  $A$  and  $V$  and the restriction  $A_r$  of  $A$  to  $\mathcal{L}$  is injective and it satisfies all the assumptions of Corollary 2.4.  $\square$

Let  $\mu$  be a Borel measure on  $\mathbb{R}^n$ . A  $\mu$ -measurable function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  is *nonnegative* if  $f(x) \geq 0$  for  $\mu$ -almost all  $x \in \mathbb{R}^n$ . Denote by  $M_f$  the operator of multiplication by  $f$  in the Hilbert space  $L^2(\mathbb{R}^n, \mu)$ .

**Lemma 2.6.** *Let  $g$  and  $h$  be nonnegative  $\mu$ -measurable functions on  $\mathbb{R}^n$ .*

(1) *The following statements are equivalent.*

- (a)  $\text{dom}(M_g) = \text{dom}(M_h)$
- (b) *There exists  $c > 0$  such that the functions  $\frac{h}{c+g}$  and  $\frac{g}{c+h}$  are  $\mu$ -essentially bounded.*

(2) *The following statements are equivalent.*

- (a)  $\text{ran}(M_g) = \text{ran}(M_h)$ .
- (b) *There exists a constant  $C > 0$  such that*

$$g \leq Ch(1+g) \quad \mu - a.e. \quad \text{and} \quad h \leq Cg(1+h) \quad \mu - a.e. \quad (2.2)$$

Proof. The statement (1) is evident. To prove (2), for a  $\mu$ -measurable function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  denote by  $N_f$  the set  $\{x \in \mathbb{R}^n \mid f(x) = 0\}$ . Note that the conditions (2.2) imply that the symmetric difference of the sets  $N_g$  and  $N_h$  has  $\mu$ -measure zero. Therefore  $\ker(M_g) = \ker(M_h)$ . Let

$$G(x) = \begin{cases} 0 & \text{if } g(x) = 0, \\ \frac{1}{g(x)} & \text{if } g(x) \neq 0 \end{cases} \quad \text{and} \quad H(x) = \begin{cases} 0 & \text{if } h(x) = 0, \\ \frac{1}{h(x)} & \text{if } h(x) \neq 0. \end{cases}$$

It follows from (1) that the condition (2.2) is equivalent to  $\text{dom}(M_G) = \text{dom}(M_H)$ . Since  $\text{dom}(M_G) = \text{ran}(M_g) \oplus \ker(M_g)$ , (2a) and (2b) are equivalent.  $\square$

We need the following simple lemma in Section 4.

**Lemma 2.7.** *Let  $A$  be a uniformly positive operator in the Krein space  $(\mathcal{K}, [\cdot, \cdot])$ . Let  $\gamma > 0$  be a lower bound of the uniformly positive operator  $B = JA$  in the Hilbert space  $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ . Then the interval  $(-\gamma, \gamma)$  is contained in the resolvent set of  $A$ .*

Proof. Clearly  $\gamma^{-1} = \|B^{-1}\| = \|A^{-1}\|$ . Let  $|\lambda| < \gamma$ . Then  $\lambda^{-1} \in \rho(A^{-1})$ , hence  $\lambda \in \rho(A)$ .  $\square$

### 3. Similarity to selfadjoint operators

In this section we reformulate and improve some results from [2] and [6]. Let  $(\mathcal{K}, [\cdot, \cdot])$  be a Krein space, let  $J$  be a fundamental symmetry in  $\mathcal{K}$  and let  $\langle \cdot, \cdot \rangle = [J\cdot, \cdot]$  be the corresponding Hilbert space inner product.

**Lemma 3.1.** *Let  $\eta > 0$ . The following statements are equivalent.*

- (a) *The operator  $JP$  is positive in  $(\mathcal{K}, [\cdot, \cdot])$ ,  $\rho(JP) \neq \emptyset$  and  $\infty$  is not a singular critical point of  $JP$ .*
- (b) *The operator  $JP^\eta$  is positive in  $(\mathcal{K}, [\cdot, \cdot])$ ,  $\rho(JP^\eta) \neq \emptyset$  and  $\infty$  is not a singular critical point of  $JP^\eta$ .*

Proof. Assume (a). Then  $J(P + I)$  is a uniformly positive operator in  $(\mathcal{K}, [\cdot, \cdot])$ . Since  $\text{dom}(J(P + I)) = \text{dom}(JP)$ , [2, Corollary 3.3] (see also [8, Theorem 1.6]) implies that  $\infty$  is not a singular critical point of  $J(P + I)$ . [2, Theorem 2.9] implies that  $\infty$  is not a singular critical point of  $J(P + I)^\eta$ . Since  $\text{dom}(J(P + I)^\eta) = \text{dom}(J(P^\eta + I))$ , and since both operators  $J(P + I)^\eta$  and  $J(P^\eta + I)$  are uniformly positive, [2, Corollary 3.3] implies that  $\infty$  is not a singular critical point of  $J(P^\eta + I)$ . By [2, Theorem 2.5] (or [8, Theorem 1.6]) the operator  $J(P^\eta + I)$  is similar to a selfadjoint operator in  $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ . Lemma 2.1 implies that  $\rho(JP^\eta) \neq \emptyset$ , so  $JP^\eta$  is a definitizable operator. As  $\text{dom}(JP^\eta) = \text{dom}(J(P^\eta + I))$ , the statement (b) follows from [2, Corollary 3.3]. The implication (b)  $\Rightarrow$  (a) follows by applying (a)  $\Rightarrow$  (b) to the operator  $JP^\eta$  and the positive number  $1/\eta$ .  $\square$

**Corollary 3.2.** *Let  $\eta > 0$ . The following statements are equivalent.*

- (a) *The operator  $JP$  is positive in  $(\mathcal{K}, [\cdot, \cdot])$ ,  $0$  is not an eigenvalue of  $P$ ,  $\rho(JP) \neq \emptyset$  and  $0$  is not a singular critical point of the operator  $JP$ .*
- (b) *The operator  $JP^\eta$  is positive in  $(\mathcal{K}, [\cdot, \cdot])$ ,  $0$  is not an eigenvalue of  $P^\eta$ ,  $\rho(JP^\eta) \neq \emptyset$  and  $0$  is not a singular critical point of the operator  $JP^\eta$ .*

**Corollary 3.3.** *Let  $\eta \neq 0$ . The following statements are equivalent:*

- (a) *The operator  $JP$  is positive in  $(\mathcal{K}, [\cdot, \cdot])$ ,  $0$  is not an eigenvalue of  $P$  and  $JP$  is similar to a selfadjoint operator in  $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ .*
- (b) *The operator  $JP^\eta$  is positive in  $(\mathcal{K}, [\cdot, \cdot])$ ,  $0$  is not an eigenvalue of  $P^\eta$  and  $JP^\eta$  is similar to a selfadjoint operator in  $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ .*

The following theorem is an improvement of [6, Theorem 1.4] since it does not require a priori knowledge of nonemptiness of the resolvent set of the operator  $Jh(S)$ . It also can be considered as an abstract version of [6, Theorem 2.3].

**Theorem 3.4.** *Let  $S$  be a selfadjoint operator in the Hilbert space  $(\mathcal{K}, \langle \cdot, \cdot \rangle)$  and let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a nonnegative continuous function.*

- (1) *Assume that there exists  $\eta > 0$  such that the functions  $g(t) = |t|^\eta$  and  $h$  satisfy the conditions (1b) of Lemma 2.6. The following statements are equivalent.*
- (a)  $\infty$  *is not a singular critical point of  $J(S^2 + I)$ .*
  - (b)  $\rho(Jh(S)) \neq \emptyset$  *and  $\infty$  is not a singular critical point of  $Jh(S)$ .*
- (2) *Assume that  $0$  is not an eigenvalue of  $S$  and that there exists  $\eta > 0$  such that the functions  $g(t) = |t|^\eta$  and  $h$  satisfy the condition (2.2). Then the following statements are equivalent.*
- (a)  $\rho(JS^2) \neq \emptyset$  *and  $0$  is not a singular critical point of  $J(S^2)$ .*
  - (b)  $\rho(Jh(S)) \neq \emptyset$  *and  $0$  is not a singular critical point of  $Jh(S)$ .*

*Proof.* The proof combines ideas used in the proofs of Lemma 3.1 and [6, Theorem 1.4]. We prove (2). The proof of (1) is similar. Note that Lemma 2.6 (2), with  $n = 1$ , implies that for any Borel measure  $\mu$  the multiplication operators  $M_g$  and  $M_h$  in  $L^2(\mathbb{R}, \mu)$  have the same range. The Spectral Theorem, see [25, Theorem 7.18], implies  $\text{ran}(|S|^\eta) = \text{ran}(h(S))$ . Therefore,  $\text{ran}(J|S|^\eta) = \text{ran}(Jh(S))$ . Assume (2a). Corollary 3.2 implies that  $0$  is not an eigenvalue of  $J|S|^\eta$ ,  $\rho(J|S|^\eta) \neq \emptyset$  and  $0$  is not a singular critical point of  $J|S|^\eta$ . Therefore  $\infty$  is not a singular critical point of  $(J|S|^\eta)^{-1}$ . Since  $(Jh(S))^{-1} + J$  is uniformly positive and since its domain coincides with the domain of  $(J|S|^\eta)^{-1}$  we conclude that  $\infty$  is not a singular critical point of  $(Jh(S))^{-1} + J$ , that is  $(Jh(S))^{-1} + J$  is similar to a selfadjoint operator in  $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ . Lemma 2.1 implies that  $\rho((Jh(S))^{-1}) \neq \emptyset$ . Consequently,  $\rho((Jh(S))) \neq \emptyset$ . The equality  $\text{ran}(J|S|^\eta) = \text{ran}(Jh(S))$  implies that  $0$  is not an eigenvalue of  $Jh(S)$  and  $0$  is not a singular critical point of  $Jh(S)$ . This proves (2b). The proof of the converse is similar.  $\square$

The combination of parts (1) and (2) of Theorem 3.4 gives sufficient conditions under which the similarity to a selfadjoint operator of  $JS^2$  is equivalent to the similarity to a selfadjoint operator of  $Jh(S)$ . If the function  $h$  is a polynomial this takes a particularly simple form which we state in the following corollary.

**Corollary 3.5.** *Let  $S$  be a selfadjoint operator in the Hilbert space  $(\mathcal{K}, \langle \cdot, \cdot \rangle)$  and let  $p$  be a nonnegative polynomial on  $\mathbb{R}$  with  $0$  being its only root. The following statements are equivalent.*

- (a)  $JS^2$  *is similar to a selfadjoint operator in the Hilbert space  $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ .*
- (b)  $Jp(S)$  *is similar to a selfadjoint operator in the Hilbert space  $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ .*

*Proof.* Let  $2k, k > 0$ , be the degree of  $p$  and let  $2j, j > 0$ , be the multiplicity of the root  $0$  of  $p$ . Let  $g_1(t) = t^{2k}$  and  $g_2(t) = t^{2j}$ . Then  $g_1$  and  $p$  satisfy the

conditions (1b) in Lemma 2.6 and  $g_2$  and  $p$  satisfy the conditions (2.2) in Lemma 2.6. Therefore the equivalence of (a) and (b) follows from Theorem 3.4.  $\square$

**Example 3.6.** Let  $w(t) = |t|^\tau \operatorname{sgn} t$ ,  $\tau > -1$ , and  $S = -i|t|^{-\tau/2} \frac{d}{dt}$ . Let  $\mathcal{K} = \overline{L^2(\mathbb{R}, |w|)}$  be a Krein space with the indefinite inner product  $[f, g] = \int_{\mathbb{R}} f(t) \overline{g(t)} w(t) dt$ . The operator  $(Jf)(t) = (\operatorname{sgn} t) f(t)$  is a fundamental symmetry on  $\mathcal{K}$ . By [13, Theorem 2.7] the operator  $JS^2$  is similar to a selfadjoint operator in the Hilbert space  $L^2(\mathbb{R}, |w|)$ . Let  $p$  be a nonnegative polynomial on  $\mathbb{R}$  with 0 being its only root. Corollary 3.5 implies that the operator  $Jp(S)$  is similar to a selfadjoint operator in  $L^2(\mathbb{R}, |w|)$ . Using [6, Proposition 2.4] we can extend this result to nonnegative polynomials with exactly one real root.

#### 4. Partial differential operators with constant coefficients

In this section  $\mathcal{K}$  denotes the Krein space  $L^2(\mathbb{R}^n)$  with the inner product  $[f, g] = \int_{\mathbb{R}^n} f(x) \overline{g(x)} \operatorname{sgn} x_n dx$ , where  $x = (x_1, \dots, x_n)$ . The multiplication operator

$$(Jy)(x) = (\operatorname{sgn} x_n) y(x)$$

is a fundamental symmetry on  $(L^2(\mathbb{R}^n), [\cdot, \cdot])$  and the corresponding Hilbert space inner product is  $\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) \overline{g(x)} dx$ . The points  $x \in \mathbb{R}^n$  are denoted by  $x = (\hat{x}, t)$ , where  $\hat{x} = (x_1, \dots, x_{n-1})$ ,  $t = x_n$ . The partial Fourier transform with respect to  $\hat{x}$  is denoted by  $F$ . It is a unitary operator in  $L^2(\mathbb{R}^n)$ .

We study partial differential operators with constant coefficients. Let  $p$  be a nonconstant polynomial of degree  $m$  in  $n$  variables,

$$p(x) = \sum_{|\alpha| \leq m} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where  $(x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multiindex,  $c_\alpha \in \mathbb{R}$  and  $|\alpha| = \sum \alpha_j$ . Denote by  $D^\alpha$  the partial differential expression

$$\left(\frac{1}{i}\right)^{|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

and let  $B$  be the closed operator associated with the differential expression

$$p(D) = \sum_{|\alpha| \leq m} c_\alpha D^\alpha$$

in the Hilbert space  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ . Instead of  $B$  we will often write  $p(D)$  to emphasize its dependence on  $p$ . The operator  $B$  is selfadjoint in the Hilbert



space  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ . The operator  $A = JB$  is selfadjoint in the Krein space  $(L^2(\mathbb{R}^n), [\cdot, \cdot])$ . We will prove that, under certain assumptions on the polynomial  $p$ , the operator  $A$  is similar to a selfadjoint operator in  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ .

**Definition 4.1.** Let  $p$  be a nonnegative polynomial in  $n$  variables, let  $q(\hat{x}) = p(\hat{x}, 0)$ , let  $a_0 t^{2k}$ ,  $a_0 \geq 0$ , be the leading term of the polynomial  $p(0, t) - p(0, 0)$  and put

$$p = p_1 + p_2 \quad \text{with} \quad p_1(x) = a_0 t^{2k} + q(\hat{x}) \quad \text{and} \quad p_2(x) = p(x) - p_1(x). \quad (4.1)$$

The polynomial  $p$  is *weakly separated* if there exist  $\gamma_1, \gamma_2, \beta \geq 0$ ,  $\gamma_1 < 1$  such that

$$-\gamma_1 p_1(x) - \beta \leq p_2(x) \leq \gamma_2 p_1(x) + \beta \quad (4.2)$$

The polynomial  $p$  is *strongly separated* if (4.2) holds with  $\beta = 0$ .

**Lemma 4.2.** Let  $p(y, t) = ay^2 + byt + ct^2 + \mu t + \nu$ , with  $a, c > 0$ ,  $\delta := \frac{|b|}{2\sqrt{ac}} < 1$  and not both  $\mu$  and  $\nu$  equal 0. Then

- (i)  $p$  is weakly separated.
- (ii) If  $4c\nu \leq \mu^2$ , then  $p$  is not strongly separated.
- (iii) If  $4(1 - \delta)^2 c\nu > \mu^2$ , then  $p$  is strongly separated.

*Proof.* By Definition 4.1  $p_1(y, t) = ct^2 + ay^2 + \nu$  and  $p_2(y, t) = byt + \mu t$ . To prove (i) note that  $|byt| < \frac{|b|}{2\sqrt{ac}}(ct^2 + ay^2)$ . Since  $\frac{|b|}{2\sqrt{ac}} < 1$ , there exists  $\epsilon > 0$  such that  $\frac{|b|}{2\sqrt{ac}} + \epsilon < 1$ . Choosing  $\tau \geq \frac{\mu^2}{4\epsilon c}$ , we get that  $|\mu t| \leq \epsilon ct^2 + \tau$ . Therefore

$$|p_2(y, t)| \leq \left( \frac{|b|}{2\sqrt{ac}} + \epsilon \right) (ct^2 + ay^2) + \tau = \left( \frac{|b|}{2\sqrt{ac}} + \epsilon \right) p_1(y, t) + \beta$$

for some real number  $\beta$ . Thus  $p$  is weakly separated.

To prove (ii) assume that  $p$  is strongly separated. Then for some  $0 \leq \gamma_1 < 1$  we have  $-\gamma_1(ct^2 + ay^2 + \nu) \leq byt + \mu t$ . With  $y = 0$ , this inequality implies  $\mu^2 - 4\gamma_1^2 c\nu \leq 0$ , and therefore  $\mu^2 < 4c\nu$ .

To prove (iii) assume that  $4(1 - \delta)^2 c\nu > \mu^2$ . Then  $\nu > 0$  and there exists  $\epsilon > 0$  such that  $\mu^2 - 4(1 - \delta - \epsilon)^2 c\nu < 0$ . Consequently  $|\mu t| \leq (1 - \delta - \epsilon)(ct^2 + \nu)$ . Together with the first inequality used in the proof of (i), this yields

$$|byt + \mu t| \leq \delta(ct^2 + ay^2) + (1 - \delta - \epsilon)(ct^2 + ay^2 + \nu) \leq (1 - \epsilon)p_1(y, t).$$

Thus  $p$  is strongly separated.  $\square$

**Lemma 4.3.** Let  $p$  be a nonnegative polynomial in  $n$  variables and let  $p_1$  be the polynomial introduced in Definition 4.1.

- (a) Assume that  $p$  is weakly separated. Then  $p$  does not depend on  $t$  if and only if  $p_1$  does not depend on  $t$ .
- (b) If  $p$  is weakly separated, then the multiplication operators  $M_p$  and  $M_{p_1}$  have the same domain in  $\mathcal{K}$ .
- (c) If  $p$  is strongly separated, then  $p(x) = 0$  if and only if  $p_1(x) = 0$ .
- (d) If  $p$  is strongly separated, then the multiplication operators  $M_p$  and  $M_{p_1}$  have the same range in  $\mathcal{K}$ .

Proof. The statements in (a), (c) follow directly from Definition 4.1. Note that  $p_1$  does not depend on  $t$  if and only if  $a_0 = 0$ . Assume that  $p$  is weakly separated. Then  $p$  and  $p_1$  satisfy the conditions in (1b) of Lemma 2.6 with  $c = \beta + 1 > 0$ . Indeed, the condition (4.2) yields

$$\frac{p_1}{\beta + 1 + p} \leq \frac{1}{1 - \alpha_1} \quad \text{and} \quad \frac{p}{\beta + 1 + p_1} \leq 1 + \alpha_2 + \beta .$$

Therefore (b) follows from Lemma 2.6. If  $p$  is strongly separated, then (4.2), with  $\beta = 0$ , implies

$$\frac{p_1}{p(1 + p_1)} \leq \frac{1}{1 - \alpha_1} \quad \text{and} \quad \frac{p}{p_1(1 + p)} \leq 1 + \alpha_2 ,$$

and (d) is a consequence of Lemma 2.6. □

Denote by  $P$  the operator  $-\frac{d^2}{dt^2}$  in  $L^2(\mathbb{R})$  on  $H^2(\mathbb{R})$ . By [6, Theorem 2.5] (see also Example 3.6) for any  $b \geq 0$  and  $k$  a natural number the operator  $(\operatorname{sgn} t)(P^k + bI)$  is similar to a selfadjoint operator in  $L^2(\mathbb{R})$ .

**Lemma 4.4.** *Let  $p$  be a nonnegative polynomial such that  $p_2 = 0$ . Assume that  $\rho(A) \neq \emptyset$ . Then  $A = Jp(D)$  is similar to a selfadjoint operator in the Hilbert space  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ .*

Proof. Let  $p(x) = q(\hat{x}) + a_0 t^{2k}$ ,  $a_0 \geq 0$ . If  $a_0 = 0$ , the operator  $A$  commutes with the fundamental symmetry  $J$  and consequently  $A$  is similar to a selfadjoint operator in the Hilbert space  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ .

If  $a_0 > 0$  without loss of generality we can assume that  $a_0 = 1$ . Since we assume that  $\rho(A) \neq \emptyset$ , we only have to prove that the points 0 and  $\infty$  are not singular critical points of  $A$ . Let  $y \in \operatorname{dom}(A)$  and  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . It follows from the basic properties of the partial Fourier transform  $F$  that

$$((A - \lambda I)y)(x) = (F^{-1}(JP^k + q(\hat{x})J - \lambda I)Fy)(x). \quad (4.3)$$

Denote by  $E$  the spectral function of  $A$  and by  $G_\alpha$  the spectral function of the operator  $J(P^k + \alpha I)$ . Consider an interval  $\iota = (a, b)$  with  $0 < a < b$ . It follows

from the definition of the spectral function and (4.3) that

$$(E(\iota)y)(x) = (F^{-1}G_{q(\hat{x})}(\iota)Fy)(x) . \quad (4.4)$$

Let  $\alpha > 0$ . The operator  $J(P^k + \alpha I)$  is uniformly positive in the Krein space  $(L^2(\mathbb{R}^n), [\cdot, \cdot])$  and the lower bound of  $P^k + \alpha I$  is  $\alpha$ . Lemma 2.7 implies that the interval  $(-\alpha, \alpha)$ , belongs to the resolvent set of  $J(P^k + \alpha I)$  and consequently  $G_\alpha(\iota) = 0$  for  $b < \alpha$ . Thus, it follows from (4.4) that

$$\|E(\iota)y\|^2 = \int_{q(\hat{x}) \leq b} \|(G_{q(\hat{x})}(\iota)Fy)(\hat{x}, \cdot)\|^2 d\hat{x} . \quad (4.5)$$

Denote by  $U(\delta)$ ,  $\delta \in \mathbb{R} \setminus \{0\}$  the dilation operator:  $(U(\delta)f)(x) = f(\delta x)$ ,  $x \in \mathbb{R}^n$ . Then  $U(\delta)$  is a bounded operator with the bounded inverse  $U(1/\delta)$ . We have

$$\langle U(\delta)f, U(\delta)f \rangle = |\delta|^{-n} \langle f, f \rangle \quad (4.6)$$

and

$$U(\delta)^{-1}P^kU(\delta) = a^{2k}P^k . \quad (4.7)$$

From

$$J(P^k + \alpha I) = \alpha U\left(\alpha^{\frac{1}{2k}}\right) J(P^k + I) U\left(\alpha^{-\frac{1}{2k}}\right) ,$$

it follows that

$$G_\alpha(\iota) = U\left(\alpha^{\frac{1}{2k}}\right) G_1(\iota_\alpha) U\left(\alpha^{-\frac{1}{2k}}\right) , \quad (4.8)$$

where  $\iota_\alpha = \left(\frac{a}{\alpha}, \frac{b}{\alpha}\right)$ . From (4.5) and (4.8) we conclude

$$\|E(\iota)y\|^2 = \int_{q(\hat{x}) \leq b} \left\| \left( U\left(\alpha^{\frac{1}{2k}}\right) G_1(\iota_\alpha) U\left(\alpha^{-\frac{1}{2k}}\right) Fy \right) (\hat{x}, \cdot) \right\|^2 d\hat{x} . \quad (4.9)$$

Since  $U(t)$  is a multiple of an isometry, it follows from the Plancherel theorem that

$$\|E(a, b)\| \leq \sup_{q(\hat{x}) \leq b} \left\| G_1\left(\frac{a}{q(\hat{x})}, \frac{b}{q(\hat{x})}\right) \right\| \leq \sup_{0 < \alpha \leq b} \left\| G_1\left(\frac{a}{\alpha}, \frac{b}{\alpha}\right) \right\| .$$

A similar formula holds for  $a < b < 0$ . Since  $J(P^k + I)$  is similar to a selfadjoint operator in the Hilbert space  $L^2(\mathbb{R})$ , it follows that both 0 and  $\infty$  are not singular critical points of  $A$ .  $\square$

**Corollary 4.5.** *Let  $p$  be a nonnegative polynomial and assume that  $p_2 = 0$ . Then  $A = Jp(D)$  is similar to a selfadjoint operator in the Hilbert space  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ .*

Proof. The polynomial  $p+1$  is strictly positive. The operator  $p(D)+I$  is uniformly positive in  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ . Therefore the operator  $J(p(D)+I)$  is uniformly positive in  $(L^2(\mathbb{R}^n), [\cdot, \cdot])$  and consequently  $0 \in \rho(J(p(D)+I))$ . Lemma 4.4 implies that  $J(p(D)+I)$  is similar to a selfadjoint operator in  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ . By Lemma 2.1 the operator  $Jp(D)$  has a nonempty resolvent set. Applying Lemma 4.4 again yields that  $Jp(D)$  is similar to a selfadjoint operator in  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ .  $\square$

**Theorem 4.6.** *Let  $p$  be a nonnegative polynomial and  $A = Jp(D)$ .*

- (a) *If  $p$  is a weakly separated polynomial, then  $A$  is a positive operator in the Krein space  $L^2(\mathbb{R}^n)$ ,  $\rho(A) \neq \emptyset$  and  $\infty$  is not a singular critical point of  $A$ .*
- (b) *If  $p$  is a strongly separated polynomial, then  $0$  is not a singular critical point of  $A$ . The operator  $A$  is similar to a selfadjoint operator in  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ .*

Proof. If  $p$  does not depend on  $t$  the operator  $A$  commutes with the fundamental symmetry  $J$  and consequently  $A$  is similar to a selfadjoint operator in  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ . By Lemma 4.3 (a)  $p$  does not depend on  $t$  if and only if  $a_0 = 0$ . Thus, in the rest of the proof we can assume that  $p_1(x) = a_0 t^{2k} + q(\hat{x})$ , where  $a_0 > 0$ . Put  $A_1 = Jp_1(D)$ . By Corollary 4.5 the operator  $A_1$  is similar to a selfadjoint operator in  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ . The operator  $A = Jp(D)$  is positive in  $(L^2(\mathbb{R}^n), [\cdot, \cdot])$ . Lemma 4.3 implies that  $\text{dom}(M_p) = \text{dom}(M_{p_1})$ . Applying the inverse Fourier transform we conclude that  $\text{dom}(A) = \text{dom}(A_1)$ . Clearly the operator  $J(p(D)+I) = A+J$  is uniformly positive in  $(L^2(\mathbb{R}^n), [\cdot, \cdot])$  and  $\text{dom}(A+J) = \text{dom}(A) = \text{dom}(A_1)$ . Since  $\infty$  is not a singular critical point of  $A_1$ , [2, Corollary 3.3] implies that  $\infty$  is not a singular critical point of  $A+J$ . Therefore  $A+J = J(B+I)$  is similar to a selfadjoint operator in  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ . Lemma 2.1 implies that  $\rho(A) \neq \emptyset$  and consequently  $A$  is a definitizable operator. Since  $\text{dom}(A+J) = \text{dom}(A)$ , [2, Corollary 3.3] implies that  $\infty$  is not a singular critical point of  $A$ . This proves part (a).

We prove part (b) for a strongly separated polynomial  $p$ . It remains to prove that  $0$  is not a singular critical point of  $A$ . By Lemma 4.3 the ranges of the multiplication operators  $M_p$  and  $M_{p_1}$  coincide. Applying the inverse Fourier transform we conclude that  $\text{ran}(A) = \text{ran}(A_1)$ . Note that  $0$  is not an eigenvalue neither of  $A$  nor of  $A_1$ . Since  $0$  is not a singular critical point of  $A_1$ , we conclude that  $0$  is not a singular critical point of  $A$ . This proves the theorem.  $\square$

**Proposition 4.7.** *Let  $q$  be a nonnegative polynomial in  $n-1$  variables,  $r$  a nonnegative and nonconstant polynomial in one variable and  $p(x) = q(\hat{x}) + r(t)$ . Let  $A = Jp(D)$ . Then:*

- (a) *The operator  $A$  has no eigenvalues.*
- (b) *The spectrum of  $A$  is given by*

$$\sigma(A) = (-\infty, -m_p] \cup [m_p, +\infty) , \quad (4.10)$$

where  $m_p = \min\{p(x) : x \in \mathbb{R}^n\}$ .

Proof. (a) The operator  $A$  is definitizable by Theorem 4.6. Let  $\lambda \in \mathbb{R}$  and  $y \in \text{dom}(A)$  satisfy  $Jp(D)y = \lambda y$ . Let  $z = Fy$  be the partial Fourier transform of  $y$ . Then

$$J \left( r \left( \frac{1}{i} \frac{d}{dt} \right) + q(\hat{x})I \right) z(\hat{x}, t) = \lambda z(\hat{x}, t) .$$

[6, Theorem 2.2 (b)] implies that  $z(\hat{x}, \cdot) = 0$  for all  $\hat{x} \in \mathbb{R}^{n-1}$ . Thus  $y = 0$ .

To prove (b) we extend the argument of Lemma 4.4. Denote by  $E$  the spectral function of  $A$  and by  $G_\alpha$  the spectral function of the operator  $J(r(-i\frac{d}{dt}) + \alpha I)$ . The equalities (4.4) and (4.5) hold true for newly defined  $G_\alpha$ .

We prove that for all positive  $a, b$  such that  $b > m_p$  and  $a < b$  we have  $E(a, b) \neq 0$ . Note that  $m_p = m_r + m_q$ . Let  $\hat{x}_0$  be such that  $m_r + q(\hat{x}_0) < b$ . By [6, Theorem 2.2] the spectrum of the operator  $J(r(-i\frac{d}{dt}) + q(\hat{x}_0)I)$  is  $(-\infty, -m_r - q(\hat{x}_0)] \cup [m_r + q(\hat{x}_0), +\infty)$ . Therefore there exists  $h \in L^2(\mathbb{R})$  such that  $G_{q(\hat{x}_0)}(a, b)h \neq 0$ . The function  $\alpha \mapsto \|G_\alpha(a, b)h\|$  is continuous on  $\mathbb{R}_+$  by [18, Theorem 3.1. part 3)]. Therefore the function

$$\hat{x} \mapsto \|G_{q(\hat{x})}(a, b)h\|$$

is continuous on  $\mathbb{R}^{n-1}$ . Hence the set

$$\mathcal{O} = \{\hat{x} \in \mathbb{R}^{n-1} : \|G_{q(\hat{x})}(a, b)h\| > 0\}$$

is open. This set is nonempty since  $\hat{x}_0 \in \mathcal{O}$ . The set  $\mathcal{O}$  is contained in  $\{\hat{x} \in \mathbb{R}^{n-1} : q(\hat{x}) \leq b\}$ . Choose  $z \in L^2(\mathbb{R}^{n-1})$  such that  $z \neq 0$  almost everywhere. Let  $y(x) = h(t)(F^{-1}z)(\hat{x})$ . From (4.5) it follows

$$\begin{aligned} \|E(a, b)y\|^2 &= \int_{q(\hat{x}) \leq b} |z(\hat{x})|^2 \|G_{q(\hat{x})}(a, b)h\|^2 d\hat{x} \\ &\geq \int_{\mathcal{O}} |z(\hat{x})|^2 \|G_{q(\hat{x})}(a, b)h\|^2 d\hat{x} > 0 . \end{aligned}$$

We have proved that for arbitrary  $b > m_p$  and  $0 < a < b$  we have  $E(a, b) \neq 0$ .

This implies that the spectrum of  $A$  in  $\mathbb{R}_+$  contains  $[m_p, +\infty)$ . If  $m_p > 0$  and  $0 < \lambda < m_p$ , then (4.5) implies that  $\lambda \in \rho(A)$ . In this case  $0 \in \rho(A)$  since  $A$  is a uniformly positive operator. Therefore the spectrum of  $A$  in  $\mathbb{R}_+$  coincides with  $[m_p, +\infty)$ . Similarly one proves that the spectrum of  $A$  in  $\mathbb{R}_-$  coincides with  $(-\infty, -m_p]$ .  $\square$

**Corollary 4.8.** *Let  $q$  be a nonnegative polynomial in  $n - 1$  variables,  $r$  a nonnegative and nonconstant polynomial in one variable and  $p(x) = q(\hat{x}) + r(t)$ . Let  $A = Jp(D)$ .*

(a) *The point  $\infty$  is a regular critical point of  $A = Jp(D)$ .*

(b) *Assume that the polynomial  $r$  has at most one root. The following statements are equivalent:*

- (i)  *$p$  has a zero.*
- (ii)  $0 \in \sigma(A)$ .
- (iii)  $0$  *is a regular critical point of  $A$ .*

## 5. Variable coefficients<sup>1</sup>

In this section we use Corollary 2.5 to extend results from Section 4. To illustrate the method, we consider the Schrödinger operator with indefinite weight  $(\operatorname{sgn} x_n)(-\Delta + q)$  on  $\mathbb{R}^n$ .

Let  $H = -\Delta$  be defined on its natural domain in  $L^2(\mathbb{R}^n)$ . Its inverse is an unbounded integral operator.

**Proposition 5.1.** *Let  $5 \leq n \leq 8$  and  $q \in L^{n/2}(\mathbb{R}^n)$ . There exists  $\kappa_0 > 0$  such that for all real  $\kappa$  with  $|\kappa| < \kappa_0$  the operator  $(\operatorname{sgn} x_n)(-\Delta + \kappa q)$  is similar to a selfadjoint operator in  $L^2(\mathbb{R}^n)$ .*

*Proof.* Since  $n \leq 8$ , it follows from the Sobolev embedding theorem that

$$\operatorname{dom}(H) = H^2(\mathbb{R}^n) \subseteq \operatorname{dom}(q) .$$

We show that the operator  $qH^{-1}$  is bounded by a constant multiple of  $\|q\|_{n/2}$ .

Note that  $H^{-1} = h(-i\nabla)$  with  $h(x) = |x|^{-2}$ . Therefore  $h \in L_w^{n/2}(\mathbb{R}^n)$ , see [21, Example IX.4.2]. By [22, Theorem 4.2]  $q(x)h(-i\nabla) \in L_w^{n/2}(\mathbb{R}^n)$ , and moreover

$$\|q(x)h(-i\nabla)\|_{n/2,w} \leq C\|q\|_{n/2}\|h\|_{n/2,w} ,$$

where  $\|\cdot\|_{p,w}$  are the functions defined in [22, p. 13] and [21, Definition IX.4]. Hence (see [22, p. 13])  $\|q(x)h(-i\nabla)\|_{n/2,w} \leq C_1\|q\|_{n/2}$ . Next we can use the inequalities on p. 13 of [22] to conclude

$$\|q(x)h(-i\nabla)\| = \|q(x)h(-i\nabla)\|_\infty \leq C_2\|q(x)h(-i\nabla)\|_{n/2,w} \leq C_3\|q\|_{n/2} .$$

It follows from [16, Theorems IV.1.1, IV.2.14, IV.3.1 and VI.3.1] that for  $|\kappa|$  sufficiently small we have that the operator  $J(H + \kappa q)$  is positive in  $(L^2(\mathbb{R}^n), [\cdot, \cdot])$ ; the resolvent set  $\rho(J(H + \kappa q))$  is nonempty and  $\operatorname{dom}(J(H + \kappa q)) = \operatorname{dom}(H)$ . The

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<sup>1</sup>For the case of a more general elliptic operator with indefinite weight  $\frac{1}{r}(L + q)$  we can use the results from [9].

conclusion of the proposition follows from Theorem 4.6, Lemma 2.2, Corollary 2.5 and [2, Corollary 3.3].  $\square$

Note that we needed  $n \leq 8$  only to prove that the operator  $qH^{-1}$  is densely defined. However, the Gagliardo-Nirenberg inequality implies that  $\text{dom}(H) \subseteq \text{dom}(q)$  (and also that  $qH^{-1}$  is bounded) as soon as  $n \geq 5$ . This shows that the assumption  $n \leq 8$  is in fact redundant.

We prove a strengthening of Proposition 5.1 .

**Theorem 5.2.** *Let  $n \geq 5$  and*

$$B = \sum_{0 \leq i+j \leq 2} b_{ij} D^{ij}$$

*be a partial differential operator with the coefficients  $b_{ij}$  satisfying*

$$b_{ij} \in L^\infty \quad \text{if } i+j=2, \quad b_{ij} \in L^n(\mathbb{R}^n) \quad \text{if } i+j=1, \quad \text{and } b_{00} \in L^{n/2}(\mathbb{R}^n).$$

*Further assume that  $B$  is symmetric in  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ .*

*Then the operator  $B$ , defined on  $\text{dom}(B) = H^2(\mathbb{R}^n)$  is a closed operator in  $L^2(\mathbb{R}^n)$ . There exists  $\kappa_0$  such that if*

$$\sum_{1 \leq i \leq n} \|b_{ii} - \text{sgn } x_n\|_\infty + \sum_{i+j=2, j \neq i} \|b_{ij}\|_\infty + \sum_{i+j=1} \|b_{ij}\|_n + \|b_{00}\|_{n/2} \leq \kappa_0,$$

*then  $(\text{sgn } x_n)B$  is similar to a selfadjoint operator in  $(L^2(\mathbb{R}^n), \langle \cdot, \cdot \rangle)$ .*

*Proof.* The first statement easily follows from the Sobolev embedding theorem. Let  $A_0 = (\text{sgn } x_n)(-\Delta) = JH$ , defined on  $\text{dom}(A_0) = \text{dom}(B)$ ,  $V = JB - A_0$ . By Theorem 4.6 the operator  $A_0$  is similar to a selfadjoint operator in  $L^2(\mathbb{R}^n)$ . Note that

$$JV = \sum_{0 \leq i+j \leq 2} v_{ij} D^{ij}$$

with  $v_{ii} = b_{ii} - \text{sgn } x_n$ ,  $1 \leq i \leq n$ ,  $v_{ij} = b_{ij}$  for all other  $i, j$ . It is sufficient to show that  $JVA_0^{-1}$  or equivalently  $VH^{-1}$  is a bounded densely defined operator. To this end, we show that  $v_{ij}D^{ij}H^{-1}$  is bounded and densely defined. In fact, it is sufficient to show that

$$\|v_{ij}D^{ij}u\| \leq C_{ij}\|\Delta u\| \quad u \in H^2(\mathbb{R}^n) \quad (5.1)$$

for all  $i, j$  with  $i+j \leq 2$ . If  $i+j=2$ , the estimate (5.1) is evident. If  $i=j=0$ , let  $\frac{1}{p} = \frac{1}{2} - \frac{2}{n}$ . Hölder's inequality yields

$$\|qu\|_2 \leq \|q\|_{n/2}\|u\|_p \quad (u \in L^p(\mathbb{R}^n)),$$

From the Gagliardo-Nirenberg inequality, see [20, p. 125], it follows that

$$\|u\|_p \leq C\|\Delta u\|_2 \quad (u \in H^2(\mathbb{R}^n)) .$$

This implies (5.1) if  $i + j = 0$ . Finally, if  $i + j = 1$ , then  $v_{ij}D^{ij} = b_k D^k$  for some  $k \in \{1, \dots, n\}$ , where  $b_k = b_{k0}$  or  $b_{0k}$ . Let  $p = \frac{2n}{n-2}$ . From Hölder's and Gagliardo-Nirenberg inequality we again find

$$\|v_{ij}D^{ij}u\| = \|b_k D^k u\| \leq \|b_k\|_n \|D^k u\|_p \leq C\|b_k\|_n \|\Delta u\| ,$$

and this proves (5.1). □

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