Discreteness of the Spectrum of Second-Order Differential Operators and Associated Embedding Theorems

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Necessary and sufficient conditions and also simple sufficient conditions are given for the self-adjoint operators associated with the second-order linear differential expression

$$\tau(y) = \frac{1}{w}\left(-(py')' + qy\right)$$

on $[a,b)$ to have discrete spectrum. Here the coefficients of τ are non-negative and satisfy minimal smoothness conditions. These results follow from compact embedding theorems from a weighted one-dimensional Sobolev space with norm $(\int_a^b (p|f'|^r + q|f|^r))^{1/r}$ into a weighted Banach space with norm $(\int_a^b w|f|^s)^{1/s}$. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

We will give a new set of conditions necessary and sufficient for the self-adjoint operators associated with the second-order linear differential expression

$$\tau(y) = \frac{1}{w}\left(-(py')' + qy\right) \tag{1}$$

to have a discrete spectrum on the interval $[a,b)$, $-\infty < a < b \leqslant +\infty$. A simplified version of these conditions also serves as a rather effective criterion for a discrete spectrum which can be quite easy to verify.

It turns out to be no more difficult to present our results in terms of embedding theorems between weighted spaces, from which the spectral theory results then follow immediately. Thus, we will give first sufficient, and then necessary and sufficient conditions for the identity map from the

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weighted Sobolev space $W_{p,q}^{1,r}(a,b)$ with norm

$$\|f\|_{p,q} = \left(\int_a^b (p|f'|^r + q|f|^r) \right)^{1/r} \tag{2}$$

into the weighted Banach space $L_w^s(a,b)$ with norm $\|f\|_{w,s} = (\int_a^b w|f|^s)^{1/s}$ to be continuous. We also estimate its measure of non-compactness in terms of a quantity depending on p, q, and w. Here $1 < r \leqslant s < +\infty$, $\frac{1}{r} + \frac{1}{r'} = 1$ and p, q, w are non-negative functions such that $w > 0$ a.e. and

$$w, p^{1-r'}, q \in L_{\text{loc}}^1[a,b]. \tag{3}$$

We recall that when $r = s = 2$, there is a minimal operator T_m associated with τ whose domain is the closure in the weighted Hilbert space $L_w^2(a,b)$ of all compactly supported, locally absolutely continuous functions f for which $\tau(f) \in L_w^2(a,b)$. We assume that the differential expression τ is singular at b. The hypotheses imply that it is otherwise regular. The self-adjoint extensions of T_m are all bounded below, all have the same essential spectrum, and have a discrete spectrum if and only if this essential spectrum is empty. The property of having a discrete spectrum may therefore be associated with T_m rather than with any particular self-adjoint operator. For a discussion of differential operators with these weak smoothness assumptions on the coefficients, see [12, Chapt. V]. For details concerning the essential spectrum, see the classical treatment by Glazman [6].

The connection between the embedding theorems and the spectral theoretic results when $r = s = 2$ can be made in several ways. One is to recall the definition of the Friedrichs extension, T_F, of T_m in terms of quadratic forms (see [9, VI.3] or [4, IV.2]) from which it is clear that $(T_F f, f) = \|f\|_{p,q}$ for all f in the domain of T_F. It is then clear that the essential spectrum of T_F is bounded away from the origin if and only if there is an embedding, and Rellich's theorem [16] (see also [12, Sect. 24.5]) states that the essential spectrum is empty if and only if the set $\{f : (T_F f, f) \leqslant 1\}$ has compact closure in $L_w^2(a,b)$, that is, if and only if the embedding is compact. (For a similar argument, see [4, IV.2.9 and remarks preceding VIII.4.1].) Another route is to remember the "decomposition principle" that properties of the essential spectrum depend only on the behavior of the Dirichlet integral (2) for $r = 2$ in arbitrarily small neighborhoods of the singular point b (see [6, Sects. 7 and 10]) and to observe that the proofs of the principal embedding theorems below (Theorems 4.1, 5.1 and 5.2) amount to establishing the necessary inequalities.

Many authors have given sufficient conditions for the self-adjoint operators associated with (1) to have discrete spectrum on $[a,b)$ under

various restrictions on the coefficients. The earliest criterion, due to Weyl [17] for the special case $p(x) = w(x) = 1$ on the interval $[a, +\infty)$, states that the spectrum is discrete if $q(x) \to +\infty$ as $x \to +\infty$. See [8, 10, 15] for some representative relatively recent work which considers the behavior of certain averages of the coefficients on a family of compact subintervals. This more recent work has the character of prescribing that $q(x)$ grow large in some suitable sense as x approaches the singular endpoint. (This means large compared to $w(x)$ if w is not identically equal to 1.) In this spirit, the following simplified version of our conditions will be shown to be sufficient for a discrete spectrum in Section 4, and will be shown there by example to be quite effective in cases when discreteness arises from the rapid growth of q:

Each self-adjoint extension of T_{m} has discrete spectrum if for each $x \in [a, b)$ there is a bounded interval $I(x)$ centered at x such that

$$\left(\int_{I(x)} p^{-1} + \left(\int_{I(x)} q \right)^{-1} \right) \int_{I(x)} w \to 0 \quad \text{as } x \to b.$$

In the special case $p(x) = w(x) = 1$, it turned out that a rather simple and natural extension of the Weyl condition was, in fact, both necessary and sufficient for a discrete spectrum. In this situation, Molchanov [11] (see also [6, Sect. 28]) showed that the condition

$$\text{for each } \theta > 0, \qquad \int_x^{x+\theta} q \to +\infty \quad \text{as } x \to b \qquad (4)$$

is necessary and sufficient for the self-adjoint extensions of T_{m} to have discrete spectrum.

Much more recently, Oinarov and Otelbaev [13] have given necessary and sufficient conditions for the self-adjoint operators associated with the general second-order expression (1) to have discrete spectrum. This work is also phrased in terms of necessary and sufficient conditions for the embedding mentioned above and for it to be compact. Unfortunately, their conditions are complex and rather difficult to verify in specific cases. They are also, at least superficially, quite different in character from (4). The basic idea is to use the coefficients p and q to define a family of intervals $I(x)$ of "unit length" with $I(x)$ centered at x in a suitable sense. The condition is then, very roughly, that on such intervals I, $(\int_I w)/(\int_I q)$ should approach 0 as $x \to b$. However, in order to make the condition necessary as well as sufficient, Oinarov and Otelbaev replace this simple quotient with a complicated expression involving integrals over several linked intervals and a supremum over the points $t \in I$.

One reason for this complication is that if the leading coefficient p and the weight function w are also allowed to vary, then other possible routes for discreteness of the spectrum (or compactness of the embedding) appear—for instance, it is also possible for the spectrum to be discrete even when q is comparable to w provided $p(x)$ increases sufficiently fast as $x \to b$. Thus a condition which is necessary as well as sufficient must be flexible enough to take into account rapid growth by either p or q or some combination of the two. Few of the many published criteria for discrete spectrum have this dual ability, although the well-known criterion of Friedrichs [5], for instance, does have this character.

While our conditions have some similarity to that of Oinarov and Otelbaev, they differ in two respects likely to make them easier to apply in a specific situation. First, instead of prescribing a particular family of intervals to use, considerable latitude is allowed in choosing a family. Several examples illustrating this process are given in Sections 4 and 6. Second, we are able to avoid the calculation of a supremum for each individual interval. The combined effect is often to allow rather loose estimates to suffice to demonstrate properties of the spectrum (or the embedding).

The key ingredient in allowing a fairly free choice of intervals will be a variant of the Besicovitch covering theorem [2]. This is developed in Section 2. In Section 3, we review briefly necessary information about the ball measure of non-compactness of a map between Banach spaces. We establish our first embedding theorems in Section 4 (Theorem 4.1 and Corollary 4.1) and thus obtain as Corollary 4.2, a simple condition sufficient for a discrete spectrum. More refined results on embeddings (Theorems 5.1 and 5.2 and Corollary 5.2) and discreteness of spectrum (Corollaries 5.1 and 5.3) are proved in Section 5. Two examples that illustrate their use are discussed in Section 6.

2. THE BESICOVITCH COVERING THEOREM

We will need a variant of the Besicovitch covering theorem [2] (see also [7]) for a family of bounded intervals contained in a not necessarily bounded subset of the line and for which x does not necessarily lie near the center of $I(x)$. We begin by stating a one-dimensional version of the theorem as it appears in [7, Theorem 1.1, Remark 4].

THEOREM 2.1. (Besicovitch [2]). *There are fixed positive integers θ and ξ such that for any bounded subset A of $\mathbb{R}$ and any family $\{I(x): x \in A\}$ of closed intervals with x in the middle third of $I(x)$, one can choose from $\{I(x): x \in A\}$ a*

sequence $\{I_k\}$, *possibly finite, such that*

   (i) $A \subset \bigcup_k I_k$,
   (ii) *no point of* $\mathbb{R}$ *is in more than* θ *intervals of* $\{I_k\}$,
   (iii) *the sequence* $\{I_k\}$ *can be partitioned into* ξ *families of pairwise disjoint intervals.*

We will need to allow A to be unbounded, and also to loosen the requirement that x lie in the middle third of $I(x)$. We deal with these extensions one at a time, beginning with possibly un-bounded A.

THEOREM 2.2.   *Let* $A = [a, b)$, $-\infty < a < b \leqslant +\infty$, *be an interval in* $\mathbb{R}$. *Let* $\{I(x): x \in A\}$ *be any family of closed, bounded intervals with x in the middle third of* $I(x) = [x^-, x^+]$. *If $b = +\infty$, suppose that $x^- \to +\infty$ as $x \to +\infty$. Then there is a sequence* $\{I_k\}$, *possibly finite, such that*

   (i) $A \subset \bigcup_k I_k$,
   (ii) *no point of* $\mathbb{R}$ *is in more than* 3θ *intervals of* $\{I_k\}$,
   (iii) *the sequence* $\{I_k\}$ *can be partitioned into* 2ξ *families of pairwise disjoint intervals.*

Proof.   We may assume that $b = +\infty$. Partition A into a family $\{A_k\}$ of bounded intervals with common endpoints as follows. Set $A_1 = [a, a + 1]$. If $A_n = [a_n, b_n]$ has been chosen, set $x_{n+1} = \sup\{x: x^- \leqslant b_n^+\}$. Since $x^- \to +\infty$ as $x \to +\infty$, x_{n+1} is finite. Set $a_{n+1} = b_n$, $b_{n+1} = x_{n+1} + 1$. It is then clear that if $x \in A_m$, $y \in A_n$, $n \geqslant m + 2$, then $I(x) \cap I(y) = \emptyset$. From Theorem 2.1 we can select for each k a sequence $\{I_{k,n}\}$ of intervals covering A_k. The union of these sequences covers A, and it is clear from the construction that if $x \in A_k$, then x can lie in intervals from at most the three sequences $\{I_{k-1,n}\}$, $\{I_{k,n}\}$, and $\{I_{k+1,n}\}$. This establishes (i) and (ii). Finally, since $I_{k,n} \cap I_{k+2,m} = \emptyset$, the intervals from the sequences $\{I_{k,n}\}$ for $k = 1, 3, 5, \ldots$, and for $k = 2, 4, 6, \ldots$, can each be partitioned into ξ families of pairwise disjoint intervals.   ∎

Next, we wish to loosen somewhat the requirement that x lie in the middle third of $I(x)$.

DEFINITION 2.1.   Let A be an interval in $\mathbb{R}$. A family $\{I(x): x \in A\}$ of bounded closed intervals $I(x) = [x^-, x^+]$ is *centralizable* if $x^- < x < x^+$ for each $x \in A$ and if there is an increasing, continuous function f such that for each $x \in A$, $f(x)$ is in the middle third of the interval $f[I(x)]$.

COROLLARY 2.1. *Let $A = [a, b), -\infty < a < b \leqslant +\infty$, be an interval in $\mathbb{R}$, and let $\{I(x): x \in A\}$ be a centralizable family of closed, bounded intervals. Then, the conclusion of Theorem 2.2 holds for $\{I(x): x \in A\}$.*

Proof. There is an increasing, continuous function f that transforms the family $\{I(x): x \in A\}$ to a family $\{J(t): t \in B\}$, $B = f[A]$, to which Theorem 2.2 applies. Thus, there is a subfamily $\{J_k\}$ with properties (i)–(iii) relative to B. But f preserves all of the intersection properties of intervals. Thus, the subfamily $I_k = f^{-1}[J_k]$ also has properties (i)–(iii) relative to A. ∎

Remark 2.1. Of course, a family is centralizable if x is in the middle third of $I(x)$. Another possibility is that the change of variables $t = \int_a^x p^{-1}$ acts on the family $I(x) = [x^-, x^+]$ to produce a new family $[t^-, t^+]$ with $t^+ - t = t - t^-$, that is, $\int_{x^-}^x p^{-1} = \int_x^{x^+} p^{-1}$. Here p^{-1} could be replaced by any positive locally integrable function g defined on A. As yet another alternative, the function $f(x) = 1 - 1/(1 + \int_a^x w)$ will be used for one of the examples in Section 6.

3. MEASURES OF NON-COMPACTNESS

We list some definitions and results that will be needed below. For more details see [4].

DEFINITION 3.1. Let A be a bounded subset of a Banach space X. Then,

$$\tilde{\psi}(A) = \inf\{\delta > 0: A \text{ has a finite cover by balls of radius } \delta\}.$$

Remark 3.1. The closure of A is compact if and only if $\tilde{\psi}(A) = 0$.

Remark 3.2. If X is infinite dimensional and B_1 is the closed unit ball in X, then $\tilde{\psi}(B_1) = 1$. (Of course, if $\dim X < +\infty$, then $\tilde{\psi}(A) = 0$ for all bounded A.)

DEFINITION 3.2. Let X and Y be Banach spaces, and let $T: X \to Y$ be a bounded linear map. Then,

$$\tilde{\beta}(T) = \tilde{\beta}_{X,Y}(T)$$
$$= \sup\{\tilde{\psi}_Y(T[A]): A \text{ bounded}, \ \tilde{\psi}_X(A) = 1\}$$
$$= \tilde{\psi}_Y(T[B_1]).$$

Remark 3.3. The last equality in the definition is valid because T is a linear map.

Remark 3.4. $\tilde{\beta}(T) \leqslant \|T\|.$

Remark 3.5. T is compact if and only if $\tilde{\beta}(T) = 0.$

Remark 3.6.   If S is compact and T is bounded, $\tilde{\beta}(S + T) = \tilde{\beta}(T).$

The last remark will be used in connection with the following theorem. Although this result is well known, it is difficult to find a statement in this form in the literature. Since the proof has several elements in common with arguments from later sections, we sketch it briefly here. Here and subsequently we will write r' for the exponent conjugate to r, $\frac{1}{r} + \frac{1}{r'} = 1.$

THEOREM 3.1.   *Let* $[a, b]$ *be a compact interval in* $\mathbb{R}$, *let* $1 < r < +\infty$, $1 \leqslant s < +\infty$, *and let* $p, q, w,$ *be non-negative functions such that* $p^{1-r'}, q, w \in L^1[a, b]$, $\int_a^b q > 0$, *and* $w > 0$ *a.e. Then* $W_{p,q}^{1,r}[a, b]$ *is compactly embedded in* $L_w^s[a, b]$.

Proof.   For any $f \in W_{p,q}^{1,r}[a, b]$, if $|f(c)| = \min\{|f(x)|: x \in [a, b]\}$, then

$$|f(c)| \left(\int_a^b q \right)^{1/r} \leqslant \left(\int_a^b q|f|^r \right)^{1/r} \leqslant \|f\|_{p,q}.$$

Here $\| \cdot \|_{p,q}$ denotes norm (2). Also, for any $x, y \in [a, b]$,

$$|f(x) - f(y)| \leqslant \int_x^y |f'|$$
$$\leqslant \left(\int_x^y p^{1-r'} \right)^{1/r'} \left(\int_x^y p|f'|^r \right)^{1/r}$$
$$\leqslant \left(\int_x^y p^{1-r'} \right)^{1/r'} \|f\|_{p,q}.$$

It follows that the functions in the unit ball of $W_{p,q}^{1,r}[a, b]$ are uniformly bounded and (by a standard measure theory result) uniformly equicontinuous. The result then follows from Ascoli's theorem.   ∎

4. AN UPPER BOUND AND A SUFFICIENT CONDITION

In this section, we establish an upper bound for $\tilde{\beta}(E)$, with E the identity map $E: W_{p,q}^{1,r}(a, b) \rightarrow L_w^s(a, b)$, valid for $1 < r \leqslant s < +\infty$. Specialized to the case $r = s = 2$, this yields in Corollary 4.2 a simple but surprisingly effective

condition sufficient for discreteness of the spectrum of self-adjoint extensions of T_m. We conclude with an example that illustrates its power.

THEOREM 4.1.   *Let p, q, w be as in (3), and let $1 < r \leqslant s < +\infty$. Let $a < c < b$ and   let $\{I(x): x \in [c, b)\}$, $I(x) = [x^-, x^+]$, $a < x^- < x < x^+ < b$,   be   a   centralizable family of bounded intervals such that $x^- \to b$ as $x \to b$. For $x \in [c, b)$ define*

$$U_{r,s}(x) = \left(\int_{I(x)} w \right)^{1/s} \left[\left(\int_{I(x)} p^{1-r'} \right)^{1/r'} + \left(\int_{I(x)} q \right)^{-1/r} \right]. \qquad (5)$$

If $\mathscr{U}_{r,s} = \limsup_{x \to b} U_{r,s}(x)$ is finite, then the identity map $E: W^{1,r}_{p,q}[a, b] \to L^s_w[a, b]$ is a continuous embedding, and $\tilde{\beta}(E) \leqslant (2\xi)^{1/r} \mathscr{U}_{r,s}$, where ξ is as in Theorem 2.2.

Proof.   Let χ_A denote the characteristic function of the set A. For any $a < t < b$, we can write $E(f) = S_t(f) + T_t(f)$, where $S_t(f) := f\chi_{(a,t)}$, $T_t(f) := f\chi_{[t,b)}$. From Remarks 3.4 and 3.6 and Theorem 3.1, $\tilde{\beta}(E) = \tilde{\beta}(T_t) \leqslant \|T_t\|$. Thus it suffices to show that for any $\varepsilon > 0$ there is t such that

$$\left(\int_t^b |f|^s w \right)^{1/s} \leqslant (2\xi)^{1/r} (\mathscr{U}_{r,s} + \varepsilon) \left(\int_t^b (p|f'|^r + q|f|^r) \right)^{1/r}.$$

Given ε, choose $t \in [c, b)$ so that for $x \geqslant t$, $U_{r,s}(x) < \mathscr{U}_{r,s} + \varepsilon$. For any such x, any f supported on (t, b), and any $y \in I(x)$, it follows as in the proof of Theorem 3.1 that

$$|f(y)| \leqslant \left[\left(\int_{I(x)} p^{1-r'} \right)^{1/r'} + \left(\int_{I(x)} q \right)^{-1/r} \right] \left(\int_{I(x)} (p|f'|^r + q|f|^r) \right)^{1/r}.$$

Since the right-hand side of this inequality is independent of y, computing the norm of f in $L^s_w(I(x))$ yields

$$\left(\int_{I(x)} |f|^s w \right)^{1/s} \leqslant (\mathscr{U}_{r,s} + \varepsilon) \left(\int_{I(x)} (p|f'|^r + q|f|^r) \right)^{1/r}.$$

Now we must obtain a similar inequality on the interval (t, b). By Theorem 2.2, there is a sequence $\{I_k\}$ chosen from the $\{I(x): x \in [t, b)\}$ that cover $[t, b)$ with the property that they can be partitioned into 2ξ families of

pairwise disjoint intervals. Thus,

$$\int_t^b |f|^s w \leqslant \sum_k \int_{I_k} |f|^s w < (\mathscr{U}_{r,s} + \varepsilon)^s \sum_k \left(\int_{I_k} (p|f'|^r + q|f|^r) \right)^{s/r}$$

$$\leqslant 2\xi(\mathscr{U}_{r,s} + \varepsilon)^s \left(\int_t^b (p|f'|^r + q|f|^r) \right)^{s/r}. \qquad \blacksquare$$

Remark 4.1.   Clearly, $U_{r,s}(x)$ in Theorem 4.1 depends on the choice of family $\{I(x)\}$. Indeed, we will see in the example at the end of this section that the finiteness of $\mathscr{U}_{r,s}$ can depend on the choice of $\{I(x): x \in [c, b)\}$.

COROLLARY 4.1.   *Let* p, q, w *be as in* (3) *and let* $1 < r \leqslant s < +\infty$. *Let* $a < c < b$ *and let* $\{I(x): x \in [c, b)\}$, $I(x) = [x^-, x^+]$, $a < x^- < x < x^+ < b$, *be a centralizable family of bounded intervals such that* $x^- \to b$ *as* $x \to b$. *Then,*

$$\left(\int_{I(x)} w \right)^{1/s} \left[\left(\int_{I(x)} p^{1-r'} \right)^{1/r'} + \left(\int_{I(x)} q \right)^{-1/r} \right] \to 0$$

as $x \to b$, *implies that* $W_{p,q}^{1,r}[a, b)$ *is compactly embedded in* $L_w^s[a, b)$.

Proof.   This follows immediately from Remark 3.5. $\blacksquare$

Specializing to $r = s = 2$, we have the following spectral result.

COROLLARY 4.2.   *Let* p, q, w *be as in* (3), *and let* $1 < r \leqslant s < +\infty$. *Let* $a < c < b$ *and let* $\{I(x): x \in [c, b)\}$, $I(x) = [x^-, x^+]$, $a < x^- < x < x^+ < b$, *be a centralizable family of bounded intervals such that* $x^- \to b$ *as* $x \to b$. *Then,*

$$\mathbf{U}^2 = \limsup_{x \to b} \left[\left(\int_{I(x)} p^{-1} \right) + \left(\int_{I(x)} q \right)^{-1} \right] \int_{I(x)} w < +\infty,$$

implies that the essential spectrum of T_{m} *is contained in* $[(8\xi\mathbf{U}^2)^{-1}, +\infty)$. *If*

$$\left[\left(\int_{I(x)} p^{-1} \right) + \left(\int_{I(x)} q \right)^{-1} \right] \int_{I(x)} w \to 0, \qquad (6)$$

then each self-adjoint extension of T_{m} *has discrete spectrum.*

Proof.   Note that $\frac{1}{2} \mathscr{U}_{2,2} \leqslant \mathbf{U} \leqslant \mathscr{U}_{2,2}$. $\blacksquare$

Remark 4.2.   Note that unlike Molchanov's discrete spectrum criterion or any of the results in [8, 10, 15], there is a single family of intervals rather than a parametrized collection of families where the parameter must be allowed to approach 0. (However, it is easy to see that a family satisfying (6) can be chosen if Molchanov's condition (4) is satisfied, so that Corollary 4.2 contains the sufficiency part of his criterion.) A second difference, organizing the intervals according to their "centers" rather than their left endpoint, will be discussed in connection with the following example.

EXAMPLE 4.1.   Consider the differential expression on $[0, +\infty)$ with $p = 1$ and q and w defined on $[n, n + 1)$ for $n = 0, 1, \ldots$, by

$$
q(x) = \begin{cases} n, & n \leqslant x < n + 1 - (n + 1)^{-2}, \\ (n + 1)^4, & n + 1 - (n + 1)^{-2} \leqslant x < n + 1, \end{cases}
$$

$$
w(x) = \begin{cases} n^{4-\varepsilon}, & n \leqslant x < n + n^{-2}, \\ 1, & n + n^{-2} \leqslant x < n + 1. \end{cases}
$$

Here it is approximately true that $q(x)/w(x) \to +\infty$ as $x \to +\infty$, but the intervals where q and w are large are adjacent rather than identical, so it is not quite true. It will be clear from the construction to follow that we could actually separate the intervals where q and w are large, say by setting $q(x) = (n + 1)^4$ on

$$
n + 1 - (k + 1)(n + 1)^{-2} \leqslant x < n + 1 - k(n + 1)^{-2},
$$

without disturbing the discreteness of the spectrum.

To choose the family $\{I(x)\}$ that demonstrates this, if $n \leqslant x \leqslant n + n^{-2}$, $n \geqslant 1$, choose $I(x)$ to be centered at x with left endpoint $x^- = n - n^{-2}$. Then, $(\int_I w)/(\int_I q) \sim n^{-\varepsilon}$, $\int_I p^{-1} \int_I w = |I| \int_I w \sim 2n^{-\varepsilon}$. (Here and in what follows $|I|$ denotes the length of the interval I.) If $n + n^{-2} < x < n + 1$, choose $I(x)$ to be centered at x with length at most $\sqrt{n}$ and contained in the interval where $w(x) = 1$. Then $(\int_I w)/(\int_I q) \leqslant n^{-1}$, $\int_I p^{-1} \int_I w \leqslant n^{-1}$. Thus, it follows from Corollary 4.2 that each self-adjoint extension of T_{m} has discrete spectrum.

Remark 4.3.   Note that the ease of use of Corollary 4.2 derives in part from the possibility of choosing each interval $I(x)$ individually, without reference to the intervals at other points. We will have to compromise this independence somewhat in the next section to obtain more refined criteria.

EXAMPLE 4.2.   To demonstrate that the finiteness of $\mathcal{U}_{2,2}$ depends on the choice of $\{I(x)\}$, consider the family $\{J(x)\}$ defined for Example 4.1 so that x

is the center of $J(x)$, and so that $|J(x)| \int_{J(x)} q = 1$ for each x. Then,

$$U_{2,2}^2(x) = \int_{J(x)} w \left[|J(x)|^{1/2} + \left(\int_{J(x)} q \right)^{-1/2} \right]^2$$

$$= 4|J(x)| \int_{J(x)} w.$$

Let $x_n = n + 1/2\sqrt{n}$ for each n. Then $J(x_n) = [n, n + 1/\sqrt{n}]$, and $\int_{J(x_n)} w \geqslant n^{2-\varepsilon}$ so that

$$|J_n(x)| \int_{J_n(x)} w \geqslant n^{3/2-\varepsilon} \to +\infty \qquad \text{as } n \to +\infty.$$

Remark 4.4. It appears that Example 4.1 is not covered by any of the results in [8, 10] or [15]. The crucial difference is that although each of these results also considers averaged behavior on a family of intervals, the families of intervals are organized by their left endpoints x rather than by their centers as here. This means that on an interval with left endpoint satisfying $n \leqslant x \leqslant n + n^{-2}$, there is no way to "see" the nearby interval where q is large.

Remark 4.5. It should, however, be pointed out that condition (6) is not necessary for the self-adjoint extensions of T_m to have a discrete spectrum, even when $p = 1$. In particular, condition (6) cannot be satisfied in Example 6.2 (to be given in Section 6) although the expression there will be shown to have discrete spectrum.

5. NECESSARY AND SUFFICIENT CONDITIONS

We begin by presenting in Theorem 5.1 a more refined upper bound for $\tilde{\beta}(E)$ when $r \leqslant s$, defined in terms of a more complicated cousin $\mathscr{V}_{r,s}$ of $\mathscr{U}_{r,s}$, and a criterion (Corollary 5.1) for discreteness of the spectrum that follows from it. We will, however, have to restrict the allowable families $\{I(x)\}$ of intervals somewhat. We will then see that by further restricting $\{I(x)\}$, $\mathscr{V}_{r,s}$ can also be used to give a lower bound for $\tilde{\beta}(E)$, that is we will have $c_1 \mathscr{V}_{r,s} \leqslant \tilde{\beta}(E) \leqslant c_2 \mathscr{V}_{r,s}$ (Theorem 5.2). Finally, using $\mathscr{V}_{2,2}$, we construct a necessary and sufficient condition for the self-adjoint extensions of T_m to have discrete spectrum (Corollary 5.3). In Section 6, two examples will be given to illustrate the flexibility and relative ease of use of the criteria of this section.

To make the notation slightly less cumbersome in the next theorem and in what follows, we abbreviate $(x^-)^+$ to x^{-+}, $(x^+)^-$ to x^{+-}, and so forth.

THEOREM 5.1. *Let p, q, w be as in (3), and let $1 < r \leqslant s < +\infty$. Let $a < c < b$ and let $\{I(x): x \in [c, b)\}$, $I(x) = [x^-, x^+]$, be a centralizable family of bounded intervals such that*

(a) $a < x^- < x < x^+ < b$ *for each* $x \in [c, b)$,

(b) $x \mapsto x^-$ *and* $x \mapsto x^+$, $x \in [c, b)$, *are non-decreasing functions of x,*

(c) *the intervals* $\{(x^-, x^{-+}): x \in [c, b)\}$ *and* $\{(x^{+-}, x^+): x \in [c, b)\}$ *each cover* $[c, b)$,

(d) $x^- \to b$ *as* $x \to b$.

Choose $c_1 \in [c, b)$ so that $c_1^{--} \geqslant c$. For $x \geqslant c_1$ define

$$V_{r,s}(x) = \left(\int_{x^-}^{x} w \right)^{1/s} \left[\left(\int_{I(x^-)} p^{1-r'} \right)^{1/r'} + \left(\int_{I(x^-)} q \right)^{-1/r} \right]$$
$$+ \left(\int_{x}^{x^+} w \right)^{1/s} \left[\left(\int_{I(x^+)} p^{1-r'} \right)^{1/r'} + \left(\int_{I(x^+)} q \right)^{-1/r} \right].$$

If $\mathscr{V}_{r,s} = \limsup_{x \to b} V_{r,s}(x)$ is finite, then the identity map $E: W_{p,q}^{1,r}[a, b) \to L_w^s[a, b)$ is a continuous embedding, and $\tilde{\beta}(E) \leqslant K\mathscr{V}_{r,s}$. In particular, if $\mathscr{V}_{r,s} = 0$ then $W_{p,q}^{1,r}[a, b)$ is compactly embedded in $L_w^s[a, b)$.

Proof. It will be convenient for the proof to adopt the notation

$$\|f\|_{p,q,I} = \left(\int_I (p|f'|^r + q|f|^r) \right)^{1/r}.$$

As in the proof of Theorem 4.1, it suffices to show that for $\varepsilon > 0$ there is $a < c < b$ such that for f supported in (c, b),

$$\left(\int_c^b |f|^s w \right)^{1/s} \leqslant K(\mathscr{V}_{r,s} + \varepsilon) \|f\|_{p,q,(c,b)}.$$

Given $\varepsilon > 0$, choose $x_0 \geqslant c_1$ so that $V_{r,s}(x) < \mathscr{V}_{r,s} + \varepsilon$ for $x > x_0$. Fix $x \in [x_0, b)$. By the argument in the proof of Theorem 3.1 applied to the interval $I(x^-)$ and the point $t = x^-$,

$$|f(x^-)| \leqslant \left[\left(\int_{I(x^-)} p^{1-r'} \right)^{1/r'} + \left(\int_{I(x^-)} q \right)^{-1/r} \right] \|f\|_{p,q,I(x^-)}. \tag{7}$$

We assert first that for any z with $x^- < z < x$, there is y so that $[x^-, z] \subset I(y^-)$ and $[z, x] \subset [y^-, y] \subset I(y)$. Indeed, for $x^- \leqslant z \leqslant x^{-+}$ we can take $y = x$. Suppose that $x^{-+} < x$. Then, the assumption that the intervals $\{(u^-, u^{-+}): u \in$

$[c,b)\}$ cover $[c,b)$ implies that there is a finite collection $\{(u_j^-, u_j^{-+}): j = 1, \ldots, n\}$ that covers $[x^{-+}, x]$. We may assume $x < u_1 < \cdots < u_n$ and then $x^- < u_1^- < \cdots < u_n^-$ and $x^{-+} < u_1^{-+} < \cdots < u_n^{-+}$. (If any of the inequalities are not strict, the corresponding interval can simply be omitted.) Thus for any z with $x^{-+} < z < x$, $z \in (u_k^-, u_k^{-+})$ for some k. For this $k, [z, x] \subset [u_k^-, u_k]$. Also $u_k^- < x$ implies $u_k^{-} \leqslant x^-$. Thus $[x^-, z] \subset [u_k^{-}, u_k^{-+}] = I(u_k^-)$ and we can take $y = u_k$.

Now for any z with $x^- < z < x$ and the corresponding y selected as above, we have

$$\left(\int_{x^-}^z p^{1-r'} \right)^{1/r'} \left(\int_z^x w \right)^{1/s} \leqslant \left(\int_{I(y^-)} p^{1-r'} \right)^{1/r'} \left(\int_{y^-}^y w \right)^{1/s}$$
$$\leqslant V_{r,s}(y) < \mathscr{V}_{r,s} + \varepsilon.$$

Thus by the generalized Hardy inequality (see [3] or [14, Theorem 1.3.4]), using the fact that $f - f(x^-)$ vanishes at x^- and that $r \leqslant s$,

$$\left(\int_{x^-}^x |f - f(x^-)|^s w \right)^{1/s} \leqslant K_0(\mathscr{V}_{r,s} + \varepsilon) \left(\int_{x^-}^x p|f'|^r \right)^{1/r}. \tag{8}$$

Now, using (7) and (8),

$$\left(\int_{x^-}^x |f|^s w \right)^{1/s}$$
$$\leqslant |f(x^-)| \left(\int_{x^-}^x w \right)^{1/s} + \left(\int_{x^-}^x |f - f(x^-)|^s w \right)^{1/s}$$
$$\leqslant \left(\int_{x^-}^x w \right)^{1/s} \left[\left(\int_{I(x^-)} p^{1-r'} \right)^{1/r'} + \left(\int_{I(x^-)} q \right)^{-1/r} \right] \|f\|_{p,q,I(x^-)}$$
$$+ K_0(\mathscr{V}_{r,s} + \varepsilon) \|f\|_{p,q,[x^-,x]}$$
$$\leqslant K_0(\mathscr{V}_{r,s} + \varepsilon)(\|f\|_{p,q,I(x^-)} + \|f\|_{p,q,[x^-,x]}).$$

Similarly,

$$\left(\int_x^{x^+} |f|^s w \right)^{1/s} \leqslant K_0(\mathscr{V}_{r,s} + \varepsilon)(\|f\|_{p,q,I(x^+)} + \|f\|_{p,q,[x,x^+]}),$$

so that

$$\left(\int_{I(x)} |f|^s w \right)^{1/s}$$
$$\leqslant 2K_0(\mathscr{V}_{r,s} + \varepsilon)(\|f\|_{p,q,I(x^-)} + \|f\|_{p,q,I(x^+)} + \|f\|_{p,q,[x^-,x^+]}).$$

Now by Theorem 2.2 there is a subfamily $\{I(x)\}$ covering (c, b) which can be partitioned into 2ξ families of pairwise disjoint intervals. We assert that each of the corresponding families $\{I(x_k^-)\}$, $\{I(x_k^+)\}$ where $I(x_k) = [x_k^-, x_k^+]$ can be partitioned into 4ξ families of pairwise disjoint intervals. Assuming for the moment that this is true, we have

$$\left(\int_c^b |f|^s w\right)^{1/s} \leqslant \sum_{k=1}^{+\infty} \left(\int_{I(x_k)} |f|^s w\right)^{1/s}$$

$$\leqslant 8\xi K_0(\mathscr{V}_{r,s} + \varepsilon)\left(\int_c^b (p|f'|^r + q|f|^r)\right)^{1/r},$$

and the theorem will be proved with $K = 8\xi K_0$.

To establish the claim, let $\{I(y_n)\}$ be one of the subfamilies of pairwise disjoint intervals obtained from $\{I(x_k)\}$, ordered so that $y_1 < y_2 < \cdots$. It suffices to show that for any n, $I(y_n^-) \cap I(y_{n+2}^-) = \emptyset$ and $I(y_n^+) \cap I(y_{n+2}^+) = \emptyset$, since then the families $\{I(y_{2n}^-)\}$, $\{I(y_{2n+1}^-)\}$, $\{I(y_{2n}^+)\}$, and $\{I(y_{2n+1}^+)\}$ will consist of pairwise disjoint intervals. We show $I(y_n^-) \cap I(y_{n+2}^-) = \emptyset$, that is, $y_n^{-+} < y_{n+2}^-$ by exploiting the non-decreasing property of x^- and x^+. Since $y_n^- < y_n$, $y_n^{-+} \leqslant y_n^+$. Since $I(y_{n+1}) \cap I(y_{n+2}) = \emptyset$, $y_{n+1} < y_{n+2}^-$. Thus, $y_n^{-+} \leqslant y_n^+ < y_{n+1} \leqslant y_{n+2}^-$. The argument that $y_n^{+-} < y_{n+2}^{+-}$ is similar. ∎

Remark 5.1. If a centralizable family $\{I(x): x \in [c, b)\}$, $I(x) = [x^-, x^+]$, of bounded intervals satisfies assumption (b) of Theorem 5.1 and if the functions $x \mapsto x^-$ and $x \mapsto x^+$, $x \in [c, b)$, are continuous functions of x, then assumption (c) of Theorem 5.1 is trivially satisfied.

Remark 5.2. If a family $\{I(x): x \in [c, b)\}$, $I(x) = [x^-, x^+]$, of bounded intervals satisfies assumption (b) of Theorem 5.1 and $x = (x^- + x^+)/2$ for all $x \in [c, b)$, then it is not difficult to see that the functions $x \mapsto x^-$ and $x \mapsto x^+$, $x \in (c, b)$, are continuous functions of x.

COROLLARY 5.1. *Let p, q, w be as in* (3) *and let $\{I(x): x \in [c, b)\}$, $I(x) = [x^-, x^+]$, be as in Theorem 5.1. If*

$$\limsup_{x \to b} \left(\left[\int_{I(x^-)} p^{-1} + \left(\int_{I(x^-)} q\right)^{-1}\right] \int_{x^-}^x w \right.$$

$$\left. + \left[\int_{I(x^+)} p^{-1} + \left(\int_{I(x^+)} q\right)^{-1}\right] \int_x^{x^+} w \right) \tag{9}$$

is finite, then the essential spectrum of T_m is bounded away from 0. *If* (9) *is* 0, *then each self-adjoint extension of T_m has discrete spectrum.*

Proof.   The corollary follows from the following inequality:

$$\frac{1}{2}(V_{2,2}(x))^2 \leqslant \left[\int_{I(x^-)} p^{-1} + \left(\int_{I(x^-)} q \right)^{-1} \right] \int_{x^-}^x w$$

$$+ \left[\int_{I(x^+)} p^{-1} + \left(\int_{I(x^+)} q \right)^{-1} \right] \int_x^{x^+} w$$

$$\leqslant (V_{2,2}(x))^2. \quad \blacksquare$$

DEFINITION 5.1.   A centralizable family $\{I(x): x \in [c, b)\}$, with $I(x) = [x^-, x^+]$, of bounded intervals is *p-centralizable* if in addition to assumptions (a)–(d) of Theorem 5.1, for all $x \in (c, b)$ it satisfies

$$\gamma_1 \int_{x^-}^x p^{1-r'} \leqslant \int_x^{x^+} p^{1-r'} \leqslant \gamma_2 \int_{x^-}^x p^{1-r'} \tag{10}$$

and

$$\delta_1 \leqslant \left(\int_{I(x)} p^{1-r'} \right)^{1/r'} \left(\int_{I(x)} q \right)^{1/r} \leqslant \delta_2 \tag{11}$$

with some positive constants $\gamma_1, \gamma_2, \delta_1, \delta_2$.

Remark 5.3.   The restriction in Section 2 that $f(x)$ lie in the middle third of $f[I(x)]$ can be loosened to the requirement that for some $\theta < 1$, $f(x)$ lie in the "middle θ" of $f[I(x)]$. Thus, (10) is really the requirement that $\{I(x)\}$ be centralizable by the function $f = p^{1-r'}$.

Remark 5.4.   The significance of (10) and (11), as we see in the following lemma, is that they allow us to estimate $\int p^{1-r'}$ and $\int q$ on the intervals $I(x^-)$ and $I(x^+)$ in terms of their value on $I(x)$.

Remark 5.5.   Conditions (10) and (11) represent a relaxed form of the specification of intervals in [13] where $\gamma_1 = \gamma_2 = 1$ and $\delta_1 = \delta_2$. We are motivated here by the desire to have a condition which is easier to achieve in practice.

It will be convenient for the next several proofs to adopt the notation $P(\alpha, \beta) = \int_\alpha^\beta p^{1-r'}$ and $P(I(x)) = P(x^-, x^+)$.

LEMMA 5.1.   *Let p and q be as in* (3), $1 < r < +\infty$, *and let $\{I(x): x \in [c, b)\}$, $I(x) = [x^-, x^+]$, be a p-centralizable family of bounded*

intervals. Then,

$$\max\left\{\int_{I(x^-)} p^{1-r'}, \int_{I(x^+)} p^{1-r'}\right\} \leqslant \eta \int_{I(x)} p^{1-r'}$$

and

$$\int_{I(x)} q \leqslant \mu \min\left\{\int_{I(x^-)} q, \int_{I(x^+)} q\right\},$$

where $\eta = \max\{1 + \gamma_1^{-1}, 1 + \gamma_2\}$, $\mu = \frac{\delta_2}{\delta_1} \max\{(1 + \gamma_2)^{-1}, \gamma_1(1 + \gamma_1)^{-1}\}$.

Proof. Since $x^{-+} \leqslant x^+$ we have $[x^-, x^{-+}] \subset [x^-, x^+] = I(x)$. Thus $P(x^-, x^{-+}) \leqslant P(x^-, x^+)$. Thus from (10), $P(I(x^-)) \leqslant (1 + \gamma_1^{-1})P(I(x))$. Similarly, $P(I(x^+)) \leqslant (1 + \gamma_2)P(I(x))$. Also,

$$\int_{I(x)} q \leqslant \delta_2 P(I(x))^{-1} \leqslant \frac{\delta_2 \gamma_1}{1 + \gamma_1} P(I(x^-))^{-1}$$
$$\leqslant \left(\frac{\delta_2}{\delta_1}\right)^r \left(\frac{\gamma_1}{1 + \gamma_1}\right)^{1-r} \int_{I(x^-)} q.$$

Similarly, $\int_{I(x)} q \leqslant \delta_2 P(I(x))^{-1} \leqslant (\frac{\delta_2}{\delta_1})^r (1 + \gamma_1)^{r-1} \int_{I(x^+)} q$.   ∎

We are now ready to identify $\tilde{\beta}(E)$ with $\mathcal{V}_{r,s}$ as defined via a p-centralizable family of intervals.

THEOREM 5.2. *Let* p, q, w *be as in* (3), *and let* $1 < r \leqslant s < +\infty$. *Let* $\{I(x): x \in [c, b)\}$, $a < c < b$, $I(x) = [x^-, x^+]$, *be a* p-*centralizable family of bounded intervals. Choose* $c_1 \in [c, b)$ *so that* $c_1^{--} \geqslant c$. *For* $x \geqslant c_1$ *define*

$$V_{r,s}(x) = \left(\int_{x^-}^x w\right)^{1/s} \left[\left(\int_{I(x^-)} p^{1-r'}\right)^{1/r'} + \left(\int_{I(x^-)} q\right)^{-1/r}\right]$$
$$+ \left(\int_x^{x^+} w\right)^{1/s} \left[\left(\int_{I(x^+)} p^{1-r'}\right)^{1/r'} + \left(\int_{I(x^+)} q\right)^{-1/r}\right]. \quad (12)$$

Then, $\mathcal{V}_{r,s} = \limsup_{x \to b} V_{r,s}(x)$ *is finite if and only if the identity map* $E: W_{p,q}^{1,r}[a,b) \to L_w^s[a,b)$ *is a continuous embedding, and in that case* $k\mathcal{V}_{r,s} \leqslant \tilde{\beta}(E) \leqslant K\mathcal{V}_{r,s}$. *Here* k, K *depend only on* $\gamma_1, \gamma_2, \delta_1, \delta_2, r, s$.

Proof. This follows from Theorem 5.1 except for the assertion that E being a continuous embedding implies $\mathcal{V}_{r,s} < +\infty$, and that then $k\mathcal{V}_{r,s} \leqslant$

$\tilde{\beta}(E)$. Denote the first term in (12) by $V_{r,s}^{\mathrm{L}}(x)$ with limit superior $\mathscr{V}_{r,s}^{\mathrm{L}}$ and the second term by $V_{r,s}^{\mathrm{R}}(x)$ with limit superior $\mathscr{V}_{r,s}^{\mathrm{R}}$. It will be convenient to bound these separately.

Let $\{x_n\}$ be any sequence in (c_1, b) such that $x_n \to b$ and such that $x_n^+ < x_{n+1}^-$ for each n. Define a sequence $\{f_n\}$ of functions with disjoint supports as follows:

$$f_n(t) = \begin{cases} \dfrac{P(x_n^{--}, t)}{P(x_n^{--}, x_n^-)^{1/r}}, & x_n^{--} \leqslant t \leqslant x_n^-, \\[2ex] P(x_n^{--}, x_n^-)^{1/r'}, & x_n^- \leqslant t \leqslant x_n, \\[2ex] P(t, x_n^+) \dfrac{P(x_n^{--}, x_n^-)^{1/r'}}{P(x_n, x_n^+)}, & x_n \leqslant t \leqslant x_n^+, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Note that

$$P(x_n^{--}, x_n^-)^{1/r'} \geqslant (1 + \gamma_2^{-1})^{1/r'} P(I(x_n^-))^{1/r'}$$

$$\geqslant (1 + \gamma_2^{-1})^{1/r'} \delta_1 \left(\int_{I(x_n^-)} q \right)^{-1/r},$$

so that

$$(1 + \delta_1) P(x_n^{--}, x_n^-)^{1/r'}$$

$$\geqslant \delta_1 (1 + \gamma_2^{-1})^{1/r'} \left[\left(\int_{I(x_n^-)} p^{1-r'} \right)^{-1/r'} + \left(\int_{I(x_n^-)} q \right)^{-1/r} \right].$$

Thus,

$$\left(\int_a^b |f_n|^s w \right)^{1/s} \geqslant \left(\int_{x_n^-}^{x_n} |f_n|^s w \right)^{1/s}$$

$$= \left(\int_{x_n^-}^{x_n} w \right)^{1/s} P(x_n^{--}, x_n^-)^{1/r'}$$

$$\geqslant \frac{\delta_1 (1 + \gamma_2^{-1})^{1/r'}}{1 + \delta_1} \left(\int_{x_n^-}^{x_n} w \right)^{1/s}$$

$$\times \left[\left(\int_{I(x_n^-)} p^{1-r'} \right)^{-1/r'} + \left(\int_{I(x_n^-)} q \right)^{-1/r} \right]$$

$$= \frac{\delta_1}{\delta_1 + 1} (1 + \gamma_2^{-1})^{1/r'} V_{r,s}^{\mathrm{L}}(x_n).$$

On the other hand, we have

$$\int_{x_n^{--}}^{x_n^{-}} p^{1-r'} \leqslant \int_{I(x_n^{-})} p^{1-r'} \leqslant \eta \int_{I(x_n)} p^{1-r'}$$

$$\leqslant \eta(1+\gamma_1^{-1}) \int_{x_n}^{x_n^{+}} p^{1-r'}.$$

Thus,

$$\int_a^b p|f_n'|^r = \frac{1}{P(x_n^{--}, x_n^{-})} \int_{x_n^{--}}^{x_n^{-}} p^{1-r'} + \frac{P(x_n^{--}, x_n^{-})^{r/r'}}{P(x_n, x_n^{+})^r} \int_{x_n}^{x_n^{+}} p^{1-r'}$$

$$= 1 + \left[\frac{P(x_n^{--}, x_n^{-})}{P(x_n, x_n^{+})}\right]^{1-r}$$

$$\leqslant 1 + [\eta(1+\gamma_1^{-1})]^{1-r}.$$

Finally, from Lemma 5.1,

$$\int_a^b q|f_n|^r \leqslant P(x_n^{--}, x_n^{-})^{r/r'} \int_{x_n^{--}}^{x_n^{+}} q \leqslant P(x_n^{--}, x_n^{-})^{r/r'} \left[\int_{I(x^{-})} q + \int_{I(x)} q\right]$$

$$\leqslant (1+\mu)P(I(x_n^{-}))^{r/r'} \int_{I(x^{-})} q \leqslant (1+\mu)\delta_2^r.$$

Thus, we have constants k_1, k_2 depending only on the constants in Definition 5.1, r, and s such that for each n,

$$\|f_n\|_w \geqslant k_1 V_{r,s}^{\mathrm{L}}(x_n) \quad \text{and} \quad \|f_n\|_{p,q} \leqslant k_2.$$

Thus if E is bounded,

$$V_{r,s}^{\mathrm{L}}(x_n) \leqslant k_1^{-1}\|E(f_n)\|_w \leqslant (k_2/k_1)\|E\|.$$

Since these estimates hold for any sequence $\{x_n\}$ as described above, $\mathscr{V}_{r,s}^{\mathrm{L}} \leqslant (k_2/k_1)\|E\|$. A similar argument, using the sequence g_n defined by

$$g_n(t) = \begin{cases} P(x_n^{-}, t) \frac{P(x_n^{+}, x_n^{++})^{1/r'}}{P(x_n^{-}, x_n)}, & x_n^{-} \leqslant t \leqslant x_n, \\ P(x_n^{+}, x_n^{++})^{1/r'}, & x_n \leqslant t \leqslant x_n^{+}, \\ \frac{P(t, x_n^{++})}{P(x_n^{+}, x_n^{++})^{1/r}}, & x_n^{+} \leqslant t \leqslant x_n^{++}, \\ 0 & \text{otherwise} \end{cases}$$

yields a similar estimate for $\mathscr{V}_{r,s}^{\mathrm{R}}$. Thus, $\mathscr{V}_{r,s} < +\infty$ if E is bounded.

To show that $k\mathscr{V}_{r,s} \leqslant \tilde{\beta}(E)$, let $\varepsilon > 0$ be given, and suppose that the sequence $\{x_n\}$ has the additional property that $V_{r,s}(x_n) > \mathscr{V}_{r,s} - \varepsilon$. Passing to a subsequence if necessary, we may assume that, say, $V_{r,s}^{L}(x_n) > (\mathscr{V}_{r,s} - \varepsilon)/2$. Since E is linear,

$$\tilde{\beta}(E) \geqslant \frac{\tilde{\psi}_L(\{f_n\})}{\tilde{\psi}_W(\{f_n\})} \geqslant k_2^{-1}\tilde{\psi}_L(\{f_n\}),$$

where $\tilde{\psi}_L$ denotes the quantity $\tilde{\psi}$ from Definition 3.1 acting in $L_w^s[a, b]$ and $\tilde{\psi}_W$ denotes $\tilde{\psi}$ acting in $W_{p,q}^{1,r}[a, b]$. Since the functions $\{f_n\}$ have pairwise disjoint supports and $\|f_n\|_w \geqslant k_1 V_{r,s}^{L}(x_n) > (k_1/2)(\mathscr{V}_{r,s} - \varepsilon)$,

$$\|f_n - f_m\|_w > 2^{1/s}(k_1/2)(\mathscr{V}_{r,s} - \varepsilon) \qquad \text{when } m \neq n.$$

It follows that $\{f_n\}$ cannot be covered by finitely many balls of radius less than half the quantity on the right, that is, $\tilde{\beta}(E) > k(\mathscr{V}_{r,s} - \varepsilon)$ with $k = 2^{(1-2s)/s}k_1/k_2$. Since ε is arbitrary, the proof is complete. ∎

COROLLARY 5.2.   *Let p, q, w be as in* (3), *and let $1 < r \leqslant s < +\infty$. Then, $W_{p,q}^{1,r}[a, b]$ is compactly embedded in $L_w^s[a, b]$ if and only if for some (and thus for every) p-centralizable family of intervals $\{I(x)\}$ it is the case that $\mathscr{V}_{r,s} = \limsup_{x \to b} V_{r,s}(x) = 0$.*

Specializing to $r = s = 2$ and replacing $V_{r,s}(x)$ by an equivalent quantity without fractional exponents, we obtain our characterization of discreteness of the spectrum of a second-order differential operator.

COROLLARY 5.3.   *Let p, q, w be defined on $A = [a, b]$ as in* (3) *and assume that τ from* (1) *is singular at b. Then each self-adjoint extension of T_m has discrete spectrum if and only if there is a p-centralizable family $\{I(x) : x \in [c, b)\}$, $a < c < b$, of closed, bounded intervals $I(x) = [x^-, x^+]$ such that*

$$\left[\int_{I(x^-)} p^{-1} + \left(\int_{I(x^-)} q\right)^{-1}\right] \int_{x^-}^{x} w$$

$$+ \left[\int_{I(x^+)} p^{-1} + \left(\int_{I(x^+)} q\right)^{-1}\right] \int_{x}^{x^+} w \to 0 \qquad (13)$$

as $x \to b$.

Remark 5.6.   It should be pointed out that by Corollary 5.1 one can demonstrate discreteness of the spectrum using a family $\{I(x)\}$ that is merely centralizable, and that, in fact, it often suffices to use the simpler setting described in Corollary 4.2.

6. TWO EXAMPLES

We conclude with two examples that illustrate the versatility of the criteria.

EXAMPLE 6.1.   Consider   a   one-term   expression $\tau(y) = -w^{-1}(py')'$ on $[0, +\infty)$. Here $q = 0$. It is known that if $p^{-1} \in L^1(0, +\infty)$, then the self-adjoint operators associated with τ have discrete spectrum if and only if

$$\int_0^x w \int_x^{+\infty} p^{-1} \to 0 \qquad \text{as } x \to b.$$

This is a special case of Theorem 4.1 in [1] for $2n$th order equations with matrix coefficients. The proof there is based on oscillation theory. We will use Corollary 5.1 to show that this condition suffices when $w \notin L^1(0, +\infty)$. (Otherwise the expression is regular, and it is clear that the spectrum is discrete.) We may assume that $q = w$, since this just shifts the entire spectrum to the right by one unit.

Let $x_0 > 0$ be such that $1 + \int_0^{x_0} w = 2$. For each $x > x_0$ set $I(x) = [x^-, x^+]$, where

$$1 + \int_0^{x^-} w = \frac{1}{2}\left(1 + \int_0^x w\right), \qquad 1 + \int_0^{x^+} w = \left(1 + \int_0^x w\right)^2.$$

Note that x^{--} is defined for all $x > 0$ for which $\int_0^x w > 3$. The function $f(x) = 1 - 1/(1 + \int_0^{x^-} w)$ maps $[x_0, +\infty)$ onto $[1/2, 1)$. It is not difficult to verify that $f(x)$ is in the middle third of the interval $[f(x^-), f(x^+)]$ for all $x \geqslant x_0$. Thus, $\{I(x): x \in [x_0, +\infty)\}$ is a centralizable family, although it may not be p-centralizable. We verify that the limit superior in (9) is 0 so that we can apply Corollary 5.1.

Given $\varepsilon > 0$, choose X so that for $x \geqslant X$,

$$\left(1 + \int_0^{x^{--}} w\right)\int_{x^{--}}^{+\infty} p^{-1} + \left(\frac{1}{2} + \int_0^{x^-} w\right)^{-1} < \varepsilon.$$

Now $\int_{x^-}^x w = (1 + \int_0^x w) - (1 + \int_0^{x^-} w) = 1 + \int_0^{x^-} w = 2(1 + \int_0^{x^{--}} w)$, so for $x \geqslant X$,

$$\int_{x^-}^x w \int_{I(x^-)} p^{-1} \leqslant 2\left(1 + \int_0^{x^{--}} w\right)\int_{x^{--}}^{+\infty} p^{-1} < 2\varepsilon.$$

Similarly, $\int_x^{x^+} w \leqslant 1 + \int_0^{x^+} w = 2(1 + \int_0^{x^{+-}} w)$, so

$$\int_x^{x^+} w \int_{I(x^+)} p^{-1} \leqslant 2\left(1 + \int_0^{x^{+-}} w\right) \int_{x^{+-}}^{+\infty} p^{-1} < 2\varepsilon.$$

Since $q = w$,

$$\int_{I(x^-)} q = \int_{x^{--}}^{x^{-+}} w$$

$$= \left(1 + \int_0^{x^-} w\right)^2 - \frac{1}{2}\left(1 + \int_0^{x^-} w\right)$$

$$= \left(1 + \int_0^{x^-} w\right)\left(\frac{1}{2} + \int_0^{x^-} w\right).$$

Thus,

$$\left(\int_{I(x^-)} q\right)^{-1} \int_{x^-}^x w = \left(\int_{I(x^-)} q\right)^{-1}\left(1 + \int_0^{x^-} w\right)$$

$$= \left(\frac{1}{2} + \int_0^{x^-} w\right)^{-1} < \varepsilon.$$

By a similar calculation, $(\int_{I(x^+)} q)^{-1} \int_x^{x^+} w \leqslant (\frac{1}{2} + \int_0^{x^+} w)^{-1} < \varepsilon$. Thus, (13) holds and the spectrum is discrete.

EXAMPLE 6.2. We consider an expression on $[0, +\infty)$ where $p = 1$ and for which $q/w \to +\infty$ for the most part, but such that $q = w$ on a sequence of intervals of length 2. That the expression has discrete spectrum is thus because of the growth of q in some places, and because of the relatively large size of p, compared to q and w in others. Specifically, for $n = 1, 2, \ldots$, define

$$q(x) = \begin{cases} (n+1)(x-n^2)^n & \text{if } n^2 - 1 \leqslant x < n^2 + 1, \\ x & \text{otherwise,} \end{cases}$$

$$w(x) = \begin{cases} q(x) & \text{if } n^2 - 1 \leqslant x < n^2 + 1, \\ 1 & \text{otherwise.} \end{cases}$$

Note first that Corollary 4.2 cannot be used here, since for $x = n^2$, if $I = I(x) \subset [n^2 - 1, n^2 + 1]$, then $\int_I w = \int_I q$, while if not, then $|I| \int_I w \geqslant 1$. Thus, for any family $\{I(x)\}$, $\mathbf{U} \geqslant 1$.

We use instead Corollary 5.3 with $I(x)$ the interval centered at x defined as follows:

(i) if $(n-1)^2 + 1 + \frac{1}{2(n-1)} \leqslant x \leqslant n^2 - 1 - \frac{1}{2n}$, then $I(x) = [x - \phi_n(x), x + \phi_n(x)]$, where ϕ_n is the linear function equal to $\frac{1}{2(n-1)}$ when $x = (n-1)^2 + 1 + \frac{1}{2(n-1)}$ and equal to $\frac{1}{2n}$ when $x = n^2 - 1 - \frac{1}{2n}$,

(ii) if $n^2 - 1 - \frac{1}{2n} \leqslant x \leqslant n^2 - \frac{1}{2}$, say $x = n^2 - \frac{1}{2} - \frac{\alpha}{2}(1 + \frac{1}{n})$ for some $0 \leqslant \alpha \leqslant 1$, then $I(x) = [n^2 - 1 - \frac{\alpha}{n}, n^2 - \alpha]$,

(iii) if $n^2 - \frac{1}{2} \leqslant x \leqslant n^2$, then $I(x) = [n^2 - 1, 2x - n^2 + 1]$,

(iv) if $n^2 \leqslant x \leqslant n^2 + \frac{1}{2}$, then $I(x) = [2x - (n^2 + 1), n^2 + 1]$,

(v) if $n^2 + \frac{1}{2} \leqslant x \leqslant n^2 + 1 + \frac{1}{2n}$, $x = n^2 + \frac{1}{2} + \frac{\alpha}{2}(1 + \frac{1}{n})$, then $I(x) = [n^2 + \alpha, n^2 + 1 + \frac{\alpha}{n}]$.

We will verify (13) in case (ii). The argument for (v) is parallel, and that for the other parts similar but much easier. Suppose first that $x \leqslant n^2 - 1$, that is, $1 \geqslant \alpha \geqslant \frac{n}{n+1}$. Then with I denoting any one of $I(x), I(x^-), I(x^+)$, it is easy to see that $|I| \leqslant \frac{2}{n}, \int_I q \geqslant \frac{n}{2}$ (just considering the part of I that lies outside $[n^2 - 1, n^2 + 1]$), and $\int_I w \leqslant 2$. Thus (13) is $O(n^{-1})$ for $n^2 - 1 - \frac{1}{2n} \leqslant x \leqslant n^2 - 1$.

Next, suppose $n^2 - 1 \leqslant x \leqslant n^2 - \frac{1}{2}$, that is, $\frac{n}{n+1} \geqslant \alpha \geqslant 0$. Then $x^- \leqslant n^2 - 1$ and the estimates just made apply to the left half of (13), so that again it is $O(n^{-1})$.

For the second term in (13), suppose first $\alpha \geqslant 1 - (n+1)^{-1/2}$. Then $|I(x^+)| \leqslant 3(n+1)^{-1/2}$. For large values of n, $x^{+-} \leqslant n^2 - 1 - \frac{1}{2n}$, so considering the part of $I(x^+)$ lying outside of $[n^2 - 1, n^2 + 1]$, $\int_{I(x^+)} q \geqslant \frac{n}{3}$. Since $\int_x^{x^+} w \leqslant 1$, the second term in (13) is $O(n^{-1/2})$ for these values of α.

Finally, suppose $\alpha < 1 - (n+1)^{-1/2}$. Then $x \geqslant n^2 - 1 + \frac{1}{4\sqrt{n+1}}$. Thus,

$$\int_x^{x^+} w \leqslant \int_x^{n^2} w = (n^2 - x)^{n+1} \leqslant \left(1 - \frac{1}{4\sqrt{n+1}}\right)^{n+1}$$
$$\leqslant \exp\left(-\frac{\sqrt{n+1}}{4}\right).$$

Since $n^2 - 1 \in I(x^+)$,

$$\int_{I(x^+)} q \geqslant \int_{n^2-1}^x q = 1 - \int_x^{n^2} w \geqslant 1 - \exp\left(-\frac{\sqrt{n+1}}{4}\right).$$

Also $|I(x^+)| \leqslant 2$. Thus for these values of α, the second term in (13) is $O(e^{-\sqrt{n+1}/4})$. This completes the verification for (ii).

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