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Standard symmetric operators in Pontryagin spaces: a generalized von Neumann formula and minimality of boundary coefficients

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Abstract

Certain meromorphic matrix valued functions on $\mathbb{C}\backslash\mathbb{R}$, the so-called boundary coefficients, are characterized in terms of a standard symmetric operator S in a Pontryagin space with finite (not necessarily equal) defect numbers, a meromorphic mapping into the defect subspaces of S, and a boundary mapping for S. Under some simple assumptions the boundary coefficients also satisfy a minimality condition. It is shown that these assumptions hold if and only if for S a generalized von Neumann equality is valid.

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1. Introduction

Let Q be an invertible self-adjoint $d\times d$, matrix with d_+ positive and d_- negative eigenvalues, so $d=d_-+d_+$, and let $\varkappa\in\{0,1,2,\dots\}$. In this paper a Q-*boundary coefficient* with $\varkappa$ negative squares is a matrix valued function $\mathscr{U}$ defined on $\mathrm{dom}(\mathscr{U})$, where $\mathrm{dom}(\mathscr{U})\subset\mathbb{C}\backslash\mathbb{R}$, the set $\mathbb{C}\backslash(\mathbb{R}\cup\mathrm{dom}(\mathscr{U}))$ is finite, and $\mathrm{dom}(\mathscr{U})$ is symmetric with respect to the real axis, and the function $\mathscr{U}$ has the following properties:

- ($\mathscr{U}1$) $\mathscr{U}(z)$ is a $d_+\times d$ matrix if $z\in\mathrm{dom}(\mathscr{U})\cap\mathbb{C}^+$ and $\mathscr{U}(z)$ is a $d_-\times d$ matrix if $z\in\mathrm{dom}(\mathscr{U})\cap\mathbb{C}^-$.
- ($\mathscr{U}2$) $\mathscr{U}(z)$ is holomorphic on $\mathrm{dom}(\mathscr{U})$ and meromorphic on $\mathbb{C}\backslash\mathbb{R}$.
- ($\mathscr{U}3$) The matrices $\mathscr{U}(z)$, $z\in\mathrm{dom}(\mathscr{U})$, have maximal rank.
- ($\mathscr{U}4$) $\mathscr{U}(z)\mathsf{Q}^{-1}\mathscr{U}(z^*)^*=0$, $z\in\mathrm{dom}(\mathscr{U})$.
- ($\mathscr{U}5$) The limit

$$\lim_{w\to z^*}\frac{\mathscr{U}(z)\mathsf{Q}^{-1}\mathscr{U}(w)^*}{z-w^*}$$

exists for each $z\in\mathrm{dom}(\mathscr{U})$ and the kernel

$$K_{\mathscr{U}}(z,w)=\begin{cases} i\dfrac{\mathscr{U}(z)\mathsf{Q}^{-1}\mathscr{U}(w)^*}{z-w^*}, & z\neq w^*,\ z,w\in\mathrm{dom}(\mathscr{U}),\\ \lim_{\zeta\to z^*} i\dfrac{\mathscr{U}(z)\mathsf{Q}^{-1}\mathscr{U}(\zeta)^*}{z-\zeta^*}, & z=w^*,\ z\in\mathrm{dom}(\mathscr{U})\end{cases}$$

has $\varkappa$ negative squares.

The kernel condition ($\mathscr{U}5$) means that for any choice of the natural number n and $\lambda_1,\dots,\lambda_n\in\mathrm{dom}(\mathscr{U})$, the self-adjoint block matrix

$$[K_{\mathscr{U}}(\lambda_j,\lambda_k)]_{j,k=1}^n$$

has at most $\varkappa$ negative eigenvalues and for at least one such a choice it has exactly $\varkappa$ negative eigenvalues. If $d_-=0$, then a Q-boundary coefficient $\mathscr{U}$ is not defined on $\mathbb{C}^-$ and all the requirements in the above definition that relate to the numbers in $\mathbb{C}^-$ need to be dropped in this case. The modifications needed to cover this case in the proofs are straightforward and are omitted. The same remark applies to the case $d_+=0$. See Example 6.6 for more information about these cases.

To characterize boundary coefficients we use standard symmetric linear relations in Pontryagin spaces. For basic terminology related to linear relations, Pontryagin and Krein spaces see [6] or [13,4,5,7]. Recall only that a *linear relation* T in a normed vector space $\mathscr{H}$ is a linear subset of $\mathscr{H}^2=\mathscr{H}\oplus\mathscr{H}$. For brevity, we will relate to linear relations simply as relations. A subspace of a normed vector space $\mathscr{H}$ is a closed linear manifold of $\mathscr{H}$. We use the standard notation: $\mathbb{C}$ for the set of complex numbers, $\mathbb{R}$ for the set of real numbers, $\mathbb{C}^+$ and $\mathbb{C}^-$ for the open upper and the open lower half-plane of $\mathbb{C}$, $\mathbb{T}$ for the unit circle in $\mathbb{C}$, and $\mathbb{D}$ for the open unit disk in $\mathbb{C}$.

In this paper we show, see Theorem 4.2, that for a given Q a $(-\mathsf{Q})$-boundary coefficient can be constructed as follows:

(A) Let S be a standard symmetric linear operator with a not necessarily dense domain $\operatorname{dom} S$ and with finite defect indices (d_+, d_-) in a Pontryagin space $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ with negative index $\varkappa$.

(B) Let $\Phi(z)$ be a holomorphic basis for $\ker(S^{[*]} - z)$. This is short for: Let $\Phi : \mathbb{C}^{\pm}\backslash\gamma \to \mathscr{H} \times \mathscr{H} \times \cdots \times \mathscr{H}$ ($d_{\pm}$ copies) be a holomorphic row vector function, such that the components $\phi_j(z)$ of $\Phi(z)$:

$$\Phi(z) = (\phi_1(z), \phi_2(z), \ldots, \phi_{d_{\pm}}(z))$$

constitute a basis for $\ker(S^{[*]} - z)$, $z \in \mathbb{C}^{\pm}\backslash\gamma$. Here γ is a finite subset of $\mathbb{C}\backslash\mathbb{R}$.

(C) Let $\mathsf{b} : S^{[*]} \to \mathbb{C}^d$ be a boundary mapping for S with Gram matrix Q, for the definition see Section 4.

Then

$$\mathscr{U}(z) := (\mathsf{Q}[\mathsf{b}(\phi_1(z^*)) \quad \mathsf{b}(\phi_2(z^*)) \quad \cdots \quad \mathsf{b}(\phi_{d_{\mp}}(z^*)])^*, \tag{1.1}$$

where $\mathsf{b}(\phi_j(z^*))$ is short for $\mathsf{b}(\{\phi_j(z^*), z^*\phi_j(z^*)\})$, is a $(-\mathsf{Q})$-boundary coefficient.

This construction is similar as in the Hilbert space case considered in [7]; that is the case corresponding to $\varkappa = 0$ here. Then $\mathscr{U}(z)$ is holomorphic on $\mathbb{C}^+ \cup \mathbb{C}^-$ and the kernel $K_{\mathscr{U}}(z, w)$ in $(\mathscr{U}5)$ is non-negative. Moreover, in this case all boundary coefficients constructed in this way have the property that the $d \times d$ matrix

$$\begin{bmatrix} \mathscr{U}(z) \\ \mathscr{U}(z^*) \end{bmatrix} \text{ is invertible,} \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

When $\varkappa > 0$ this minimality property does not hold in general. The reason is that for the symmetric operator S considered in (A) the defect subspaces $\ker(S^{[*]} - z)$, $z \in \mathbb{C}\backslash\mathbb{R}$, need not be regular subspaces of $\mathscr{H}$. Instead, we shall use the following more general definition of minimality. A Q-boundary coefficient $\mathscr{U}(z)$ is said to be *minimal* if, with $\varkappa$ as in $(\mathscr{U}5)$,

($\mathscr{U}6$) There exist distinct $z_0, \ldots, z_\varkappa \in \mathbb{C}^+ \cap \operatorname{dom}(\mathscr{U})$ or, equivalently, distinct $z_0, \ldots, z_\varkappa \in \mathbb{C}^- \cap \operatorname{dom}(\mathscr{U})$ such that the matrix

$$[\mathscr{U}(z_0)^* \quad \mathscr{U}(z_0^*)^* \quad \mathscr{U}(z_1^*)^* \quad \cdots \quad \mathscr{U}(z_\varkappa^*)^*] \tag{1.2}$$

has the maximal rank d.

The equivalence between the two statements in $(\mathscr{U}6)$ is proved in the appendix. See Corollary A.6 which provides a list of equivalent statements.

It turns out that the $(-\mathsf{Q})$-boundary coefficients constructed via (A)–(C) have the additional property $(\mathscr{U}6)$ if and only if S satisfies

$$(\operatorname{dom} S)^{[\perp]} \cap \operatorname{span}\{\ker(S - \lambda)^\varkappa : \lambda \in \sigma_p(S)\} = \{0\}. \tag{1.3}$$

Surprisingly, this condition is equivalent to a *generalized von Neumann equality*: Eq. (1.3) holds if and only if

$$S^{[*]} = S + S^{[*]} \cap \mu_0 I + \sum_{j=0}^{\varkappa} S^{[*]} \cap \mu_j^* I$$

holds for one (and then for any) set of distinct complex numbers $\mu_0, \dots, \mu_\varkappa$ from $\mathbb{C}\backslash(\mathbb{R} \cup \sigma_p(S))$ such that $\mu_j \neq \mu_k^*$, $j,k = 0, \dots, \varkappa$.

If S is a simple symmetric operator then (1.3) holds and so $\mathscr{U}$ in (1.1) constructed via (A)–(C) satisfies $(\mathscr{U}1)$–$(\mathscr{U}6)$. In this paper we show that the converse also holds, that is let $\mathscr{U}$ be a Q-boundary coefficient satisfying $(\mathscr{U}1)$–$(\mathscr{U}6)$. The reproducing kernel Pontryagin space $\mathscr{H}(K_\mathscr{U})$ with reproducing kernel $K_\mathscr{U}(z,w)$ consists of functions which are holomorphic on $\mathrm{dom}(\mathscr{U})$ and in this space the operator $S_\mathscr{U}$ of multiplication by the independent variable z is a simple symmetric operator with defect index (d_-, d_+). There exist a holomorphic row vector function $\Phi : \mathbb{C}^\pm\backslash\gamma \to \mathscr{H}(K_\mathscr{U}) \times \mathscr{H}(K_\mathscr{U}) \times \cdots \times \mathscr{H}(K_\mathscr{U})$ ($d_\mp$ copies) as in (B) and a boundary mapping $\mathsf{b} : S_\mathscr{U}^{[*]} \to \mathbb{C}^d$ for $S_\mathscr{U}$ with Gram matrix $-\mathsf{Q}$ such that (1.3) holds.

As to the contents of the paper: In Section 2 we define standard symmetric linear relations and show that for these relations the defect indices can be defined in the same way as for symmetric relations in a Hilbert space. The generalized von Neumann formula is studied in detail in Section 3. The definition of a boundary mapping and the construction of boundary coefficients can be found in Section 4. In Section 5 we show that a Q-boundary coefficient satisfying $(\mathscr{U}1)$–$(\mathscr{U}5)$ can be reduced to a minimal one. The proof makes use of a geometric interpretation of the maximum modulus principle for generalized Schur functions, which we explain in the appendix. Finally, in Section 6 we derive a representation of a minimal Q-boundary coefficient in terms of a closed simple symmetric operator in a Pontryagin space. This result is a generalization of [7, Theorem 4.4]. The proof given here is simpler than the proof in [7].

Q-boundary coefficients $\mathscr{U}$ with $\varkappa$ negative squares occur in the study of boundary-eigenvalue problems with eigenvalue boundary conditions of the form

$$\mathscr{U}(\lambda)\mathsf{b}(\{f,g\}) = 0, \quad \{f,g\} \in S^{[*]},$$

where λ denotes the eigenvalue. In [7] the linearization of such problems were studied for the case $\varkappa = 0$, in a sequel [3] to this paper we will consider the linearization problem in the Pontryagin space setting.

2. Standard symmetric relations in Pontryagin spaces

Let $S \subset \mathscr{H}^2 = \mathscr{H} \oplus \mathscr{H}$ be a closed symmetric relation in a Pontryagin space $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ with negative index $\varkappa$. Then for all $\mu \in \mathbb{C}\backslash\mathbb{R}$, the ranges $\mathrm{ran}(S-\mu)$ and $\mathrm{ran}(S^{[*]}-\mu)$ are closed. If for some $\mu \in \mathbb{C}\backslash\mathbb{R}$, we have that $\mu \notin \sigma_p(S)$, then each of the

sets $\mathbb{C}^{\pm}\cap\sigma_p(S)$ contains at most $\varkappa$ points [12, Propositions 4.3 and 4.4]. S with this property will be called *standard*. A standard symmetric relation is said to have *finite defect* if for some $\mu\in\mathbb{C}^+$,

$$\mu,\mu^*\notin\sigma_p(S),\quad \dim(\ker(S^{[*]}-\mu))<\infty,\ \dim(\ker(S^{[*]}-\mu^*))<\infty. \tag{2.1}$$

Note that the kernels $\ker(S^{[*]}-\lambda)=\operatorname{ran}(S-\lambda^*)^{[\perp]}$, $\lambda\in\mathbb{C}$, maybe degenerate subspaces. We show that the dimensions of these spaces are the same for essentially all (that is, with the exception of at most finitely many points) $\lambda\in\mathbb{C}^+$, and the dimensions of these spaces are the same for essentially all $\lambda\in\mathbb{C}^-$. See Theorem 2.3.

In our proof of the theorem we use two lemmas. The first one concerns the existence of a maximal standard isometric extension of a standard isometric operator with finite defect. Following the terminology of [4, Definition 5.2.1] we call an isometry $V\subset\mathscr{H}^2$ a *standard isometry* if V is a closed bounded operator whose inverse is also a bounded operator. By definition, $\operatorname{dom}V$ and $\operatorname{ran}V$ are closed (but maybe degenerate). Sorjonen [16] studies rectangular symmetric and isometric relations in Krein spaces. Such relations in Pontryagin spaces are special cases of their standard counter parts defined here. A standard isometry has *finite defect* if

$$\dim(\operatorname{dom}V)^{[\perp]}<\infty,\quad \dim(\operatorname{ran}V)^{[\perp]}<\infty.$$

Lemma 2.1. *Let $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$ be a Pontryagin space and let $V\subset\mathscr{H}^2$ be a standard isometry in $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$. Assume that*

$$\dim(\operatorname{dom}V)^{[\perp]}=d_-,\quad \dim(\operatorname{ran}V)^{[\perp]}=d_+,\quad d_-+d_+<+\infty.$$

Then there exists a maximal isometry $\tilde V$ in $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$ such that $\tilde V$ extends V and $\tilde V$ is a standard isometry. If $d_-<d_+$ ($d_+<d_-$, respectively) for each such $\tilde V$ we have

$$\operatorname{dom}\tilde V=\mathscr{H}\quad(\operatorname{ran}\tilde V=\mathscr{H}), \tag{2.2}$$

$$\dim(\operatorname{ran}\tilde V)^{[\perp]}=d_+-d_-\quad(\dim(\operatorname{dom}\tilde V)^{[\perp]}=d_--d_+), \tag{2.3}$$

$$\dim\tilde V/V=d_-\quad(\dim\tilde V/V=d_+). \tag{2.4}$$

Proof. It follows from [4, Theorem 5.2.2], which concerns a Krein space version of this lemma, that there exists a maximal isometry $\tilde V$ in $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$ such that $\tilde V$ extends V and $\tilde V$ is a standard isometry. By Azizov and Iokhvidov [4, Corollary 5.2.3], $\tilde V$ is a maximal isometry in $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$ if and only if at least one of the following conditions hold:

(a) $(\operatorname{dom}\tilde V)^{[\perp]}=\{0\}$,
(b) $(\operatorname{ran}\tilde V)^{[\perp]}=\{0\}$,

(c) $(\operatorname{dom}\tilde{V})^{[\perp]}\neq\{0\}$ and $(\operatorname{ran}\tilde{V})^{[\perp]}\neq\{0\}$ are uniformly definite subspaces of different signs.

Condition (c) implies that $\operatorname{dom}\tilde{V}$ and $\operatorname{ran}\tilde{V}$ are regular subspaces of the Pontryagin space $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$ and $\tilde{V}$ acts as a unitary operator from $\operatorname{dom}\tilde{V}$ onto $\operatorname{ran}\tilde{V}$. Since $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$ is a Pontryagin space, the maximal uniformly negative subspaces of $(\operatorname{dom}\tilde{V})^{[\perp]}$ and $(\operatorname{ran}\tilde{V})^{[\perp]}$ have the same dimension. Hence (c) is impossible.

Assume that $d_-<d_+$. It follows from the construction in the proof of [4, Theorem 5.2.2] that

$$(\operatorname{dom}\tilde{V})^{[\perp]}=\{0\}\quad\text{and}\quad(\operatorname{ran}\tilde{V})^{[\perp]}=d_+-d_-.$$

Equality (2.4) follows from (2.2) and $\dim(\operatorname{dom}V)^{[\perp]}=d_-$. If $d_+<d_-$, apply the previous case to V^{-1} and obtain the statement of the lemma within the brackets. □

In the next lemma we use the Potapov–Ginzburg transform on a Pontryagin space $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$. Let $\mathscr{H}=\mathscr{H}_+[\dotplus]\mathscr{H}_-$ be a fundamental decomposition of $\mathscr{H}$ and let P_+, P_-, $J=P_+-P_-$, and $\langle\cdot,\cdot\rangle=[J\cdot,\cdot]$ be the corresponding fundamental projections, fundamental symmetry, and corresponding Hilbert space inner product, respectively. Simplifying the notation of [4, Chapter V] we denote by $\omega:\mathscr{H}^2\to\mathscr{H}^2$ the *Potapov–Ginzburg* transform which is the linear involution defined by

$$\omega(\{f,g\}):=\{P_+f+P_-g,P_-f+P_+g\},\quad\{f,g\}\in\mathscr{H}^2.$$

If T is a subspace of $\mathscr{H}^2$, then $\omega(T)$ denotes the image of T under ω. It follows from the definition that $\omega(T^{[*]})=\omega(T)^{\langle*\rangle}$ and $\omega(T^{-1})=\omega(T)^{-1}$. If V is an operator, then

$$\omega(V):=(P_-+P_+V)(P_++P_-V)^{-1}.$$

Lemma 2.2. *Let V be as in Lemma* 2.1. *Then with the above notation, the Potapov–Ginzburg transform $W=\omega(V)$ of V is a (standard) isometry in $(\mathscr{H},\langle\cdot,\cdot\rangle)$ with* $\dim(\operatorname{dom}W)^{\langle\perp\rangle}=d_-$ *and* $\dim(\operatorname{ran}W)^{\langle\perp\rangle}=d_+$.

Proof. It follows from the definition of the Potapov–Ginzburg transform that W is an isometry in the Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$; see also [4, Corollary 5.1.7]. Since W acts in a Hilbert space, W is a standard isometry. Set $\dim(\operatorname{dom}W)^{\langle\perp\rangle}=\delta_-$ and $\dim(\operatorname{ran}W)^{\langle\perp\rangle}=\delta_+$. We will prove the lemma by showing that $d_++d_-=\delta_++\delta_-$ and $d_+-d_-=\delta_+-\delta_-$.

In the notation used above, $\mathscr{H}^2=(\mathscr{H}_+\oplus\mathscr{H}_+)[\dotplus](\mathscr{H}_-\oplus\mathscr{H}_-)$ is a fundamental decomposition of $\mathscr{H}^2$. Denote the corresponding fundamental projections by P_1^+ and P_1^-, the corresponding Hilbert space by $(\mathscr{H}^2,\langle\cdot,\cdot\rangle)$ and the Potapov–Ginzburg transform on this space by ω_1. Consider the block matrix operators (with respect to

the decomposition $\mathscr{H}^2 = \mathscr{H} \oplus \mathscr{H}$)

$$V_1 = \begin{bmatrix} V & 0 \\ 0 & V^{-1} \end{bmatrix} \quad \text{and} \quad W_1 = \begin{bmatrix} W & 0 \\ 0 & W^{-1} \end{bmatrix}.$$

Then $\dim(\operatorname{dom} V_1)^{[\perp]} = \dim(\operatorname{ran} V_1)^{[\perp]} = d_+ + d_-$ and

$$W_1 = \begin{bmatrix} \omega(V) & 0 \\ 0 & \omega(V^{-1}) \end{bmatrix} = \omega_1(V_1).$$

The operator V_1 admits a unitary extension $\tilde{V}_1$ in the Pontryagin space $\mathscr{H}^2$ and Lemma 2.1 implies that $\dim \tilde{V}_1/V_1 = d_+ + d_-$. By Azizov and Iokhvidov [4, Corollary 5.1.7] $\tilde{W}_1 := \omega_1(\tilde{V}_1)$ is a unitary operator in the Hilbert space $(\mathscr{H}^2, \langle \cdot, \cdot \rangle)$. Since ω_1 is a linear involution, we have

$$\dim \tilde{W}_1/W_1 = \dim\, \omega_1(\tilde{V}_1)/\omega_1(V_1) = \dim \tilde{V}_1/V_1 = d_+ + d_-. \tag{2.5}$$

Since $\tilde{W}_1$ is a unitary operator in $(\mathscr{H}_1, \langle \cdot, \cdot \rangle)$, it is a maximal isometry which extends W_1 and which is clearly standard. Since $\dim(\operatorname{dom} W_1)^{\langle \perp \rangle} = \delta_+ + \delta_-$, Lemma 2.1 implies that $\dim \tilde{W}_1/W_1 = \delta_+ + \delta_-$. This and (2.5) imply $d_+ + d_- = \delta_+ + \delta_-$.

If we apply the above reasoning to the operator $\tilde{V}$ from Lemma 2.1 instead of V, we get $d_+ - d_- = \delta_+ - \delta_-$. Consequently, $\dim(\operatorname{dom} W)^{\langle \perp \rangle} = d_-$ and $\dim(\operatorname{ran} W)^{\langle \perp \rangle} = d_+$ and the proposition is proved. □

To prove the theorem below with the help of these two lemmas we use, for $\mu \in \mathbb{C}^+$, the *Cayley transform* V of S defined by

$$V = (S - \mu^*)(S - \mu)^{-1}.$$

This formula establishes a bijective correspondence between the standard closed symmetric relations S with $\mu, \mu^* \notin \sigma_p(S)$ and the standard isometries V. Its inverse is given by $S = (\mu V - \mu^*)(V - I)^{-1}$. Under this correspondence V has finite defect if and only if (2.1) holds and then

$$\dim(\ker(S^{[*]} - \mu)) = \dim(\operatorname{ran} V)^{[\perp]},$$

$$\dim(\ker(S^{[*]} - \mu^*)) = \dim(\operatorname{dom} V)^{[\perp]}.$$

Theorem 2.3. *Let $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ be a Pontryagin space, J a fundamental symmetry on $\mathscr{H}$ and $\langle \cdot, \cdot \rangle$ the corresponding Hilbert space inner product. Let S be a standard symmetric relation with finite defect in $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$. Then*

(a) $\dim \ker(S^{[*]} - \lambda) = \dim \ker((SJ)^{\langle * \rangle} - \lambda)$ *whenever λ is a non-real number such that $\lambda, \lambda^* \notin \sigma_p(S)$,*

(b) *the number* $\dim\ker(S^{[*]}-\lambda)$, $\lambda,\lambda^*\notin\sigma_p(S)$ *is independent of* $\lambda\in\mathbb{C}^+\backslash\sigma_p(S)$ ($\lambda\in\mathbb{C}^-\backslash\sigma_p(S)$, *respectively*),

(c) $\dim S^{[*]}/S=d_++d_-$, *where* $d_+:=\dim\ker(S^{[*]}-\lambda)$ *and* $d_-:=\dim\ker(S^{[*]}-\lambda^*)$ *for some* $\lambda\in\mathbb{C}^+$ *such that* $\lambda,\lambda^*\notin\sigma_p(S)$, *and*

(d) *there exists a closed symmetric extension* $\tilde{S}$ *in* $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$ *of* S *such that at least one of the sets* $\mathbb{C}^\pm\backslash\rho(\tilde{S})$ *is finite*.

Proof. Since S is standard there exists a $\mu\in\mathbb{C}^+$ such that $\mu,\mu^*\notin\sigma_p(S)$. Without loss of generality we can assume that $\mu=i$. Consider the Cayley transforms of S and SJ:

$$V=(S+i)(S-i)^{-1},\quad W=(SJ+i)(SJ-i)^{-1}. \tag{2.6}$$

It follows from [12, Proposition 4.1] and basic properties of closed linear relations that V is a standard isometry in $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$ which satisfies the assumptions of Lemma 2.1. From [4, Theorem 5.1.14] we conclude that $W=\omega(V)$. Since

$$\dim\ker(S^{[*]}-i)=\dim(\operatorname{ran}V)^{[\perp]},$$

$$\dim\ker(S^{[*]}+i)=\dim(\operatorname{dom}V)^{[\perp]},$$

$$\dim\ker((SJ)^{\langle*\rangle}-i)=\dim(\operatorname{ran}W)^{\langle\perp\rangle},$$

$$\dim\ker((SJ)^{\langle*\rangle}+i)=\dim(\operatorname{dom}W)^{\langle\perp\rangle},$$

an application of Lemma 2.2 yields (a).

Statement (b) then follows from (a) and the fact that, since SJ is symmetric in the Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$, the numbers

$$\dim\ker((SJ)^{\langle*\rangle}-\lambda),\quad \lambda\in\mathbb{C}^+,$$

$$\dim\ker((SJ)^{\langle*\rangle}-\lambda),\quad \lambda\in\mathbb{C}^-,$$

are independent of λ.

To prove (c) note that the mapping $\{f,g\}\mapsto\{Jf,g\}$, $\{f,g\}\in\mathscr{H}^2$, is a linear bijection between $S^{[*]}$ and $(SJ)^{\langle*\rangle}$ which maps S onto SJ. Therefore

$$\dim S^{[*]}/S=\dim(SJ)^{\langle*\rangle}/(SJ). \tag{2.7}$$

Since SJ is a symmetric relation in the Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$, the von Neumann formula implies that

$$\dim(SJ)^{\langle*\rangle}/(SJ)=\dim\ker((SJ)^*-i)+\dim\ker((SJ)^*+i). \tag{2.8}$$

Statement (c) now follows from (2.7), (2.8) and (a).

To prove (d) consider the standard operator V defined in (2.6). Assume that $d_+ := \dim\ker(S^{[*]} - i) = \dim(\operatorname{ran} V)^{[\perp]} \geqslant \dim\ker(S^{[*]} + i) = \dim(\operatorname{dom} V)^{[\perp]} = d_-$. By Lemma 2.1, there exists a standard isometry $\tilde{V}$ which extends V, $\operatorname{dom}\tilde{V} = \mathscr{H}$ and $\dim(\operatorname{ran}\tilde{V})^{[\perp]} = d_+ - d_-$. Put

$$\tilde{S} := i(\tilde{V}+1)(\tilde{V}-1)^{-1}.$$

It follows that $\tilde{S}$ is a standard symmetric relation, $i, -i \notin \sigma_p(\tilde{S})$ and $\dim\ker(\tilde{S}^{[*]} - i) = d_+ - d_-$ and $\dim\ker(\tilde{S}^{[*]} + i) = 0$. Therefore $i \in \rho(\tilde{S})$. It follows from [12, Theorem 4.6 and Corollary] that $\rho(\tilde{S}) \cap \mathbb{C}^+ = \mathbb{C}^+ \setminus \sigma_p(\tilde{S})$ and $\mathbb{C}^+ \cap \sigma_p(\tilde{S})$ consists of at most $\varkappa$ points, where $\varkappa$ is the negative index of $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$. □

Theorem 2.3 yields that if S is a standard symmetric relation with finite defect, the dimension of the subspace $\ker(S^{[*]} - z)$ is constant for essentially all points z in each of the open half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$. This constant is denoted by d_+ for $z \in \mathbb{C}^+$ and by d_- for $z \in \mathbb{C}^-$. The numbers d_+ and d_- are called the *upper* and *lower defect numbers* of S, the pair (d_+, d_-) is called the *defect index* and the number $d = d_- + d_+$ is called the *defect*.

Following [14, Section 2.2], a closed symmetric relation S in a Pontryagin space $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ will be called *simple* if it has no non-real eigenvalues and

$$\overline{\operatorname{span}}\{\ker(S^{[*]} - \lambda) : \lambda \in \mathbb{C}\setminus\mathbb{R}\} = \mathscr{H}. \tag{2.9}$$

Since

$$\overline{\operatorname{span}}\{\ker(S^{[*]} - \lambda) : \lambda \in \mathbb{C}\setminus\mathbb{R}\}^{[\perp]} = \bigcap\{\operatorname{ran}(S - \lambda) : \lambda \in \mathbb{C}\setminus\mathbb{R}\},$$

equality (2.9) is equivalent to

$$\bigcap\{\operatorname{ran}(S - \lambda) : \lambda \in \mathbb{C}\setminus\mathbb{R}\} = \{0\}. \tag{2.10}$$

Proposition 2.4. *Let S be a simple, closed symmetric relation in a Pontryagin space. Then S is a standard operator and $\sigma_p(S) = \emptyset$.*

Proof. Denote the Pontryagin space by $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$. By definition each simple relation in a Pontryagin space is standard. Let $z \in \mathbb{C}\setminus\mathbb{R}$ and $g \in \ker(S^{[*]} - z)$ be arbitrary. We first prove that S is an operator. Let $\{0, f\} \in S$. Since $\{g, zg\} \in S^{[*]}$ we have

$$0 = [f, g] - [0, zg] = [f, g].$$

So by (2.9), $f = 0$. Thus S is an operator.

Let $\alpha \in \mathbb{R}$ and $\{f, \alpha f\} \in S$. Since $\{g, zg\} \in S^{[*]}$ we have

$$0 = [\alpha f, g] - [f, zg] = (\alpha - z^*)[f, g].$$

Since $\alpha - z^* \neq 0$, and by (2.9) we have that $f = 0$. Thus $\alpha \notin \sigma_p(S)$, that is, $\sigma_p(S) \cap \mathbb{R} = \emptyset$. Since S is simple it does not have eigenvalues in $\mathbb{C}\backslash\mathbb{R}$. □

Remark 2.5. An alternative to the last part of the proof of this lemma is: Assume $\alpha \in \mathbb{R}$ and $\{f, \alpha f\} \in S$. Then for all non-real numbers λ, $f = (\alpha - \lambda)^{-1}(S - \lambda)f \in \operatorname{ran}(S - \lambda)$, hence f belongs to the intersection on the left of the equality in (2.10) and therefore $f = 0$. This proves $\sigma_p(S) \cap \mathbb{R} = \emptyset$. An argument like this will be used in the next section.

3. Von Neumann's formula

In this section we consider a generalization of the von Neumann formula

$$S^{\langle * \rangle} = S + S^{\langle * \rangle} \cap \mu I + S^{\langle * \rangle} \cap \mu^* I, \quad \mu \in \mathbb{C}\backslash\mathbb{R}, \quad \text{direct sum}, \tag{3.1}$$

for closed symmetric relations S in a Hilbert space. If S is a closed densely defined symmetric operator in a Pontryagin space, formula (3.1) holds for all μ with $|\operatorname{Im}\mu|$ sufficiently large, see [14]. By Dijksma and de Snoo [12, Proposition 4.7], if S is standard symmetric relation in a Pontryagin space then (3.1) holds if and only if $\mu \notin \sigma_p(S)$ and $\operatorname{ran}(S - \mu)$ is non-degenerate. The following example, due to Derkach (private communication, see also [8, Remark 4.1]), shows that formula (3.1) does not hold for any $\mu \in \mathbb{C}\backslash\mathbb{R}$ even if S is a simple operator in a Pontryagin space.

Example 3.1. Let $\mathscr{H} = \mathbb{C}^4$, $J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $[f, g] = g^* J f, \quad g, f \in \mathscr{H}$,

and

$$S = \left\{ \left\{ \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \\ 0 \\ 0 \end{bmatrix} \right\} : x \in \mathbb{C} \right\}.$$

The operator S is closed standard and symmetric in $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ and $\sigma_p(S) = \emptyset$. Since

$$\operatorname{ran}(S - \mu) = \operatorname{span} \begin{bmatrix} -\mu \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

by (2.10), S is simple. As $\operatorname{ran}(S-\mu)$ is degenerate for each $\mu\in\mathbb{C}$, it follows that (3.1) does not hold for any $\mu\in\mathbb{C}$. A calculation of $S^{[*]}$ yields

$$S^{[*]}=\left\{\left\{\begin{bmatrix}f_1\\ f_2\\ f_3\\ f_4\end{bmatrix},\begin{bmatrix}g_1\\ g_2\\ g_3\\ f_3\end{bmatrix}\right\}:f_j,g_k\in\mathbb{C}\right\}.$$

For arbitrary $\mu\in\mathbb{C}\setminus\mathbb{R}$ we have

$$M_\mu:=S^{[*]}\cap\mu I=\left\{\left\{\begin{bmatrix}f_1\\ f_2\\ \mu f_4\\ f_4\end{bmatrix},\begin{bmatrix}\mu f_1\\ \mu f_2\\ \mu^2 f_4\\ \mu f_4\end{bmatrix}\right\}:f_j\in\mathbb{C}\right\}.$$

One can prove that for arbitrary $\mu,\nu\in\mathbb{C}^+,\ \mu\neq\nu$ we have

$$S^{[*]}=S+M_{\mu^*}+M_\mu+M_\nu.$$

This formula can be considered as a generalization of (3.1); note that the sum on the right-hand side is not a direct sum.

In this section we give a necessary and sufficient condition for a generalized von Neumann formula to hold for a standard symmetric operator in a Pontryagin space. We start with the following lemma.

Lemma 3.2. *Let S be a linear operator in a vector space $\mathscr{H}$ and let $\mu_1,\dots,\mu_k$ be distinct complex numbers which are not eigenvalues of S. Then*

$$\bigcap_{j=1}^{k}\operatorname{ran}(S-\mu_j)=\operatorname{ran}\left(\prod_{j=1}^{k}(S-\mu_j)\right).$$

Proof. For $k=1$ the statement of the lemma is true. Assume that the statement is true for $k\geqslant 2$ and prove it for $k+1$. Let $\mu_1,\dots,\mu_k,\mu_{k+1}$ be distinct complex numbers which are not eigenvalues of S. By the induction hypothesis,

$$\begin{aligned}\bigcap_{j=1}^{k+1}\operatorname{ran}(S-\mu_j)&=\operatorname{ran}(S-\mu_1)\cap\operatorname{ran}\left(\prod_{j=2}^{k+1}(S-\mu_j)\right)\\&=\operatorname{ran}(S-\mu_{k+1})\cap\operatorname{ran}\left(\prod_{j=1}^{k}(S-\mu_j)\right).\end{aligned}\tag{3.2}$$

Let f be an arbitrary vector in subspace (3.2). Then there exist u, v such that

$$f = \left(\prod_{j=2}^{k+1}(S-\mu_j)\right)u = \left(\prod_{j=1}^{k}(S-\mu_j)\right)v.$$

Since $\mu_2, \ldots, \mu_k \notin \sigma_p(S)$,

$$g := \left(\prod_{j=2}^{k}(S-\mu_j)\right)^{-1} f = (S-\mu_{k+1})u = (S-\mu_1)v$$

and

$$v = \frac{1}{\mu_{k+1}-\mu_1}(g-(S-\mu_{k+1})v) \in \operatorname{ran}(S-\mu_{k+1}),$$

hence $g \in \operatorname{ran}(S-\mu_1)(S-\mu_{k+1})$. Consequently,

$$f \in \operatorname{ran}\left(\prod_{j=1}^{k+1}(S-\mu_j)\right).$$

This proves the inclusion $\subset$ in the lemma. The converse inclusion is evident. □

Lemma 3.3. *Let S be a standard symmetric operator in the Pontryagin space $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ of negative index $\varkappa$. For $\mu_0, \mu_1, \ldots, \mu_\varkappa \in \mathbb{C}$ put*

$$\mathscr{L} := \ker(S^{[*]} - \mu_0^*) \cap \left(\bigcap_{j=0}^{\varkappa} \operatorname{ran}(S-\mu_j)\right) \tag{3.3}$$

and

$$\mathscr{N} := (\operatorname{dom} S)^{[\perp]} \cap \operatorname{span}\{\ker(S-\lambda)^\varkappa : \lambda \in \sigma_p(S)\}.$$

(a) *If $\mu_0^*, \mu_0, \mu_1, \ldots, \mu_\varkappa$ are distinct numbers from $\mathbb{C}\backslash\sigma_p(S)$, then $\mathscr{N} \subset (S-\mu_0^*)\mathscr{L}$.*
(b) *If $\mu_0^*, \mu_0, \mu_1, \ldots, \mu_\varkappa$ are distinct numbers from $\mathbb{C}\backslash(\mathbb{R}\cup\sigma_p(S))$ such that $\mu_j \neq \mu_k^*$, $j,k = 0, \ldots, \varkappa$, then $\mathscr{N} = (S-\mu_0^*)\mathscr{L}$.*

Proof. To prove (a), let $g \in \mathscr{N}$. Since $g \in (\operatorname{dom} S)^{[\perp]}$, we have $\{0, g\} \in S^{[*]}$, and since

$$g \in \operatorname{span}\{\ker(S-\lambda)^\varkappa : \lambda \in \sigma_p(S)\},$$

we have $g \in \bigcap_{\mu \notin \sigma_p(S)} \operatorname{ran}(S-\mu)$. Let $\mu_0^*, \mu_0, \mu_1, \ldots, \mu_\varkappa \notin \sigma_p(S)$ be arbitrary distinct complex numbers. Then by Lemma 3.2,

$$g \in \operatorname{ran}\left((S-\mu_0^*)\prod_{j=0}^{\varkappa}(S-\mu_j)\right).$$

For $f := (S-\mu_0^*)^{-1}g$, we have $f \in \bigcap_{j=0}^{\varkappa} \operatorname{ran}(S-\mu_j)$. From

$$\{f, \mu_0^* f\} + \{0, g\} = \{(S-\mu_0^*)^{-1}g, S(S-\mu_0^*)^{-1}g\} \in S,$$

we conclude $\{f, \mu_0^* f\} \in S^{[*]}$. Thus $f \in \mathscr{L}$, that is, $g \in (S-\mu_0^*)\mathscr{L}$.

To prove (b), let $\mu_0^*, \mu_0, \mu_1, \ldots, \mu_\varkappa \notin \sigma_p(S)$ be arbitrary non-real distinct numbers such that $\mu_j \neq \mu_k^*$, $j,k = 0, \ldots, \varkappa$, and put

$$P_0(z) = \prod_{j=0}^{\varkappa} (z-\mu_j)$$

and

$$P_k(z) = \prod_{j=0, j\neq k}^{\varkappa} (z-\mu_j), \quad k = 1, \ldots, \varkappa.$$

If $\mathscr{L} = \{0\}$ the statement is trivial. Let $0 \neq f \in \mathscr{L}$. Since $\ker(S^{[*]} - \mu_0^*) = (\operatorname{ran}(S-\mu_0))^{[\perp]}$, definition (3.3) of $\mathscr{L}$ implies $[f,f] = 0$. By Lemma 3.2 and (3.3), there exists $0 \neq h \in \operatorname{dom} S^{\varkappa+1}$ such that $f = P_0(S)h$. Put $h_k = P_k(S)h \in \operatorname{dom} S$, $k = 1, \ldots, \varkappa$. Then $f = (S-\mu_k)h_k$, $k = 1, \ldots, \varkappa$. The assumption $\{f, \mu_0^* f\} \in S^{[*]}$ implies

$$\begin{aligned}(\mu_0 - \mu_k)[h_k, f] &= \mu_0[h_k, f] - [f + \mu_k h_k, f] \\ &= [h_k, \mu_0^* f] - [Sh_k, f] \\ &= 0, \quad k = 1, \ldots, \varkappa.\end{aligned}$$

Since $\mu_0 \neq \mu_k$ for $k = 1, \ldots, \varkappa$, we conclude that $[h_k, f] = 0$. For $j,k = 1, \ldots, \varkappa$ we have

$$\begin{aligned}(\mu_j - \mu_k^*)[h_j, h_k] &= [f + \mu_j h_j, h_k] - [h_j, f + \mu_k h_k] \\ &= [Sh_j, h_k] - [h_j, Sh_k] \\ &= 0.\end{aligned}$$

As $\mu_j \neq \mu_k^*$, we have $[h_j, h_k] = 0$ for $j,k = 1, \ldots, \varkappa$. Thus the subspace $\mathscr{M} := \operatorname{span}\{f, h_1, \ldots, h_\varkappa\}$ is neutral in $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$. Therefore $\dim \mathscr{M} \leqslant \varkappa$ implying that the vectors $h_0 := f, h_1, \ldots, h_\varkappa$ are linearly dependent. Let $\mathbb{C}^{\varkappa+1} \ni (\alpha_0, \ldots, \alpha_\varkappa) \neq (0, \ldots, 0)$ be such that

$$\sum_{k=0}^{\varkappa} \alpha_k h_k = \sum_{k=0}^{\varkappa} \alpha_k P_k(S)h = 0.$$

Since the polynomials $P_0, \ldots, P_\varkappa$ are linearly independent, $Q := \sum_{j=0}^{\varkappa} \alpha_j P_j$ is a non-zero polynomial and $Q(S)h = 0$. The number μ_0 is a root of Q which is not an

eigenvalue of S. Since $Q(S)h = 0$ and $h \neq 0$ the polynomial Q has roots which are eigenvalues of S. Let $\lambda_j,\ j = 1, \ldots, m$, be the distinct roots of Q which are eigenvalues of S and let $m_j,\ j = 0, \ldots, m$, be the corresponding multiplicities. Note that m and all m_j's are $\leqslant \varkappa$. Since $Q(S)h = 0$ and $h \neq 0$, we have

$$h \in \operatorname{span}\{\ker(S - \lambda_j)^{m_j} : j = 1, \ldots, m\} \subset \operatorname{span}\{\ker(S - \lambda)^{\varkappa} : \lambda \in \sigma_p(S)\}.$$

As $f = P_0(S)h$, we also have $f \in \operatorname{span}\{\ker(S - \lambda)^{\varkappa} : \lambda \in \sigma_p(S)\}$. This implies that $f \in \operatorname{dom} S$ and $Sf \in \operatorname{span}\{\ker(S - \lambda)^{\varkappa} : \lambda \in \sigma_p(S)\}$. Consequently,

$$(S - \mu_0^*)f \in \operatorname{span}\{\ker(S - \lambda)^{\varkappa} : \lambda \in \sigma_p(S)\}. \tag{3.4}$$

Since $\{f, Sf\} \in S \subset S^{[*]}$ and $\{f, \mu_0^* f\} \in S^{[*]}$, we conclude that $\{0, Sf - \mu_0^* f\} \in S^{[*]}$, that is,

$$(S - \mu_0^*)f \in (\operatorname{dom} S)^{[\perp]}. \tag{3.5}$$

It follows from (3.4) and (3.5) that $(S - \mu_0^*)f \in \mathscr{N}$. This proves that $\mathscr{N} \supset (S - \mu_0^*)\mathscr{L}$. As the converse inclusion was proved in (a), (b) is proved. □

Remark 3.4. It follows from Lemma 3.3 that the non-real complex numbers $\mu_0, \ldots, \mu_\varkappa$ which are not eigenvalues of S and such that $\mu_j \neq \mu_k^*,\ j, k = 0, \ldots, \varkappa$, can be chosen arbitrarily without changing $\mathscr{L}$. Therefore, for arbitrary $\mu_0, \mu_1, \ldots, \mu_\varkappa$ which satisfy these conditions we have

$$\begin{aligned} &\ker(S^{[*]} - \mu_0^*) \bigcap \left(\bigcap_{j=0}^{\varkappa} \operatorname{ran}(S - \mu_j) \right) \\ &\quad = \ker(S^{[*]} - \mu_0^*) \bigcap \left(\bigcap_{\mu \in \mathbb{C} \setminus (\sigma_p(S) \cup \mathbb{R})} \operatorname{ran}(S - \mu) \right). \end{aligned} \tag{3.6}$$

A consequence of equality (3.6) and Lemma 3.3 is that if

$$\ker(S^{[*]} - \mu_0^*) \bigcap \left(\bigcap_{\mu \in \mathbb{C} \setminus (\sigma_p(S) \cup \mathbb{R})} \operatorname{ran}(S - \mu) \right) = \{0\},$$

then $\mathscr{N} = \{0\}$.

Remark 3.5. The condition $\mathscr{N} = \{0\}$ is satisfied in any of the following cases:

(a) $\varkappa = 0$;
(b) S is a simple symmetric operator;
(c) $\ker(S^{[*]} - \lambda)$ is non-degenerate for at least one non-real complex number λ such that $\lambda, \lambda^* \notin \sigma_p(S)$;

(d) S is densely defined, which by Krein and Langer [14], is a special case of (c).

Lemma 3.6. *Let S be a closed symmetric operator in the Pontryagin space $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$. Then for $\mu_0, \ldots, \mu_k \in \mathbb{C}$ such that $\mu_0^* \notin \sigma_p(S)$ and $\mu_0^* \neq \mu_j$, $j = 1, 2, \ldots, k$, we have*

$$\ker(S^{[*]} - \mu_0^*) \bigcap \left(\bigcap_{j=0}^{k} \operatorname{ran}(S - \mu_j) \right) = \{0\} \tag{3.7}$$

if and only if

$$S^{[*]} = S + S^{[*]} \cap \mu_0 I + \sum_{j=0}^{k} S^{[*]} \cap \mu_j^* I. \tag{3.8}$$

Proof. Equality (3.7) is equivalent to the equality

$$\operatorname{ran}(S - \mu_0) + \sum_{j=0}^{k} \ker(S^{[*]} - \mu_j^*) = \mathscr{H}. \tag{3.9}$$

Now we use (3.9) to prove (3.8). It is sufficient to prove that an arbitrary $\{f, g\} \in S^{[*]}$ belongs to the right-hand side of (3.8). By (3.9), for $\{f, g\} \in S^{[*]}$ there exist $\{u, v\} \in S$ and $x_j \in \ker(S^{[*]} - \mu_j^*)$ such that

$$g - \mu_0 f = v - \mu_0 u + \sum_{j=0}^{k} (\mu_j^* - \mu_0) x_j.$$

Put

$$y = f - u - \sum_{j=0}^{k} x_j.$$

Then

$$\mu_0 y = \mu_0 f - \mu_0 u - \mu_0 \sum_{j=0}^{k} x_j = g - v - \sum_{j=0}^{k} \mu_j^* x_j.$$

Since

$$\{f, g\}, \{u, v\}, \{x_j, \mu_j^* x_j\} \in S^{[*]},$$

we have $\{y, \mu_0 y\} \in S^{[*]} \cap \mu_0 I = M_{\mu_0}$. Thus

$$f = u + y + \sum_{j=0}^{k} x_j, \quad g = v + \mu_0 y + \sum_{j=0}^{k} \mu_j^* x_j, \tag{3.10}$$

that is $\{f, g\}$ belongs to the right-hand side of (3.8).

Conversely, assume (3.8). Since $\mu_0^* \notin \sigma_p(S)$ we have $\operatorname{ran}(S^{[*]} - \mu_0) = (\ker(S - \mu_0^*))^{[\perp]} = \mathscr{H}$. Let $h \in \mathscr{H}$ be arbitrary and $\{f, g\} \in S^{[*]}$ such that $h = g - \mu_0 f$. By (3.8), there exist

$$\{u, v\} \in S, \quad \{y, \mu_0 y\}, \{x, \mu_j^* x\} \in S^{[*]}, \quad j = 0, \ldots, k,$$

such that (3.10) holds. Then

$$h = g - \mu_0 f = v - \mu_0 u + \sum_{j=0}^{k} (\mu_j^* - \mu_0) x_j.$$

Thus h belongs to the left-hand side of (3.9). The proposition is proved. □

The next theorem gives a necessary and sufficient condition for (3.8). It is a direct consequence of Lemmas 3.3 and 3.6.

Theorem 3.7. *Let S be a standard symmetric operator in the Pontryagin space $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ of negative index $\varkappa$. The generalized von Neumann formula*

$$S^{[*]} = S + S^{[*]} \cap \mu_0 I + \sum_{j=0}^{\varkappa} S^{[*]} \cap \mu_j^* I \tag{3.11}$$

holds for one and then for any set of distinct complex numbers $\mu_0, \ldots, \mu_\varkappa$ from $\mathbb{C} \setminus (\mathbb{R} \cup \sigma_p(S))$ such that $\mu_j \neq \mu_k^*$, $j, k = 0, \ldots, \varkappa$, if and only if

$$(\operatorname{dom} S)^{[\perp]} \cap \operatorname{span}\{\ker(S - \lambda)^\varkappa : \lambda \in \sigma_p(S)\} = \{0\}. \tag{3.12}$$

Proof. Assume that equality (3.11) holds for distinct complex numbers $\mu_0, \ldots, \mu_\varkappa$ from $\mathbb{C} \setminus (\mathbb{R} \cup \sigma_p(S))$ such that $\mu_j \neq \mu_k^*$, $j, k = 0, \ldots, \varkappa$. By Lemma 3.6, this is equivalent to

$$\ker(S^{[*]} - \mu_0^*) \bigcap \left(\bigcap_{j=0}^{\varkappa} \operatorname{ran}(S - \mu_j) \right) = \{0\}.$$

By Lemma 3.3(b), the last equality is equivalent to (3.12). □

The following example gives a closed symmetric relation in a Pontryagin space for which the generalized von Neumann formula (3.8) does not hold.

Example 3.8. Let $\mathscr{H} = \mathbb{C}^2$ and $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $[f,g] = g^*Jf$, $g,f \in \mathscr{H}$, and

$$S = \left\{ \left\{ \begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} : x \in \mathbb{C} \right\}.$$

Then S is a closed symmetric operator in $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$, $\sigma_p(S) = \{0\}$, and

$$S^{[*]} = \left\{ \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} u \\ u \end{bmatrix} \right\} : x, y, u \in \mathbb{C} \right\}.$$

For arbitrary $\mu \in \mathbb{C}\backslash\mathbb{R}$ we have

$$M_\mu := S^{[*]} \cap \mu I = \left\{ \left\{ \begin{bmatrix} x \\ x \end{bmatrix}, \mu \begin{bmatrix} x \\ x \end{bmatrix} \right\} : x \in \mathbb{C} \right\}.$$

Clearly $\operatorname{dom} S^{[*]} = \mathscr{H}$. Since for arbitrary $\mu_1, \ldots, \mu_k \in \mathbb{C}\backslash\mathbb{R}$ the domain of the sum of the subspaces M_{μ_j}, $j = 1, \ldots, k$, coincides with the domain of S, we conclude that

$$S^{[*]} \supsetneq S + \sum_{j=1}^{k} M_{\mu_j}.$$

4. An application to boundary coefficients

Let $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ be a Pontryagin space, J a fundamental symmetry on $\mathscr{H}$ and $\langle \cdot, \cdot \rangle$ the corresponding Hilbert space inner product. Introduce two Lagrange inner products on $\mathscr{H}^2$ by

$$[\![\{f,g\}, \{u,v\}]\!] := \frac{1}{i}([g,u] - [f,v]),$$

$$\langle\!\langle \{f,g\}, \{u,v\} \rangle\!\rangle := \frac{1}{i}(\langle g,u \rangle - \langle f,v \rangle).$$

Then $(\mathscr{H}^2, [\![\cdot,\cdot]\!])$ and $(\mathscr{H}^2, \langle\!\langle \cdot,\cdot \rangle\!\rangle)$ are Krein spaces and the mapping

$$j : \{f,g\} \mapsto \{Jf, g\}, \quad \{f,g\} \in \mathscr{H}^2$$

is a unitary mapping between these Krein spaces. A closed subspace $S \subset \mathscr{H}^2$ is a (maximal) symmetric relation in $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ (respectively $(\mathscr{H}, \langle \cdot,\cdot \rangle)$) if

and only if S is a neutral (maximal neutral) subspace of $(\mathscr{H}^2, [\![\cdot,\cdot]\!])$ (respectively $(\mathscr{H}^2, \langle\langle\cdot,\cdot\rangle\rangle)$). It is a self-adjoint relation in $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ (respectively $(\mathscr{H}, \langle\cdot,\cdot\rangle)$) if and only if it is a neutral and maximal semi-definite (a notion to be handled with care, in [4] it is called hyper-maximal neutral) subspace of $(\mathscr{H}^2, [\![\cdot,\cdot]\!])$ (respectively $(\mathscr{H}^2, \langle\langle\cdot,\cdot\rangle\rangle)$). Since $\mathscr{J}$ is a unitary mapping S is symmetric (self-adjoint) in $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ if and only if $\mathscr{J}(S)$ is symmetric (self-adjoint) in $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. Note $\mathscr{J}(S) = SJ$ and $\mathscr{J}((SJ)^{\langle * \rangle}) = S^{[*]}$.

Extensions of a standard symmetric relation that are restrictions of its adjoint can be described in terms of its boundary mapping. Let $S \subset \mathscr{H}^2$ be a standard symmetric relation in $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ with defect index (d_+, d_-). By Theorem 2.3, $\dim S^{[*]}/S = d_+ + d_- =: d$. A *boundary mapping* for S is a surjective linear operator $\mathsf{b} : S^{[*]} \to \mathbb{C}^d$ with $\ker(\mathsf{b}) = S$. If b is a boundary mapping for S then there is a unique $d \times d$ matrix Q such that for all $\{f,g\}, \{u,v\} \in S^{[*]}$,

$$[\![\{f,g\},\{u,v\}]\!] = \mathsf{b}(u,v)^* \mathsf{Q} \mathsf{b}(f,g).$$

Q is a self-adjoint and invertible matrix and has d_+ positive and d_- negative eigenvalues. The matrix Q is called the *Gram matrix* for b.

It is easy to see that a mapping $\mathsf{b} : S^{[*]} \to \mathbb{C}^d$ is a boundary mapping for S with Gram matrix Q if and only if the composition mapping $\mathsf{b}\mathscr{J} : (SJ)^{\langle * \rangle} \to \mathbb{C}^d$ is a boundary mapping for SJ with the same Gram matrix Q. All extensions of S which are restrictions of $S^{[*]}$ are described by

$$A_{\mathsf{M}} := \{\{f,g\} \in S^{[*]} : \mathsf{M}\mathsf{b}(f,g) = 0\}, \tag{4.1}$$

where M is a $k \times d$ matrix, $0 \leqslant k \leqslant d$, of rank k. Clearly

$$A_{\mathsf{M}} J = \{\{u,v\} \in (SJ)^{\langle * \rangle} : \mathsf{M}\mathsf{b}\mathscr{J}(u,v) = 0\}. \tag{4.2}$$

Equalities (4.1), (4.2), [7, Lemma 3.4] imply that the following statements are equivalent:

(a) For a closed linear relation T we have $S \subset T \subset S^{[*]}$, and $\dim(T/S) = \tau$.
(b) There exists a $(d-\tau) \times d$ matrix A of maximal rank such that

$$T = \{\{f,g\} \in S^{[*]} : \mathsf{A}\mathsf{b}(f,g) = 0\}.$$

(c) There exists a $\tau \times d$ matrix B of maximal rank such that

$$T^{[*]} = \{\{f,g\} \in S^{[*]} : \mathsf{B}\mathsf{b}(f,g) = 0\}.$$

If (a)–(c) hold, then $B\mathsf{Q}^{-1}\mathsf{A}^* = 0$ and the matrices A and B are determined uniquely up to multiplication from the left by invertible matrices.

(d) If (a)–(c) hold and if C is a $\tau \times d$ matrix of maximal rank such that $\mathsf{C}\mathsf{Q}^{-1}\mathsf{A}^* = 0$ and

$$V = \{\{f,g\} \in S^{[*]} : \mathsf{C}\mathsf{b}(f,g) = 0\},$$

then $T^{[*]} = V$.

If $d_+ = 0$ or $d_- = 0$ then S is maximal symmetric. It follows from (a)–(d) that if $0 < d_+ < d_-$, then T in (b) is maximal symmetric if and only if $\tau = d_+$ and A is a $d_- \times d$ matrix of rank d_- satisfying $\mathsf{A}\mathsf{Q}^{-1}\mathsf{A}^* = 0$; T then has defect index $(0, d_- - d_+)$. A similar result holds when $0 < d_- < d_+$. It also follows from (a)–(d) that S has *canonical* self-adjoint extensions $A = A^{[*]}$ (that is, self-adjoint extensions in the space $\mathscr{H}$ in which S is defined) if and only if $\tau = d_+ = d_-$. If $\tau = d_+ = d_-$, a relation A is a self-adjoint extension of S if and only if

$$A = \{\{f,g\} \in S^{[*]} : \mathsf{D}\mathsf{b}(f,g) = 0\},$$

where D is a $\tau \times d$ matrix of maximal rank satisfying $\mathsf{D}\mathsf{Q}^{-1}\mathsf{D}^* = 0$.

Families of subspaces between S and $S^{[*]}$ depending on the parameter $z \in \mathbb{C}\backslash\mathbb{R}$ and of the form

$$T(z) = \{\{f,g\} \in S^{[*]} : \mathscr{U}(z)\mathsf{b}(f,g) = 0\},$$

where $\mathscr{U}(z)$ is a Q-boundary coefficient satisfying conditions $(\mathscr{U}1)$–$(\mathscr{U}5)$ with $\varkappa = 0$, were studied in [7]. The so-called linearization problem considered in [7] was: When does there exist a self-adjoint Hilbert space extension A of S such that

$$(T(z) - z)^{-1} = P_{\mathscr{H}}(A - z)^{-1}|_{\mathscr{H}}.$$

A canonical self-adjoint extension A of S defined by D as above corresponds to $\mathscr{U}(z)$ which can be chosen such that $\mathscr{U}(z) \equiv \mathsf{D}$. Non-canonical self-adjoint extensions A, that is, self-adjoint extensions defined in a Hilbert space containing the Hilbert space $\mathscr{H}$ as a proper closed subspace, correspond to the more general Q-boundary coefficients. In the context of the linearization problem the parameter z is called the eigenvalue parameter. In a sequel [3] to this paper we shall consider the linearization problem for S in a Hilbert space but then with a Q-boundary coefficient satisfying $(\mathscr{U}1)$–$(\mathscr{U}5)$ with $\varkappa > 0$.

In this section we present a method to construct Q-boundary coefficients $\mathscr{U}$ satisfying $(\mathscr{U}1)$–$(\mathscr{U}5)$ with $\varkappa > 0$. In Section 6 we prove that $\mathscr{U}$ can always be obtained in this way.

Lemma 4.1. *Let S be a standard symmetric relation with finite defect index (d_+, d_-) in a Pontryagin space $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$. Then there exists a holomorphic row vector function $\Phi : \mathbb{C}^{\pm}\backslash\gamma \to \mathscr{H}^{d_\pm}$, where γ is a finite subset of $\mathbb{C}\backslash\mathbb{R}$, such that the components of $\Phi(z)$ constitute a basis for* $\ker(S^{[*]} - z)$, $z \in \mathbb{C}\backslash(\mathbb{R} \cup \gamma)$.

Proof. Let $\mu\in\mathbb{C}^+$ be such that $\mu,\mu^*\notin\sigma_p(S)$ and assume that $d_+\geqslant d_-$. By Theorem 2.3, there exists a maximal symmetric relation extension $\tilde S$ of S in $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$ such that $\mu,\mu^*\notin\sigma_p(\tilde S)$ and $\dim\ker(\tilde S^{[*]}-\mu)=d_+-d_-$ and $\ker(\tilde S^{[*]}-\mu^*)=\{0\}$. It follows from Theorem 2.3 that $\mathbb{C}^+\backslash\rho(\tilde S)$ consists of finitely many points and the function $z\mapsto(\tilde S-z)^{-1}$, $z\in\mathbb{C}^+\cap\rho(\tilde S)$, is a holomorphic function with values in $\mathscr{L}(\mathscr{H})$. As $\ker(\tilde S-\mu)=\{0\}$, we have $\mu\in\mathbb{C}^+\cap\rho(\tilde S)$. Next, we prove that the (everywhere defined) bounded operators

$$B(z):=I+(z-\mu)(\tilde S-z)^{-1},\quad z\in\mathbb{C}^+\cap\rho(\tilde S),$$

are injective. Let $z\in\mathbb{C}^+\cap\rho(\tilde S)$ and $f\in\mathscr{H}$ be such that $B(z)f=0$. Then $(z-\mu)(\tilde S-z)^{-1}f=-f$ and therefore $\{-f,(z-\mu)f\}\in\tilde S-z$, or $\{-f,-\mu f\}\in\tilde S$. Since $\mu\in\rho(\tilde S)$ we conclude that $f=0$. Next, we prove that $B(z)\ker(S^{[*]}-\mu)=\ker(S^{[*]}-z)$ for all $z\in\mathbb{C}^+\cap\rho(\tilde S)$. By Theorem 2.3, $\dim\ker(S^{[*]}-\mu)=\dim\ker(S^{[*]}-z)$ and since $B(z)$ is injective it is sufficient to show that $B(z)\ker(S^{[*]}-\mu)[\perp]\operatorname{ran}(S-z^*)$. Let $f\in\ker(S^{[*]}-\mu)=(\operatorname{ran}(S-\mu^*))^{[\perp]}$ and $\{u,v\}\in S$. Then

$$\begin{aligned}[B(z)f,v-z^*u]&=[(I+(z-\mu)(\tilde S-z)^{-1})f,v-z^*u]\\&=[f,v]-z[f,u]+(z-\mu)[(\tilde S-z)^{-1}f,v-z^*u]\\&=[f,v]-z[f,u]+(z-\mu)[f,u]\\&=[f,v]-\mu[f,u]=[f,v-\mu^*u]\\&=0.\end{aligned}$$

Thus $B(z)f[\perp]\operatorname{ran}(S-z^*)$. Define

$$B(z):=B(z^*)^{[*]},\quad z^*\in\mathbb{C}^+\cap\rho(\tilde S).$$

Then for $z\in\mathbb{C}^-$ such that $z^*\in\rho(\tilde S)$, we have $B(z)=I+(z-\mu^*)(\tilde S^{[*]}-z)^{-1}$. Therefore, if $B(z)f=0$, then $(z-\mu^*)(\tilde S^{[*]}-z)^{-1}f=-f$, that is, $\{-f,(z-\mu^*)f\}\in\tilde S^{[*]}-z$. Hence $\{-f,\mu^*f\}\in\tilde S^{[*]}$. Since $\ker(\tilde S^{[*]}-\mu^*)=\{0\}$, we have $f=0$, that is $B(z)$ is injective. In a similar way as above one shows that for $z\in\mathbb{C}^-$ such that $z^*\in\rho(\tilde S)$, we have $B(z)\ker(S^{[*]}-\mu^*)=\ker(S^{[*]}-z)$.

Let

$$\Phi(\mu)=(\phi_1(\mu),\dots,\phi_{d_+}(\mu)),$$

$$\Phi(\mu^*)=(\phi_1(\mu^*),\dots,\phi_{d_-}(\mu^*))$$

be row vectors whose entries form a basis for $\ker(S^{[*]}-\mu)$ and $\ker(S^{[*]}-\mu^*)$. Since $B(z)$ is a holomorphic operator valued function on its domain and it is a bijection

between $\ker(S^{[*]}-\mu)$ and $\ker(S^{[*]}-z)$ for $z\in\mathbb{C}^+\cap\rho(A)$ and it is a bijection between $\ker(S^{[*]}-\mu^*)$ and $\ker(S^{[*]}-z)$ for $z\in\mathbb{C}^-$ such that $z^*\in\mathbb{C}^+\cap\rho(\tilde{S})$, it follows that

$$\Phi(z) := \begin{cases} B(z)\Phi(\mu), & z\in\mathbb{C}^+\cap\rho(\tilde{S}),\\ B(z)\Phi(\mu^*), & z^*\in\mathbb{C}^+\cap\rho(\tilde{S}) \end{cases}$$

has the properties from the lemma. □

$\Phi(z)=(\phi_1(z),\dots,\phi_{d_\pm}(z))$, $z\in\mathbb{C}^\pm\backslash\gamma$, in the lemma is called a holomorphic basis for $\ker(S^{[*]}-z)$. If $\Phi(z)$ is such a basis then $\hat{\Phi}(z)$ stands for the row vector function whose entries are pairs from $M_z:\hat{\Phi}(z)=(\{\phi_1(z),z\phi_1(z)\},\dots,\{\phi_{d_\pm}(z),z\phi_{d_\pm}(z)\})$ and if b is a boundary mapping for S then $\mathsf{b}(\hat{\Phi}(z^*))$ stands for the $d\times d_\mp$ matrix whose jth column is given by $\mathsf{b}(\{\phi_j(z^*),z^*\phi_j(z^*)\})$, $j=1,\dots,d_\mp$.

Theorem 4.2. *Let $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$ be a Pontryagin space of negative index $\varkappa$ and let S be a standard symmetric operator in $\mathscr{H}$ with defect index (d_+,d_-) and $d=d_++d_-<\infty$.*

(a) *Let b be a boundary mapping for S with Gram matrix Q and $\Phi(z)$ a holomorphic basis of $\ker(S^{[*]}-z)$ defined on $\mathbb{C}\backslash(\mathbb{R}\cup\gamma)$, where γ is a finite subset of $\mathbb{C}\backslash\mathbb{R}$. Then*

$$\mathscr{U}(z) := (\mathsf{Q}b(\hat{\Phi}(z^*)))^*$$

is a $(-\mathsf{Q})$-boundary coefficient.

(b) *Let $\Phi_1(z)$ be any holomorphic basis for $\ker(S^*-z)$, $z\in\mathbb{C}\backslash(\mathbb{R}\cup\gamma_1)$, where γ_1 is a finite subset of $\mathbb{C}\backslash\mathbb{R}$, and let b_1 be any boundary mapping for S with Gram matrix Q_1 and set $\mathscr{U}_1(z):=(\mathsf{Q}_1\mathsf{b}_1(\hat{\Phi}_1(z^*)))^*$. Then*

$$\mathscr{U}(z)=\mathscr{A}(z)\mathscr{U}_1(z)\mathsf{A}$$

on $\mathbb{C}\backslash(\mathbb{R}\cup\gamma\cup\gamma_1)$ for some invertible matrix function $\mathscr{A}(z)$ of size $d_\mp\times d_\mp$ if $z\in\mathbb{C}^\pm$ and a constant invertible $d\times d$ matrix A such that $\mathsf{A}\mathsf{Q}^{-1}\mathsf{A}^*=\mathsf{Q}_1^{-1}$.

(c) *The boundary coefficient $\mathscr{U}$ is minimal if and only if*

$$(\operatorname{dom}S)^{[\perp]}\cap\operatorname{span}\{\ker(S-\lambda)^\varkappa:\lambda\in\sigma_p(S)\}=\{0\}. \tag{4.3}$$

Proof. For $z\in\mathbb{C}^\pm\backslash\gamma$ the row vector $\hat{\Phi}(z^*)$ has $d_\mp$ components which are vectors from $S^{[*]}\cap z^*I$. The mapping $\mathsf{Q}\mathsf{b}$ maps each component from $\hat{\Phi}(z^*)$ to a $d\times 1$ vector in $\mathbb{C}^d$. Thus $\mathsf{Q}\mathsf{b}(\hat{\Phi}(z^*))$ is a $d\times d_\mp$ matrix and $\mathscr{U}(z)$ is a $d_\mp\times d$ matrix. This proves $(\mathscr{U}1)$. Since $\hat{\Phi}(z)$ is holomorphic on its domain, $\hat{\Phi}(z^*)$ is anti-holomorphic, and consequently $\mathsf{Q}\mathsf{b}(\hat{\Phi}(z^*))$ is also anti-holomorphic. Therefore $\mathscr{U}(z)$ is holomorphic on its domain and $(\mathscr{U}2)$ is proved. Since the vectors in $\hat{\Phi}(z)$ ($\hat{\Phi}(z^*)$, respectively) are

linearly independent and since Qb is a bijection on $S^{[*]} \cap zI$ ($S^{[*]} \cap z^*I$, respectively) it follows that the matrix $\mathsf{Qb}(\hat{\Phi}(z^*))$ ($\mathsf{Qb}(\hat{\Phi}(z))$, respectively) has rank d_- (d_+, respectively). Thus the property ($\mathscr{U}3$) holds. We calculate $\mathscr{U}(z)(-\mathsf{Q}^{-1})\mathscr{U}(w)^*$:

$$\begin{aligned}\mathscr{U}(z)(-\mathsf{Q}^{-1})\mathscr{U}(w)^* &= \mathsf{b}(\hat{\Phi}(z^*))^*\mathsf{Q}(-\mathsf{Q}^{-1})\mathsf{Qb}(\hat{\Phi}(w^*))\\ &= \mathsf{b}(\hat{\Phi}(z^*))^*(-\mathsf{Q})\mathsf{b}(\hat{\Phi}(w^*))\\ &= -[\![\,\hat{\Phi}(w^*), \hat{\Phi}(z^*)\,]\!]\\ &= \frac{1}{i}(z-w^*)[\Phi(w^*), \Phi(z^*)].\end{aligned}$$

Thus $\mathscr{U}$ has property ($\mathscr{U}4$) and the limit in ($\mathscr{U}5$) exists. From

$$K_{\mathscr{U}}(z,w) = [\Phi(w^*), \Phi(z^*)], \tag{4.4}$$

it follows that the block matrix $[K_{\mathscr{U}}(\lambda_j, \lambda_k)]_{j,k=1}^n$ is Gram matrix of vectors in $\Phi(\lambda_1^*), \ldots, \Phi(\lambda_n^*)$ with respect to the inner product $[\,\cdot\,,\,\cdot\,]$. Therefore the function $\mathscr{U}(z)$ has property ($\mathscr{U}5$). This proves (a). The proof of (b) is identical to the proof of [7, Proposition 4.2(b)].

Now we prove (c). Assume (4.3). By Theorem 3.7, for arbitrary non-real distinct complex numbers $\mu_0^*, \mu_0, \mu_1, \ldots, \mu_\varkappa \notin \sigma_p(S)$, such that $\mu_j \neq \mu_k^*$, $j,k = 0, \ldots, \varkappa$ we have (3.11). Since by Theorem 2.3, we have $\dim S^{[*]}/S = d$, the von Neumann formula (3.11) implies that the matrix

$$[\mathscr{U}(\mu_0)^* \quad \mathscr{U}(\mu_0^*)^* \quad \mathscr{U}(\mu_1^*)^* \quad \cdots \quad \mathscr{U}(\mu_\varkappa^*)^*]$$

has the maximal rank d. Thus condition ($\mathscr{U}6$) is satisfied. Conversely, if ($\mathscr{U}6$) is satisfied, then the dimension of

$$M_{\mu_0} + \sum_{j=0}^{\varkappa} M_{\mu_j^*}$$

over S is d. Since $\dim S^{[*]}/S = d$, we conclude

$$S^{[*]} = S + M_{\mu_0} + \sum_{j=0}^{\varkappa} M_{\mu_j^*}.$$

By Theorem 3.7, this implies (4.3). The theorem is proved. □

Remark 4.3. Note that in the Hilbert space case each closed symmetric operator satisfies condition (4.3), and therefore each closed symmetric operator with finite defect indices gives rise to a minimal boundary coefficient. The same is true when S is a symmetric relation, because the multi-valued part of S can be factored out from the Hilbert space.

5. Reduction to a minimal boundary coefficient

A boundary coefficient $\mathscr{U}(z)$ is said to be *row reduced* to a boundary coefficient $\mathscr{V}(z)$ if

$$\mathscr{A}(z)\mathscr{U}(z) = \mathscr{V}(z), \quad z \in \mathrm{dom}(\mathscr{U}) \cap \mathrm{dom}(\mathscr{V}),$$

for some invertible matrix function $\mathscr{A}(z)$ on $\mathrm{dom}(\mathscr{U}) \cap \mathrm{dom}(\mathscr{V})$ which is of size $d_\pm \times d_\pm$ for $z \in \mathbb{C}^\pm$. The main result of this section Theorem 5.1, says that any boundary coefficient can be row reduced to a boundary coefficient whose top rows are independent of the eigenvalue parameter and the remaining rows are essentially determined by a minimal boundary coefficient. The theorem shows that $\mathscr{A}(z)$ can even be chosen holomorphic on its domain.

Theorem 5.1. *Let* Q *be a self-adjoint invertible* $d \times d$ *matrix with* d_+ *positive and* d_- *negative eigenvalues. Let* $\mathscr{U}(z)$ *be a* Q*-boundary coefficient function. There exist a unique integer* τ, $0 \leqslant \tau \leqslant \min\{d_+, d_-\}$, *and a holomorphic function* $\mathscr{A}(z)$ *on* $\mathrm{dom}(\mathscr{U})$ *whose values are invertible matrices of size* $d_\pm \times d_\pm$ *for* $z \in \mathbb{C}^\pm$ *such that*

$$\mathscr{A}(z)\mathscr{U}(z) = \begin{bmatrix} \mathsf{I} & 0 \\ 0 & \mathscr{U}_0(z) \end{bmatrix} \begin{bmatrix} \mathsf{U}_0 \\ \mathsf{B}_0 \end{bmatrix}, \tag{5.1}$$

where, with $\omega_\pm := d_\pm - \tau$, $\omega := d - 2\tau = \omega_+ + \omega_-$, U_0, $\mathscr{U}_0(z)$, *and* B_0 *have the following properties*:

(I) U_0 *is a constant* $\tau \times d$ *matrix of maximal rank*,

(II) B_0 *is a constant* $\omega \times d$ *matrix such that* $\mathsf{B}_0\mathsf{Q}^{-1}\mathsf{B}_0^*$ *is invertible and has* ω_+ *positive and* ω_- *negative eigenvalues*,

(III) *the following equality holds*:

$$\begin{bmatrix} \mathsf{U}_0 \\ \mathsf{B}_0 \end{bmatrix} \mathsf{Q}^{-1} \begin{bmatrix} \mathsf{U}_0 \\ \mathsf{B}_0 \end{bmatrix}^* = \begin{bmatrix} 0 & 0 \\ 0 & \mathsf{Q}_0^{-1} \end{bmatrix}, \tag{5.2}$$

where $\mathsf{Q}_0 := (\mathsf{B}_0\mathsf{Q}^{-1}\mathsf{B}_0^*)^{-1}$ *is a self-adjoint* $\omega \times \omega$ *matrix with* ω_+ *positive and* ω_- *negative eigenvalues*,

(IV) $\mathscr{U}_0(z)$ *is a minimal* Q_0*-boundary coefficient of size* $\omega_\pm \times \omega$.

The right-hand side of (5.1) *is called a minimal representation of* $\mathscr{U}(z)$.

To prove the theorem we use two lemmas. Lemmas 5.2(e) and (f) and 5.3 are consequences of a geometric interpretation of maximum modulus principle for generalized Schur functions [9, Proposition 8.1] for which we refer to the appendix. To formulate the lemmas, let $\mathscr{F}$ and $\mathscr{G}$ be Hilbert spaces and let $T : \mathbb{C}^+ \to \mathscr{L}(\mathscr{F}, \mathscr{G})$

be a meromorphic operator function such that the kernel

$$K_T(z,w)=i\frac{I-T(z)T(w)^*}{z-w^*},\quad z,w\in\mathrm{hol}(T),\tag{5.3}$$

has $\varkappa$ negative squares. Here $\mathrm{hol}(T)$ stands for the domain of holomorphy in $\mathbb{C}^+$ of T. We set

$$\mathscr{N}(T)\coloneqq\left(\bigcap_{z,v\in\mathrm{hol}(T)}\ker(T(z)-T(v))\right)\bigcap\left(\bigcap_{w\in\mathrm{hol}(T)}\ker(I-T(w)^*T(w))\right).$$

$\mathscr{N}(T)$ is the subspace of $\mathscr{F}$ on which $T(z)$ is isometric and independent of z. If we set $T^*(z)=T(z)^*$, then $\mathscr{N}(T^*)$ is the subspace of $\mathscr{G}$ on which $T^*(z)$ is isometric and independent of z.

Lemma 5.2. *There exist decompositions $\mathscr{F}=\mathscr{F}_0\oplus\mathscr{F}_1$ and $\mathscr{G}=\mathscr{G}_0\oplus\mathscr{G}_1$ such that $T(z)$, has the matrix representation*

$$T(z)=\begin{bmatrix}V & 0\\ 0 & T_0(z)\end{bmatrix}:\begin{bmatrix}\mathscr{F}_0\\ \mathscr{F}_1\end{bmatrix}\to\begin{bmatrix}\mathscr{G}_0\\ \mathscr{G}_1\end{bmatrix},\quad z\in\mathrm{hol}(T),\tag{5.4}$$

where

(a) $\mathscr{F}_0=\mathscr{N}(T)$ *and* $\mathscr{G}_0=\mathscr{N}(T^*)$;
(b) $V:\mathscr{F}_0\to\mathscr{G}_0$ *is unitary*;
(c) $T_0:\mathbb{C}^+\to\mathscr{L}(\mathscr{F}_1,\mathscr{G}_1)$ *is a meromorphic function and* $\mathrm{hol}(T_0)\supset\mathrm{hol}(T)$;
(d) *the kernel*

$$K_{T_0}(z,w)=i\frac{I-T_0(z)T_0(w)^*}{z-w^*},\quad z,w\in\mathrm{hol}(T_0),\tag{5.5}$$

has $\varkappa$ negative squares;
(e) *for any choice of distinct complex numbers $z_0,z_1,\ldots,z_\varkappa\in\mathrm{hol}(T_0)$ and each $j\in\{0,1,\ldots,\varkappa\}$ the restriction $T_0(z_j)|_{\mathscr{M}(T_0)}$ to*

$$\mathscr{M}(T_0)\coloneqq\mathscr{M}(T_0;z_0,\ldots,z_\varkappa)=\bigcap_{k=0}^{\varkappa}\ker(T_0(z_k)-T_0(z_0))$$

is a strict contraction, or equivalently,
(f) *for any $j\in\{0,1,\ldots,\varkappa\}$*

$$\mathscr{M}(T_0)\bigcap\ker(I-T_0(z_j)^*T_0(z_j))=\{0\}.$$

Proof. Put $\mathscr{F}_0 = \mathscr{N}(T)$, $\mathscr{G}_0 = \mathscr{N}(T^*)$. From the definition of $\mathscr{N}(T)$ it follows that $T(z)|_{\mathscr{N}(T)}$ is independent of $z \in \mathrm{hol}(T)$. Put $V := T(z_0)|_{\mathscr{N}(T)}$, with $z_0 \in \mathrm{hol}(T)$. We prove that $V\mathscr{F}_0 = \mathscr{G}_0$. Let $f \in \mathscr{F}_0$. Since for arbitrary $z \in \mathrm{hol}(T)$, $f \in \ker(I - T(z)^*T(z))$ we have that $T(z)^*Vf = f$. Therefore $Vf \in \ker(T(z)^* - T(v)^*)$ for arbitrary $z, v \in \mathrm{hol}(T)$. Moreover, for arbitrary $z \in \mathrm{hol}(T)$, $T(z)T(z)^*Vf = T(z)f = Vf$, and consequently $Vf \in \mathscr{N}(T^*) = \mathscr{G}_0$. Thus $V\mathscr{N}(T) \subset \mathscr{N}(T^*)$. We now prove $\mathscr{N}(T^*) \subset V\mathscr{N}(T)$. First note that $T(z)^*|_{\mathscr{N}(T^*)}$ is independent of $z \in \mathrm{hol}(T)$ and define $V_1 := T(z_0)^*|_{\mathscr{N}(T^*)}$ for some $z_0 \in \mathrm{hol}(T)$. Consider an arbitrary $g \in \mathscr{N}(T^*)$. Reasoning as above we obtain that $V_1 g \in \mathscr{N}(T)$ and $VV_1 g = T(z_0)T(z_0)^* g = g$. Therefore $g \in V\mathscr{N}(T)$, $V : \mathscr{F}_0 \to \mathscr{G}_0$ is one to one and $V^{-1} = V_1 = V^*$. This proves (b).

Define $\mathscr{F}_1 := \mathscr{F} \ominus \mathscr{F}_0$, $\mathscr{G}_1 := \mathscr{G} \ominus \mathscr{G}_0$ and $T_0(z) := T(z)|_{\mathscr{F}_1}$, $z \in \mathbb{C}^+$. As a restriction of a meromorphic function, T_0 is meromorphic and $\mathrm{hol}(T_0) \supset \mathrm{hol}(T)$. The matrix representation (5.4) follows from the equalities

$$\begin{aligned} \langle T(z)\mathscr{F}_1, V\mathscr{F}_0 \rangle_{\mathscr{G}} &= \langle T(z)\mathscr{F}_1, T(z)\mathscr{F}_0 \rangle_{\mathscr{G}} \\ &= \langle \mathscr{F}_1, T(z)^*T(z)\mathscr{F}_0 \rangle_{\mathscr{F}} \\ &= \langle \mathscr{F}_1, \mathscr{F}_0 \rangle_{\mathscr{F}} \\ &= 0, \quad z \in \mathrm{hol}(T). \end{aligned}$$

Representation (5.4) implies

$$T(z)^* = \begin{bmatrix} V^* & 0 \\ 0 & T_0(z)^* \end{bmatrix} : \begin{bmatrix} \mathscr{G}_0 \\ \mathscr{G}_1 \end{bmatrix} \to \begin{bmatrix} \mathscr{F}_0 \\ \mathscr{F}_1 \end{bmatrix}, \quad z \in \mathrm{hol}(T). \tag{5.6}$$

Using the matrix representations (5.4) and (5.6) we calculate

$$K_T(z, w) = \begin{bmatrix} 0 & 0 \\ 0 & K_{T_0}(z, w) \end{bmatrix},$$

and therefore kernel (5.5) has the same number $\varkappa$ of negative squares as kernel (5.3).

Since $T_0(z) : \mathscr{F}_1 \to \mathscr{G}_1$ for all $z \in \mathrm{hol}(T)$, we conclude that $T_0(z)^* = T(z)^*|_{\mathscr{G}_1}$ for all $z \in \mathrm{hol}(T)$. Therefore for all $z, v \in \mathrm{hol}(T)$ we have

$$\ker(I - T_0(z)^*T_0(z)) = \ker(I - T(z)^*T(z)) \cap \mathscr{F}_1,$$

$$\ker(T_0(z) - T_0(v)) = \ker(T(z) - T(v)) \cap \mathscr{F}_1,$$

and consequently,

$$\mathcal{N}(T_0) = \mathcal{N}(T) \cap \mathcal{F}_1 = \mathcal{F}_0 \cap \mathcal{F}_1 = \{0\}.$$

Now (e) and (f) follow from Remark A.3 in the appendix below. □

The last part of Lemma 5.2 can be formulated geometrically in terms of subspaces of the Krein space $\mathcal{K}$ defined by

$$(\mathcal{K}, [\,\cdot\,,\,\cdot\,]) := (\mathcal{F}, \langle\cdot,\cdot\rangle_{\mathcal{F}}) \oplus (\mathcal{G}, -\langle\cdot,\cdot\rangle_{\mathcal{G}}).$$

In $\mathcal{K}$ we consider the graph of the operator $T(z)$:

$$\mathcal{L}(z) := G[T(z)] = \{\{f, T(z)f\} : f \in \mathcal{F}\} \subset \mathcal{K}, \quad z \in \mathbb{C}^+,$$

Recall that the *isotropic part* of a subspace $\mathcal{L}$ of $(\mathcal{K}, [\,\cdot\,,\,\cdot\,])$ is defined by $\mathcal{L}^\circ := \mathcal{L} \cap \mathcal{L}^{[\perp]}$.

Lemma 5.3. *Each* $\mathcal{L}(z),\ z \in \mathrm{hol}(T)$, *can be decomposed as*

$$\mathcal{L}(z) = \mathcal{L}_0[\dot{+}]\mathcal{L}_1(z),$$

where $\mathcal{L}_0$ *is a neutral subspace of* $\mathcal{K}$ *and the intersection the isotropic parts of* $\mathcal{L}_1(z)$ *over* $z \in \mathrm{hol}(T)$ *is* $\{0\}$, *or equivalently, for arbitrary distinct complex numbers* $z_0, \ldots, z_\varkappa$ *in* $\mathrm{hol}(T)$

$$\mathcal{L}_1(z_0)^\circ \bigcap \left(\bigcap_{j=1}^{\varkappa} \mathcal{L}_1(z_j) \right) = \{0\}.$$

Indeed, let $\mathcal{L}_0 := G[V]$ and $\mathcal{L}_1(z) = G[T_0(z)],\ z \in \mathbb{C}^+$. Then by Lemma 5.2(b), $\mathcal{L}_0$ is a neutral subspace of $\mathcal{K}$ and

$$\mathcal{L}(z) = \mathcal{L}_0[\dot{+}]\mathcal{L}_1(z).$$

The equality in Lemma 5.2(f) with $j = 0$ and the last equality in the lemma are the same. That these equalities are equivalent to the intersection of isotropic parts of $\mathcal{L}_1(z)$ being $\{0\}$ follows from Theorem A.5 in the appendix.

Proof of Theorem 5.1. In this proof we consider $\mathbb{C}^d$ equipped with the indefinite inner product

$$[x, y] = y^* \mathsf{Q}^{-1} x, \quad x, y \in \mathbb{C}^d.$$

The space $(\mathbb{C}^d, [\,\cdot\,,\,\cdot\,])$ is a Krein space. Let $\mathbb{C}^d = \mathcal{Q}_+[\dot{+}]\mathcal{Q}_-$ be a fundamental decomposition of $\mathbb{C}^d$. For example, $\mathcal{Q}_+$ ($\mathcal{Q}_-$) can be the subspace of $\mathbb{C}^d$ generated by the eigenvectors of Q corresponding to its positive (negative) eigenvalues. Whatever

the choice of the fundamental decomposition we have that $\dim(\mathcal{Q}_\pm) = d_\pm$. Denote by P_+ and P_- the orthogonal projections onto $\mathcal{Q}_+$ and $\mathcal{Q}_-$. We consider the subspaces

$$\mathcal{R}(z) := \operatorname{ran}(\mathcal{U}(z)^*), \quad z \in \operatorname{dom}(\mathcal{U}).$$

The next argument was used in [10]. Assume that there exist $\varkappa + 1$ distinct complex numbers $z_0, z_1, \ldots, z_\varkappa \in \mathbb{C}^+ \cap \operatorname{dom}(\mathcal{U})$ for which there exist vectors $x_0, x_1, \ldots, x_\varkappa \in \mathbb{C}^{d_+}$ such that for $j = 0, 1, \ldots, \varkappa$ we have

$$P_+\mathcal{U}(z_j)^* x_j = 0 \quad \text{and} \quad y_j := P_-\mathcal{U}(z_j)^* x_j \neq 0. \tag{5.7}$$

Then via complex contour integration and the residue theorem we find

$$\begin{aligned} x_j^* K_{\mathcal{U}}(z_j, z_k) x_k &= i x_j^* \frac{\mathcal{U}(z_j)\mathsf{Q}^{-1}\mathcal{U}(z_k)^*}{z_j - z_k^*} x_k \\ &= i \frac{[y_k, y_j]}{z_j - z_k^*} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left[\frac{y_k}{t - z_k^*}, \frac{y_j}{t - z_j^*} \right] dt, \quad j, k = 0, \ldots, \varkappa. \end{aligned}$$

Since each of the vectors y_j, $j = 0, 1, \ldots, \varkappa$, is negative in the Krein space $(\mathbb{C}^d, [\,\cdot\,,\,\cdot\,])$, the Gram matrix of the vectors $y_j/(t - z_j^*)$, $j = 0, \ldots, \varkappa$, is negative definite. Therefore the self-adjoint block matrix $[K_{\mathcal{U}}(z_j, z_k)]_{j,k=0}^{\varkappa}$ has at least $\varkappa + 1$ negative eigenvalues. Since $[K_{\mathcal{U}}(z_j, z_k)]_{j,k=0}^{\varkappa}$ has at most $\varkappa$ negative eigenvalues the assumption cannot hold. It follows that there exist at most $\varkappa$ distinct complex numbers $z_1, \ldots, z_\varkappa \in \mathbb{C}^+ \cap \operatorname{dom}(\mathcal{U})$ for which (5.7) holds for some vectors $x_1, \ldots, x_\varkappa$. Denote by γ the set of the exceptional complex numbers z in $\mathbb{C}^+ \cap \operatorname{dom}(\mathcal{U})$ for which

$$P_+\mathcal{U}(z)^* x = 0 \quad \text{and} \quad P_-\mathcal{U}(z)^* x \neq 0$$

holds for some $x \in \mathbb{C}^{d_+}$. As we have just proved γ has at most $\varkappa$ elements. For each $z \in \mathbb{C}^+ \cap (\operatorname{dom}(\mathcal{U}) \setminus \gamma)$ we have that

$$P_+\mathcal{U}(z)^* x = 0 \;\Rightarrow\; P_-\mathcal{U}(z)^* x = 0 \text{ and } \mathcal{U}(z)^* x = 0.$$

In other words, the restriction $P_+|_{\mathcal{R}(z)}$ is an injective operator for all $z \in \mathbb{C}^+ \cap (\operatorname{dom}(\mathcal{U}) \setminus \gamma)$. By assumption ($\mathcal{U}3$), $\dim \mathcal{R}(z) = d_+$ for all $z \in \mathbb{C}^+ \cap \operatorname{dom}(\mathcal{U})$. Consequently, the restriction $P_+|_{\mathcal{R}(z)}$ is a bijection for all $z \in \mathbb{C}^+ \cap (\operatorname{dom}(\mathcal{U}) \setminus \gamma)$. Therefore the operator $P_+\mathcal{U}(z)^* : \mathbb{C}^{d_+} \to \mathcal{Q}_+$ is a bijection for $z \in \mathbb{C}^+ \cap (\operatorname{dom}(\mathcal{U}) \setminus \gamma)$. Denote by $T(z)$ the operator from the Hilbert space $(\mathcal{Q}_-, -[\,\cdot\,,\,\cdot\,])$ to the Hilbert space $(\mathcal{Q}_+, [\,\cdot\,,\,\cdot\,])$ defined by

$$T(z)^* := P_-\mathcal{U}(z)^*(P_+\mathcal{U}(z)^*)^{-1}, \quad z \in \mathbb{C}^+ \cap (\operatorname{dom}(\mathcal{U}) \setminus \gamma).$$

In particular we have

$$\mathscr{U}(z)^*a = P_+\mathscr{U}(z)^*a + P_-\mathscr{U}(z)^*a = (I_{\mathscr{L}_+} + T(z)^*)(P_+\mathscr{U}(z)^*)a, \tag{5.8}$$

for all $a\in\mathbb{C}^{d_+}$ and so

$$\mathscr{R}(z) = \{(I_{\mathscr{L}_+} + T(z)^*)x_+ : x_+\in\mathscr{L}_+\}, \quad z\in\mathbb{C}^+\cap(\mathrm{dom}(\mathscr{U})\backslash\gamma). \tag{5.9}$$

The operator $T(z)^*$ is called the *angular operator* of $\mathscr{R}(z)$. Since $\mathscr{U}(z)$ is holomorphic on $\mathbb{C}^+\cap(\mathrm{dom}(\mathscr{U})\backslash\gamma)$, $T(z)$ is also holomorphic on this set. Note that the set $(\mathbb{C}^+\backslash\mathrm{dom}(\mathscr{U}))\cup\gamma$ is finite.

Next we verify that the function T has a finite number of negative squares on $\mathbb{C}^+\cap(\mathrm{dom}(\mathscr{U})\backslash\gamma)$. Let $z,w\in\mathbb{C}^+\cap(\mathrm{dom}(\mathscr{U})\backslash\gamma)$, $x_+ = P_+\mathscr{U}(w)^*a$ and $y_+ = P_+\mathscr{U}(z)^*b$, with $a,b\in\mathbb{C}^{d_+}$ and $x_+,y_+\in\mathscr{L}_+$. Then

$$\begin{aligned}
i\frac{b^*\mathscr{U}(z)\mathsf{Q}^{-1}\mathscr{U}(w)^*a}{z-w^*} &= \frac{i}{z-w^*}[\mathscr{U}(w)^*a, \mathscr{U}(z)^*b]\\
&= \frac{i}{z-w^*}[(I_{\mathscr{L}_+} + T(w)^*)x_+, (I_{\mathscr{L}_+} + T(z)^*)y_+]\\
&= \frac{i}{z-w^*}([x_+,y_+] + [T(w)^*x_+, T(z)^*y_+])\\
&= \frac{i}{z-w^*}([x_+,y_+] - [T(z)T(w)^*x_+, y_+])\\
&= \frac{i}{z-w^*}[(I_{\mathscr{L}_+} - T(z)T(w)^*)x_+, y_+].
\end{aligned}$$

That is,

$$b^*K_{\mathscr{U}}(z,w)a = [K_T(z,w)x_+, y_+].$$

Since the mapping $a\mapsto x_+ = P_+\mathscr{U}(w)^*a$ is a bijection between $\mathbb{C}^{d_+}$ and $\mathscr{L}_+$ for $w\in\mathbb{C}^+\cap(\mathrm{dom}(\mathscr{U})\backslash\gamma)$, the assumption $(\mathscr{U}5)$ implies that the kernel $K_T(z,w)$ also has $\varkappa$ negative squares on $\mathbb{C}^+\cap(\mathrm{dom}(\mathscr{U})\backslash\gamma)$.

It follows from Lemma 5.2 applied to $(\mathscr{F},\langle\cdot,\cdot\rangle_{\mathscr{F}}) = (\mathscr{L}_-, -[\,\cdot\,,\,\cdot\,])$ and $(\mathscr{G},\langle\cdot,\cdot\rangle_{\mathscr{G}}) = (\mathscr{L}_+, [\,\cdot\,,\,\cdot\,])$, that there exist decompositions $\mathscr{L}_\pm = \mathscr{L}_\pm^0[\dot+]\mathscr{L}_\pm^1$ such that

$$T(z) = \begin{bmatrix} V & 0\\ 0 & T_0(z)\end{bmatrix}\begin{bmatrix}\mathscr{L}_-^0\\ \mathscr{L}_-^1\end{bmatrix}\to\begin{bmatrix}\mathscr{L}_+^0\\ \mathscr{L}_+^1\end{bmatrix}, \quad z\in\mathbb{C}^+\cap(\mathrm{dom}(\mathscr{U})\backslash\gamma), \tag{5.10}$$

where $T_0(z):\mathscr{L}_-^1\to\mathscr{L}_+^1$ is such that $\mathscr{N}(T_0) = \{0\}$, $T_0(z)$ is holomorphic on $\mathbb{C}^+\cap(\mathrm{dom}(\mathscr{U})\backslash\gamma)$, and $V:\mathscr{L}_-^0\to\mathscr{L}_+^0$ is a unitary operator. Let $\tau = \dim(\mathscr{L}_+^0) = \dim(\mathscr{L}_-^0)$, $\omega_\pm = d_\pm - \tau$, and $\omega = \omega_+ + \omega_- = d - 2\tau$. The subspace $\mathscr{L}_+^1[+]\mathscr{L}_-^1$ is a Krein subspace of $(\mathbb{C}^d,[\,\cdot\,,\,\cdot\,])$ of dimension ω. The decomposition $\mathscr{L}_+^1[\dot+]\mathscr{L}_-^1$

is a fundamental decomposition of this Krein space. We have $\omega_+ = \dim(\mathcal{Q}_+^1)$ and $\omega_- = \dim(\mathcal{Q}_-^1)$.

Now, with $z \in \mathbb{C}^+ \cap (\mathrm{dom}(\mathcal{U})\backslash\gamma)$, equality (5.9) becomes

$$\begin{aligned}\mathcal{R}(z) &= \{x_0 + x_1 + V^{-1}x_0 + T_0(z)^* x_1 : x_0 \in \mathcal{Q}_+^0,\ x_1 \in \mathcal{Q}_+^1\} \\ &= \mathcal{R}_0[\dot{+}]\mathcal{R}_1(z),\end{aligned}$$

where by (5.8),

$$\begin{aligned}\mathcal{R}_0 &:= \{x_0 + V^{-1}x_0 : x_0 \in \mathcal{Q}_+^0\} \\ &= \mathcal{U}(z)^*(P_+\mathcal{U}(z)^*)^{-1}\mathcal{Q}_+^0 \subset \mathcal{Q}_+^0[\dot{+}]\mathcal{Q}_-^0\end{aligned}$$

is a neutral subspace and the subspaces

$$\begin{aligned}\mathcal{R}_1(z) &:= \{x_1 + T_0^*(z)x_1 : x_1 \in \mathcal{Q}_+^1\} \\ &= \mathcal{U}(z)^*(P_+\mathcal{U}(z)^*)^{-1}\mathcal{Q}_+^1 \subset \mathcal{Q}_+^1[\dot{+}]\mathcal{Q}_-^1\end{aligned}$$

have the property that the intersection of their isotropic parts is $\{0\}$ by Lemma 5.3.

Properties $(\mathcal{U}3)$ and $(\mathcal{U}4)$ of $\mathcal{U}$ imply that

$$(\mathrm{ran}\,\mathcal{U}(z)^*)^{[\perp]} = \mathrm{ran}\,\mathcal{U}(z^*)^*.$$

Therefore we have

$$\begin{aligned}\mathcal{R}(z^*) &= \mathcal{R}(z)^{[\perp]} \\ &= \{(I_{\mathcal{Q}_-} + T(z))x_- : x_- \in \mathcal{Q}_-\}, \quad z \in \mathbb{C}^+ \cap (\mathrm{dom}(\mathcal{U})\backslash\gamma),\end{aligned}$$

or, using decomposition (5.10), we have

$$\begin{aligned}\mathcal{R}(z^*) &= \{x_0 + x_1 + Vx_0 + T_0(z)x_1 : x_0 \in \mathcal{Q}_-^0,\ x_1 \in \mathcal{Q}_-^1\} \\ &= \mathcal{R}_0[\dot{+}]\mathcal{R}_1(z^*),\end{aligned}$$

where, as before,

$$\mathcal{R}_0 = \{x_0 + Vx_0 : x_0 \in \mathcal{Q}_-^0\} \subset \mathcal{Q}_+^0[\dot{+}]\mathcal{Q}_-^0$$

and the subspaces $\mathcal{R}_1(z^*)$ are defined by

$$\mathcal{R}_1(z^*) := \{x_1 + T_0(z)x_1 : x_1 \in \mathcal{Q}_-^1\} \subset \mathcal{Q}_+^1[\dot{+}]\mathcal{Q}_-^1$$

and they have the property that the intersection of their isotropic parts is $\{0\}$ by Lemma 5.3. Note that $\mathcal{R}_1(z^*)$ is the orthogonal complement of $\mathcal{R}_1(z)$ in the Krein

space $(\mathcal{L}^1_+[\dot{+}]\mathcal{L}^1_-, [\,\cdot\,,\,\cdot\,])$. Therefore

$$\mathscr{R}_1(z)^\circ = \mathscr{R}_1(z^*)^\circ = \mathscr{R}_1(z) \cap \mathscr{R}_1(z^*).$$

Select a basis of the ω dimensional subspace $\mathcal{L}^1_+[\dot{+}]\mathcal{L}^1_-$ of $\mathbb{C}^d$. Let the columns of the $d \times \omega$ matrix B_0^* be the vectors of this basis. The Gram matrix $\mathsf{B}_0\mathsf{Q}^{-1}\mathsf{B}_0^*$ of the columns of B_0^* with respect to the indefinite inner product $[\,\cdot\,,\,\cdot\,]$ is invertible and has ω_+ positive and ω_- negative eigenvalues. Hence B_0 has property (II). The Gram matrix $\mathsf{B}_0\mathsf{B}_0^*$ of the columns of B_0^* with respect to the Euclidean inner product is $\omega \times \omega$ and invertible and the $d \times d$ matrix $\mathsf{B}_0^*(\mathsf{B}_0\mathsf{B}_0^*)^{-1}\mathsf{B}_0$ is the orthogonal projection with respect to the Euclidean inner product of $\mathbb{C}^d$ onto $\mathcal{L}^1_+[\dot{+}]\mathcal{L}^1_-$.

Let $a_1, \ldots, a_\tau$ be a basis of the subspace $\mathcal{L}^0_+$. Then $(I_{\mathcal{L}^0_+} + V^{-1})a_j,\ j = 1, \ldots, \tau$ is a basis of $\mathscr{R}_0$. Let the columns of the $d \times \tau$ matrix U_0^* be the $d \times 1$ vectors $(I_{\mathcal{L}^0_+} + V^{-1})a_j,\ j = 1, \ldots, \tau$. Then U_0 has the property (I).

Property (III) now follows from the fact that $\mathscr{R}_0$ is a neutral subspace of $(\mathbb{C}^d, [\,\cdot\,,\,\cdot\,])$ and orthogonal to $\mathcal{L}^1_+[\dot{+}]\mathcal{L}^1_-$ in $[\,\cdot\,,\,\cdot\,]$.

For $z \in \mathbb{C}^+ \cap (\mathrm{dom}(\mathscr{U})\backslash\gamma)$ we now construct $\mathscr{U}_0(z)$. Let $b_1, \ldots, b_{\omega_+}$ be a basis of the space $\mathcal{L}^1_+$. Then $(I_{\mathcal{L}^1_+} + T_0(z)^*)b_j,\ j = 1, \ldots, \omega_+$, is a basis of $\mathscr{R}_1(z)$. Let the columns of the $d \times \omega_+$ matrix $\mathscr{W}_1(z)^*$ be the $d \times 1$ vectors $(I_{\mathcal{L}^1_+} + T_0(z)^*)b_j,\ j = 1, \ldots, \omega_+$. The rank of $\mathscr{W}_1(z)^*$ is ω_+ for all $z \in \mathbb{C}^+ \cap (\mathrm{dom}(\mathscr{U})\backslash\gamma)$. Since the function $T_0(z)^*$ is anti-holomorphic, the function $\mathscr{W}_1(z)^*$ is anti-holomorphic. Put

$$\mathscr{U}_0(z)^* = (\mathsf{B}_0\mathsf{B}_0^*)^{-1}\mathsf{B}_0\mathscr{W}_1(z)^*, \quad z \in \mathbb{C}^+ \cap (\mathrm{dom}(\mathscr{U})\backslash\gamma).$$

Clearly, $\mathscr{U}_0(z)^*$ is an $\omega \times \omega_+$ matrix and the function $z \mapsto \mathscr{U}_0(z)^*$ is anti-holomorphic on $\mathbb{C}^+ \cap (\mathrm{dom}(\mathscr{U})\backslash\gamma)$. Since the columns of the matrix $\mathscr{W}_1(z)^*$ belong to $\mathcal{L}^1_+[\dot{+}]\mathcal{L}^1_-$ we have

$$\mathsf{B}_0^*\mathscr{U}_0(z)^* = \mathsf{B}_0^*(\mathsf{B}_0\mathsf{B}_0^*)^{-1}\mathsf{B}_0\mathscr{W}_1(z)^* = \mathscr{W}_1(z)^*. \tag{5.11}$$

Thus, the columns of the matrix $[\mathsf{U}_0^*\ \ \mathsf{B}_0^*\mathscr{U}_0(z)^*]$ form an anti-holomorphic basis for $\mathscr{R}(z) = \mathrm{ran}(\mathscr{U}(z)^*)$. Another anti-holomorphic basis of this space is formed by the columns of $\mathscr{U}(z)^*$. Denote by $\mathscr{A}(z)^*$ the "change of coordinates matrix" between these two basis of $\mathscr{R}(z)$, that is, the matrix with the property

$$\mathscr{U}(z)^*\mathscr{A}(z)^* = [\,\mathsf{U}_0^* \quad \mathsf{B}_0^*\mathscr{U}_0(z)^*\,], \quad z \in \mathbb{C}^+ \cap (\mathrm{dom}(\mathscr{U})\backslash\gamma).$$

By (5.8), we have $\mathscr{A}(z)^* = (P_+\mathscr{U}(z)^*)^{-1}[a_1\ \cdots\ a_\tau\ b_1\ \cdots\ b_{\omega_+}]$. The matrix $\mathscr{A}(z)$ is a $d_+ \times d_+$ invertible matrix and the function $z \mapsto \mathscr{A}(z)$ is holomorphic on $z \in \mathbb{C}^+ \cap (\mathrm{dom}(\mathscr{U})\backslash\gamma)$. An analogous construction for $z \in \mathbb{C}^-$ leads to the extension of γ in $\mathbb{C}^-$ by at most $\varkappa$ points, the $d \times \omega_-$ matrix $\mathscr{W}_2(z)^*$ and to the $\omega \times \omega_-$ matrix $\mathscr{U}_0(z)^* := \mathsf{B}_0\mathscr{W}_2(z)^*$ and finally to the $d_- \times d_-$ matrix $\mathscr{A}_2(z)^*$ such that with the

same U_0 and B_0^* as above we have

$$\mathscr{U}(z)^*\mathscr{A}_2(z)^* = [\,\mathsf{U}_0^* \quad \mathsf{B}_0^*\mathscr{U}_0(z)^*\,], \quad z\in\mathbb{C}^-\cap(\mathrm{dom}(\mathscr{U})\backslash\gamma).$$

Thus $\mathscr{U}(z)$ has the minimal representation (5.1) for all $z\in\mathbb{C}\backslash\mathbb{R}\cap(\mathrm{dom}(\mathscr{U})\backslash\gamma)$.

It remains to show property (IV). Properties $(\mathscr{U}1)$ and $(\mathscr{U}2)$ follow from the construction of $\mathscr{U}_0(z)$. Property $(\mathscr{U}3)$ follows from (5.11) and the fact that the matrix $\mathscr{W}_1(z)^*$ has rank ω_+.

Equalities (5.1) and (5.2) yield

$$\begin{aligned}\begin{bmatrix}0 & 0\\ 0 & \mathscr{U}_0(z)\mathsf{Q}_0^{-1}\mathscr{U}_0(w)^*\end{bmatrix} &= \begin{bmatrix}\mathsf{U}_0\mathsf{Q}^{-1}\mathsf{U}_0^* & \mathsf{U}_0\mathsf{Q}^{-1}\mathsf{B}_0^*\mathscr{U}_0(w)^*\\ \mathscr{U}_0(z)\mathsf{B}_0\mathsf{Q}^{-1}\mathsf{U}_0^* & \mathscr{U}_0(z)\mathsf{B}_0\mathsf{Q}^{-1}\mathsf{B}_0^*\mathscr{U}_0(w)^*\end{bmatrix}\\ &= \begin{bmatrix}\mathsf{U}_0\\ \mathscr{U}_0(z)\mathsf{B}_0\end{bmatrix}\mathsf{Q}^{-1}\begin{bmatrix}\mathsf{U}_0\\ \mathscr{U}_0(w)\mathsf{B}_0\end{bmatrix}^*\\ &= \mathscr{A}(z)\mathscr{U}(z)\mathsf{Q}^{-1}\mathscr{U}(w)^*\mathscr{A}(w)^*. \end{aligned} \tag{5.12}$$

Properties $(\mathscr{U}4)$ and $(\mathscr{U}5)$ of $\mathscr{U}_0(z)$ follow from (5.12) and from the corresponding properties $(\mathscr{U}4)$ and $(\mathscr{U}5)$ of $\mathscr{U}(z)$.

Lemma 5.3 implies that for arbitrary distinct numbers $z_0,\ldots,z_\varkappa\in\mathbb{C}^+\cap(\mathrm{dom}(\mathscr{U})\backslash\gamma)$ we have

$$\mathscr{R}_1(z_0)^\circ\bigcap\left(\bigcap_{j=1}^{\varkappa}\mathscr{R}_1(z_j)\right) = \mathscr{R}_1(z_0)\bigcap\mathscr{R}_1(z_0^*)\bigcap\left(\bigcap_{j=1}^{\varkappa}\mathscr{R}_1(z_j)\right) = \{0\}. \tag{5.13}$$

By the definitions above, $\mathscr{R}_1(z) = \mathrm{ran}\,\mathscr{W}_1(z)^*\subset\mathscr{L}_+^1[\dot{+}]\mathscr{L}_-^1$. Since the matrix $\mathsf{B}_0^*(\mathsf{B}_0\mathsf{B}_0^*)^{-1}\mathsf{B}_0$ is the orthogonal projection with respect to the Euclidean inner product of $\mathbb{C}^d$ onto $\mathscr{L}_+^1[\dot{+}]\mathscr{L}_-^1$, the $\omega\times d$ matrix $(\mathsf{B}_0\mathsf{B}_0^*)^{-1}\mathsf{B}_0$ acts as a bijection between $\mathscr{R}_1(z)$ and $\mathrm{ran}\,\mathscr{U}_0(z)^*\subset\mathbb{C}^\omega$. Therefore (5.13) is equivalent to

$$\mathrm{ran}\,\mathscr{U}_0(z_0)^*\bigcap\mathrm{ran}\,\mathscr{U}_0(z_0^*)^*\bigcap\left(\bigcap_{j=1}^{\varkappa}\mathrm{ran}\,\mathscr{U}_0(z_j)^*\right) = \{0\}. \tag{5.14}$$

Taking the orthogonal complement in (5.14) with respect to the indefinite inner product defined by $y^*\mathsf{Q}_0^{-1}x,\ x,y\in\mathbb{C}^\omega$, on $\mathbb{C}^\omega$ and using the properties $(\mathscr{U}3)$ and $(\mathscr{U}4)$ of $\mathscr{U}_0$ one can show that (5.14) is equivalent to

$$\mathrm{ran}\,\mathscr{U}_0(z_0)^*\bigcup\mathrm{ran}\,\mathscr{U}_0(z_0^*)^*\bigcup\left(\bigcup_{j=1}^{\varkappa}\mathrm{ran}\mathscr{U}_0(z_j^*)^*\right) = \mathbb{C}^\omega, \tag{5.15}$$

and this is $(\mathscr{U}6)$. Thus $\mathscr{U}_0(z)$ is a minimal Q_0-boundary coefficient. □

6. A model for minimal boundary coefficients

In this section we provide a linear relation model for a minimal boundary coefficient by using the theory of reproducing kernel Pontryagin spaces. To be more precise, let Q be a $d\times d$ invertible self-adjoint matrix with d_+ positive and d_- negative eigenvalues. For an arbitrary minimal Q-boundary coefficient $\mathscr{U}(z)$ we construct

(a) a Pontryagin space $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$,
(b) a closed simple symmetric operator S in $\mathscr{H}$,
(c) a boundary mapping b of S with the Gram matrix $-\mathsf{Q}$,
(d) a holomorphic row vector function $\Phi:\mathbb{C}^{\pm}\backslash\gamma\to\mathscr{H}^{d_\pm}$, where γ is a finite subset of $\mathbb{C}\backslash\mathbb{R}$, and such that the components $\phi_j(z),\ j=1,\dots,d_\pm$, of

$$\Phi(z)=(\phi_1(z),\phi_2(z),\dots,\phi_{d_\pm}(z))$$

constitute a basis for $\ker(S^{[*]}-z),\ z\in\mathbb{C}\backslash(\mathbb{R}\cup\gamma)$, such that

$$\mathscr{U}(z)=(\mathsf{Qb}(\hat{\Phi}(z^*)))^*, \tag{6.1}$$

where $\hat{\Phi}$ is defined just before Theorem 4.2. If Eq. (6.1) holds, we say that $\mathscr{H},S,\mathsf{b}$ and Φ provide a *model* for the minimal boundary coefficient $\mathscr{U}(z)$.

With the kernel $K_{\mathscr{U}}(z,w)$ in $(\mathscr{U}5)$ we associate a reproducing kernel Pontryagin space $\mathscr{H}(K_{\mathscr{U}})$. It is the completion of the linear space of the holomorphic functions

$$z\mapsto\sum_{j=1}^{n}K_{\mathscr{U}}(z,w_j)x_j,\quad z\in\mathrm{dom}(\mathscr{U}),$$
$$w_j\in\mathbb{C}^{\pm}\cap\mathrm{dom}(\mathscr{U}),\ x_j\in\mathbb{C}^{d_\pm},\ j=1,\dots,n,\ n\in\mathbb{N},$$

with respect to the inner product

$$\left[\sum_{j=1}^{n}K_{\mathscr{U}}(\cdot,w_j)x_j,\sum_{k=1}^{m}K_{\mathscr{U}}(\cdot,u_k)y_k\right]=\sum_{j=1}^{n}\sum_{k=1}^{m}y_k^*K_{\mathscr{U}}(u_k,w_j)x_j. \tag{6.2}$$

This completion consists of column vector functions $f(z)$ which are holomorphic on $\mathrm{dom}(\mathscr{U})$, and are of size $d_\pm\times 1$ on $\mathbb{C}^{\pm}$. The inner product of $f(z)$ in $\mathscr{H}(K_{\mathscr{U}})$ with a function $z\mapsto K_{\mathscr{U}}(z,w)x$ reproduces the value of $f(z)$ at $z=w$ in the direction x:

$$x^*f(w)=[f(\cdot),K_{\mathscr{U}}(\cdot,w)x]. \tag{6.3}$$

Lemma 6.1. *Let Q and Q_1 be $d\times d$ invertible self-adjoint matrices with d_+ positive and d_- negative eigenvalues. Let $\mathscr{U}$ be a Q-boundary coefficient, let $\mathscr{U}_1$ be a Q_1-boundary coefficient and assume that*

$$\mathscr{U}(z)=\mathscr{A}(z)\mathscr{U}_1(z)\mathsf{A}$$

for some invertible matrix function $\mathscr{A}(z)$ *of size* $d_\mp \times d_\mp$ *if* $z \in \mathbb{C}^\pm$ *and a constant invertible* $d \times d$ *matrix* A *such that* $\mathsf{A}\mathsf{Q}^{-1}\mathsf{A}^* = \mathsf{Q}_1^{-1}$. *Then the operator of multiplication* $\mathscr{A}(\cdot): f(z) \mapsto \mathscr{A}(z)f(z)$ *is an isomorphism from* $\mathscr{H}(K_\mathscr{U})$ *onto* $\mathscr{H}(K_{\mathscr{U}_1})$ *and under this isomorphism the operators* $S_\mathscr{U}$ *and* $S_{\mathscr{U}_1}$ *of multiplication by the independent variable* z *coincide.*

In particular, if $\mathscr{U}$ *has a minimal representation* (5.1) *then the reproducing kernel spaces* $\mathscr{H}(K_\mathscr{U})$ *and* $\mathscr{H}(K_{\mathscr{U}_0})$ *are isomorphic and under the isomorphism the operators of multiplication by the independent variable* z *coincide.*

Proof. The proof of this lemma is identical to the proof of [7, Lemma 4.3]. □

Next, we study the operator $S_\mathscr{U}$ of multiplication by z in $\mathscr{H}(K_\mathscr{U})$. We assume that $\mathscr{U}(z)$ is a minimal Q-boundary coefficient. The following theorem gives a representation of a minimal boundary coefficient $\mathscr{U}(z)$ in terms of the operator $S_\mathscr{U}$ of multiplication by z in the reproducing kernel Pontryagin space $\mathscr{H}(K_\mathscr{U})$.

Theorem 6.2. *Let* Q *be a* $d \times d$ *invertible self-adjoint matrix with* d_+ *positive and* d_- *negative eigenvalues,* $d = d_+ + d_-$. *Let* $\mathscr{U}(z)$ *be a minimal* Q*-boundary coefficient.*

(a) *The operator* $S_\mathscr{U}$ *of multiplication by* z *in the reproducing kernel Pontryagin space* $\mathscr{H}(K_\mathscr{U})$ *is a closed simple symmetric operator with defect index* (d_-, d_+). *Its adjoint is given by*

$$\begin{aligned} S_\mathscr{U}^{[*]} &= \overline{\operatorname{span}}\{\{K_\mathscr{U}(\cdot, w)x, w^*K_\mathscr{U}(\cdot, w)x\} : w \in \mathbb{C}^\pm \cap \operatorname{dom}(\mathscr{U}), x \in \mathbb{C}^{d_\pm}\} \\ &= \{\{f, g\} \in \mathscr{H}(K_\mathscr{U})^2 : \exists c \in \mathbb{C}^d \text{ such that} \\ &\qquad g(z) = zf(z) - i\mathscr{U}(z)\mathsf{Q}^{-1}c, \ \forall z \in \operatorname{dom}(\mathscr{U})\}. \end{aligned} \tag{6.4}$$

The vector $c \in \mathbb{C}^d$ *in* (6.4) *is uniquely determined by* $\{f, g\} \in S_\mathscr{U}^{[*]}$ *and the mapping* $\hat{\mathsf{b}}(\{f, g\}) := c$ *is a boundary mapping for* $S_\mathscr{U}$ *with Gram matrix* $-\mathsf{Q}^{-1}$.

(b) *There exist a boundary mapping* b_1 *for* $S_\mathscr{U}$ *with Gram matrix* $-\mathsf{Q}$ *and a holomorphic basis* $\Phi_1(z)$ *for* $\ker(S_\mathscr{U}^{[*]} - z)$, $z \in \operatorname{dom}(\mathscr{U})$, *such that*

$$\mathscr{U}(z) = (\mathsf{Q}\mathsf{b}_1(\hat{\Phi}_1(z^*)))^*.$$

(c) *Let* b_2 *be any boundary mapping for* $S_\mathscr{U}$ *with Gram matrix* Q_2 *and let* $\Phi_2(z)$ *be any holomorphic basis for* $\ker(S_\mathscr{U}^{[*]} - z)$, $z \in \mathbb{C}\backslash(\mathbb{R} \cup \gamma_2)$, *where* γ_2 *is a finite subset of* $\mathbb{C}\backslash\mathbb{R}$. *Then*

$$\mathscr{U}(z) = \mathscr{A}(z)(\mathsf{Q}_2\mathsf{b}_2(\hat{\Phi}_2(z^*)))^*\mathsf{A}$$

on $\operatorname{dom}(\mathscr{U})\backslash\gamma_2$ *for some invertible matrix function* $\mathscr{A}(z)$ *of size* $d_\mp \times d_\mp$ *if* $z \in \mathbb{C}^\pm$ *and a constant invertible* $d \times d$ *matrix* A *such that* $\mathsf{A}\mathsf{Q}^{-1}\mathsf{A}^* = -\mathsf{Q}_2^{-1}$.

Proof. To prove (a) consider the following relation in $\mathscr{H}(K_{\mathscr{U}})^2$:

$$\hat{S} := \{\{f,g\} \in \mathscr{H}(K_{\mathscr{U}})^2 : \exists c \in \mathbb{C}^d \text{ such that}$$

$$g(z) = zf(z) - i\mathscr{U}(z)\mathsf{Q}^{-1}c, \ \forall z \in \mathrm{dom}(\mathscr{U})\}.$$

Note that for a given $\{f,g\} \in \hat{S}$ the vector $c \in \mathbb{C}^d$ such that $g(z) = zf(z) - i\mathscr{U}(z)\mathsf{Q}^{-1}c$ for all $z \in \mathrm{dom}(\mathscr{U})$, is uniquely determined. To show this we derive a formula for c. With $z_0, \ldots, z_\varkappa \in \mathrm{dom}(\mathscr{U})$ as in ($\mathscr{U}6$) (see (1.2)) we have

$$g(z_j) = z_j f(z_j) - i\mathscr{U}(z_j)\mathsf{Q}^{-1}c, \quad j = 0, \ldots, \varkappa,$$

$$g(z_0^*) = z_0^* f(z_0^*) - i\mathscr{U}(z_0^*)\mathsf{Q}^{-1}c,$$

or in matrix form

$$\mathsf{G}(\{f,g\}) = -i\mathsf{H}\mathsf{Q}^{-1}c,$$

where

$$\mathsf{G}(\{f,g\}) := \begin{bmatrix} g(z_0) - z_0 f(z_0) \\ \vdots \\ g(z_\varkappa) - z_\varkappa f(z_\varkappa) \\ g(z_0^*) - z_0^* f(z_0^*) \end{bmatrix} \quad \text{and} \quad \mathsf{H} := \begin{bmatrix} \mathscr{U}(z_0) \\ \vdots \\ \mathscr{U}(z_\varkappa) \\ \mathscr{U}(z_0^*) \end{bmatrix}.$$

Since the matrix H has maximal rank d, the matrix $\mathsf{H}^*\mathsf{H}$ is invertible. Therefore,

$$c = i\mathsf{Q}(\mathsf{H}^*\mathsf{H})^{-1}\mathsf{H}^*\mathsf{G}(\{f,g\}). \tag{6.5}$$

Hence if the pair $\{f,g\} = 0$, then $c = 0$, and this proves the uniqueness statement above. Since point evaluation is continuous on $\mathscr{H}(K_{\mathscr{U}})$, it also follows from (6.5) that $\hat{S}$ is a closed subspace of $\mathscr{H}(K_{\mathscr{U}})^2$. Define the mapping $\hat{\mathsf{b}} : \hat{S} \to \mathbb{C}^d$ by

$$\hat{\mathsf{b}}(\{f,g\}) := c \quad \text{for all } \{f,g\} \in \hat{S}.$$

Again since point evaluation is continuous, $\hat{\mathsf{b}}$ is a continuous linear mapping on $\hat{S}$. For arbitrary $w \in \mathbb{C}^\pm \cap \mathrm{dom}(\mathscr{U})$ and $a \in \mathbb{C}^{d_\pm}$ we have

$$\{K_{\mathscr{U}}(\cdot, w)a, w^* K_{\mathscr{U}}(\cdot, w)a\} \in \hat{S}$$

and

$$\hat{\mathsf{b}}(\{K_{\mathscr{U}}(\cdot, w)a, w^* K_{\mathscr{U}}(\cdot, w)a\}) = \mathscr{U}(w)^* a. \tag{6.6}$$

The minimality of $\mathscr{U}$ and (6.6) imply that the mapping $\hat{\mathsf{b}}$ is onto $\mathbb{C}^d$. By the definitions of $S_{\mathscr{U}}$ and $\ker\hat{\mathsf{b}}$ we have $S_{\mathscr{U}}=\ker\hat{\mathsf{b}}$. Therefore

$$\dim\hat{S}/S_{\mathscr{U}}=d. \tag{6.7}$$

As before, define the Lagrange inner product on $\mathscr{H}(K_{\mathscr{U}})^2$ by

$$[\![\{f,g\},\{u,v\}]\!]:=\frac{1}{i}([g,u]-[f,v]).$$

Put

$$T_{\max}^{\circ}:=\left\{\left\{\sum_{j=1}^{n}K_{\mathscr{U}}(\cdot,w_j)x_j,\sum_{j=1}^{n}w_j^{*}K_{\mathscr{U}}(\cdot,w_j)x_j\right\}:\right.$$
$$\left.n\in\mathbb{N},\ w_j\in\mathbb{C}^{\pm}\cap\mathrm{dom}(\mathscr{U}),\ x_j\in\mathbb{C}^{d_{\pm}}\right\}$$

and denote by $T_{\max}$ the closure of $T_{\max}^{\circ}$ in $\mathscr{H}(K_{\mathscr{U}})^2$. Note that, for $w_j\in\mathbb{C}^{\pm}\cap\mathrm{dom}(\mathscr{U})$, $x_j\in\mathbb{C}^{d_{\pm}}$,

$$z\sum_{j=1}^{n}K_{\mathscr{U}}(z,w_j)x_j-\sum_{j=1}^{n}w_j^{*}K_{\mathscr{U}}(z,w_j)x_j=\sum_{j=1}^{n}(z-w_j^{*})K_{\mathscr{U}}(z,w_j)x_j$$
$$=i\mathscr{U}(z)\mathsf{Q}^{-1}\sum_{j=1}^{n}\mathscr{U}(w_j)^{*}x_j.$$

Therefore, since $\hat{S}$ is closed, $T_{\max}^{\circ}\subset T_{\max}\subset\hat{S}$ and

$$\hat{\mathsf{b}}\left(\left\{\sum_{j=1}^{n}K_{\mathscr{U}}(\cdot,w_j)x_j,\sum_{j=1}^{n}w_j^{*}K_{\mathscr{U}}(\cdot,w_j)x_j\right\}\right)=\sum_{j=1}^{n}\mathscr{U}(w_j)^{*}x_j. \tag{6.8}$$

Let

$$T_{\min}^{\circ}:=\left\{\left\{\sum_{j=1}^{n}K_{\mathscr{U}}(\cdot,w_j)x_j,\sum_{j=1}^{n}w_j^{*}K_{\mathscr{U}}(\cdot,w_j)x_j\right\}:\right.$$
$$\left.\sum_{j=1}^{n}\mathscr{U}(w_j)^{*}x_j=0,\ n\in\mathbb{N},\ w_j\in\mathbb{C}^{\pm}\cap\mathrm{dom}(\mathscr{U}),\ x_j\in\mathbb{C}^{d_{\pm}}\right\}.$$

Denote by $T_{\min}$ the closure of $T_{\min}^{\circ}$.

It follows from property (6.3) of the inner product in the reproducing kernel space $\mathscr{H}(K_{\mathscr{U}})$ that for $\{u,v\}\in T_{\max}$ and $\{f,g\}\in\hat{S}$ we have

$$[\![\{f,g\},\{u,v\}]\!]=\hat{\mathsf{b}}(\{u,v\})^{*}(-\mathsf{Q}^{-1})\hat{\mathsf{b}}(\{f,g\}). \tag{6.9}$$

Equality (6.9) implies that $S_{\mathscr{U}} \subset T_{\max}^{[*]}$. Conversely, if $\{f,g\} \in T_{\max}^{[*]} = (T_{\max}^{\circ})^{[*]}$, then

$$\begin{aligned} 0 &= [g(\cdot), K_{\mathscr{U}}(\cdot,w)a] - [f(\cdot), w^* K_{\mathscr{U}}(\cdot,w)a] \\ &= a^*(g(w) - wf(w)), \quad \text{for all } w \in \mathbb{C}^{\pm} \cap \operatorname{dom}(\mathscr{U}),\ a \in \mathbb{C}^{d_{\pm}}. \end{aligned}$$

Hence, $g(w) = wf(w)$ for all $w \in \operatorname{dom}(\mathscr{U})$, that is, $\{f,g\} \in S_{\mathscr{U}}$. Thus $T_{\max}^{[*]} = S_{\mathscr{U}}$.

Next we prove that $T_{\min}^{[*]} = \hat{S}$. It is sufficient to prove that $(T_{\min}^{\circ})^{[*]} = \hat{S}$.

The inclusion $\hat{S} \subset (T_{\min}^{\circ})^{[*]}$ follows from (6.8) and (6.9). To prove the converse inclusion observe that

$$\left[g, \sum_{j=1}^{n} K_{\mathscr{U}}(\cdot, w_j)x_j \right] - \left[f, \sum_{j=1}^{n} w_j^* K_{\mathscr{U}}(\cdot, w_j)x_j \right] = \sum_{j=1}^{n} x_j^*(g(w_j) - w_j f(w_j)),$$

for $w_j \in \mathbb{C}^{\pm} \cap \operatorname{dom}(\mathscr{U})$, $x_j \in \mathbb{C}^{d_{\pm}}$. Therefore, for $\{f,g\} \in T_{\min}^{[*]}$,

$$\sum_{j=1}^{n} \mathscr{U}(w_j)^* x_j = 0 \ \Rightarrow \ \sum_{j=1}^{n} x_j^*(g(w_j) - w_j f(w_j)) = 0.$$

Consequently, the relation

$$\operatorname{span}\{\{\mathscr{U}(z)^* x, (g(z) - zf(z))^* x\} : z \in \mathbb{C}^{\pm} \cap \operatorname{dom}(\mathscr{U}),\ x \in \mathbb{C}^{d_{\pm}}\}$$

is an operator from $\mathbb{C}^d$ to $\mathbb{C}$. Therefore there exists an $a \in \mathbb{C}^d$ such that

$$a^* \mathscr{U}(z)^* x = (g(z) - zf(z))^* x, \quad \text{for all } z \in \mathbb{C}^{\pm} \cap \operatorname{dom}(\mathscr{U}),\ x \in \mathbb{C}^{d_{\pm}},$$

or

$$x^* \mathscr{U}(z) a = x^*(g(z) - zf(z)), \quad \text{for all } z \in \mathbb{C}^{\pm} \cap \operatorname{dom}(\mathscr{U}),\ x \in \mathbb{C}^{d_{\pm}}.$$

Thus $\mathscr{U}(z)a = g(z) - zf(z)$ and consequently $\{f,g\} \in \hat{S}$.

Since $T_{\min} \subset S_{\mathscr{U}} \subset \hat{S}$ and $T_{\min}^{[*]} = \hat{S}$, $T_{\min}$ is a symmetric operator in $(\mathscr{H}(K_{\mathscr{U}}), [\,\cdot\,,\,\cdot\,])$. Next we will prove that its defect index is (d_-, d_+).

Let $\mu \in \mathbb{C}^+ \cap \operatorname{dom}(\mathscr{U})$. We have to determine the dimension of the subspace $\hat{S} \cap \mu I$. Note that $S_{\mathscr{U}} \cap \mu I = \{\{0,0\}\}$. Therefore $\hat{\mathsf{b}}|_{\hat{S} \cap \mu I}$ is an injection. Let $\{f, \mu f\} \in \hat{S}$. Then

$$\mu f(z) = zf(z) - i\mathscr{U}(z)\mathsf{Q}^{-1}\hat{\mathsf{b}}(\{f, \mu f\}),$$

and consequently

$$\mathscr{U}(\mu)\mathsf{Q}^{-1}\hat{\mathsf{b}}(\{f, \mu f\}) = 0.$$

This equality and conditions $(\mathscr{U}4)$ and $(\mathscr{U}3)$ imply that $\hat{\mathsf{b}}(\{f,\mu f\})\in\operatorname{ran}\mathscr{U}(\mu^*)^*$. Thus, the range of the injection $\hat{\mathsf{b}}|_{\hat{S}\cap\mu I}$ is contained in the (d_-)-dimensional space $\operatorname{ran}\mathscr{U}(\mu^*)^*$. Therefore $\dim(\hat{S}\cap\mu I)\leqslant d_-$. Since

$$\{K_{\mathscr{U}}(\cdot,\mu^*)a,\mu K_{\mathscr{U}}(\cdot,\mu^*)a:a\in\mathbb{C}^{d_-}\}\subset\hat{S}\cap\mu I \tag{6.10}$$

and since the subspace on the left-hand side of (6.10) has the dimension d_-, it follows that

$$\dim\hat{S}\cap\mu I=d_-.$$

In a similar way one can prove that

$$\dim\hat{S}\cap\mu^* I=d_+.$$

Thus the defect index of $T_{\min}$ is (d_-,d_+). It follows from Theorem 2.3 that

$$\dim\hat{S}/T_{\min}=d_-+d_+=d. \tag{6.11}$$

Since $T_{\min}\subset S_{\mathscr{U}}\subset\hat{S}$, (6.7) and (6.11) imply that $T_{\min}=S_{\mathscr{U}}$. Therefore, $T_{\max}=S_{\mathscr{U}}^{[*]}=T_{\min}^{[*]}=\hat{S}$. Consequently, the operator $S_{\mathscr{U}}$ of multiplication by the independent variable is symmetric in $(\mathscr{H}(K_{\mathscr{U}}),[\,\cdot\,,\,\cdot\,])$, it has defect index (d_-,d_+), and its adjoint is $T_{\max}=\hat{S}$. The last statement in (a) now follows from (6.9).

To prove (b) put $\mathsf{b}_1=\mathsf{Q}^{-1}\hat{\mathsf{b}}$, where $\hat{\mathsf{b}}$ is the boundary mapping for $S_{\mathscr{U}}$ with Gram matrix $-\mathsf{Q}^{-1}$ introduced in the proof of part (a). Then b_1 is a boundary mapping for $S_{\mathscr{U}}$ with Gram matrix $-\mathsf{Q}$. Note that for the jth basis vector e_j of $\mathbb{C}^{d_\mp}$, $j=1,\dots,d_\mp$, the vectors $K_{\mathscr{U}}(\cdot,z^*)e_j$, $j=1,\dots,d_\mp$, form a basis of $\ker(S_{\mathscr{U}}^{[*]}-z)$, $z\in\mathbb{C}^\pm\cap\operatorname{dom}(\mathscr{U})$. Let $\Phi_1(z)$, $z\in\mathbb{C}^\pm\cap\operatorname{dom}(\mathscr{U})$, be the vector whose components are the vectors $K_{\mathscr{U}}(\cdot,z^*)e_j$, $j=1,\dots,d_\mp$. Since $\mathscr{U}(z)$ is holomorphic on $\operatorname{dom}(\mathscr{U})$, $\Phi_1(z)$ is holomorphic there too. Using the above definitions we get

$$\begin{aligned}\mathsf{b}_1(\hat{\Phi}_1(z))&=\mathsf{Q}^{-1}\hat{\mathsf{b}}(\hat{\Phi}_1(z))\\&=\mathsf{Q}^{-1}[\,\mathscr{U}(z^*)^*e_1\quad\cdots\quad\mathscr{U}(z^*)^*e_{d_\mp}\,]\\&=\mathsf{Q}^{-1}\mathscr{U}(z^*)^*.\end{aligned}$$

This readily implies (b).

Part (c) follows from Theorem 4.2(b). The theorem is proved. □

Corollary 6.3. *Let S be a closed simple symmetric operator in a Pontryagin space $(\mathscr{H},[\,\cdot\,,\,\cdot\,])$ with defect index (d_+,d_-), $d=d_++d_-<\infty$. Then there exist a $d\times d$ invertible matrix Q with d_+ positive and d_- negative eigenvalues and a minimal $(-\mathsf{Q})$-boundary coefficient $\mathscr{U}(z)$ such that S is isomorphic to the operator $S_{\mathscr{U}}$ of*

multiplication by the independent variable in the reproducing kernel Pontryagin space $\mathscr{H}(K_{\mathscr{U}})$ *and*

$$S^{[*]} = \overline{\operatorname{span}}\{\{\phi, z\phi\} : \phi \in \ker(S^{[*]} - z),\ z \in \mathbb{C}\backslash\mathbb{R}\}.$$

Proof. Assume that S is a closed simple symmetric operator in a Pontryagin space $(\mathscr{H}, [\,\cdot\,,\,\cdot\,]_{\mathscr{H}})$ with defect index (d_+, d_-), $d = d_+ + d_- < \infty$. Let $\mathscr{U}(z) = (\mathsf{Q}\mathsf{b}(\hat{\Phi}(z^*)))^*$, $z \in \mathbb{C}\backslash(\mathbb{R} \cup \gamma)$, where b is a boundary mapping for S with Gram matrix Q and $\Phi(z)$ is a holomorphic basis for $\ker(S^{[*]} - z)$, $z \in \mathbb{C}\backslash(\mathbb{R} \cup \gamma)$, where γ is a finite subset of $\mathbb{C}\backslash\mathbb{R}$. By Theorem 4.2, $\mathscr{U}(z)$ is a minimal $(-\mathsf{Q})$-boundary coefficient. It follows that the kernel

$$K_{\mathscr{U}}(z, w) = -i\frac{\mathscr{U}(z)\mathsf{Q}^{-1}\mathscr{U}(w)^*}{z - w^*}$$

has a finite number of negative squares. We show that S in $\mathscr{H}$ is isomorphic to the operator $S_{\mathscr{U}}$ of multiplication by the independent variable in the reproducing kernel space $(\mathscr{H}(K_{\mathscr{U}}), [\,\cdot\,,\,\cdot\,]_{\mathscr{H}(K_{\mathscr{U}})})$. By Theorem 6.2 the defect index of $S_{\mathscr{U}}$ is equal to that of S. Denote by $U : \mathscr{H} \to \mathscr{H}(K_{\mathscr{U}})$ the linear operator

$$U(\Phi(w^*)x) = K_{\mathscr{U}}(\cdot, w)x, \quad w \in \mathbb{C}^{\pm}\backslash\gamma,\ x \in \mathbb{C}^{d_\pm}.$$

From (4.4)

$$[\Phi(w^*)x, \Phi(z^*)y]_{\mathscr{H}} = y^*K_{\mathscr{U}}(z, w)x = [K_{\mathscr{U}}(\cdot, w)x, K_{\mathscr{U}}(\cdot, z)y]_{\mathscr{H}(K_{\mathscr{U}})}.$$

Hence U is isometric. As S is simple, $\operatorname{dom}(S^{[*]})$ is dense in $\mathscr{H}$ and as the kernel functions $K_{\mathscr{U}}(\cdot, w)x$ are total in $\mathscr{H}(K_{\mathscr{U}})$ the range of U is dense in $\mathscr{H}(K_{\mathscr{U}})$. Therefore the closure of U is a unitary operator which we also denote by U. Using Theorem 6.2 we conclude

$$\begin{aligned} S &\subset U^{-1}S_{\mathscr{U}}U \\ &\subset U^{-1}S_{\mathscr{U}}^{[*]}U \\ &= \overline{\operatorname{span}}\{\{\Phi(w^*)x, w^*\Phi(w^*)x\} : w \in \mathbb{C}^{\pm}\backslash\gamma,\ x \in \mathbb{C}^{d_\pm}\} \\ &\subset S^{[*]}. \end{aligned}$$

Since by Theorem 2.3 $\dim(S^{[*]}/S) = \dim(S_{\mathscr{U}}^{[*]}/S_{\mathscr{U}}) = d$, we have $S = U^{-1}S_{\mathscr{U}}U$ and the formula for $S^{[*]}$ holds. □

Example 6.4. Let

$$\mathsf{Q} = \begin{bmatrix} 0 & -iI \\ iI & 0 \end{bmatrix},$$

where I is the 3×3 identity matrix. The matrix valued function

$$z \mapsto \mathscr{U}(z) = \begin{bmatrix} 0 & 0 & z & -1 & 0 & 0 \\ 0 & 0 & 0 & z & -1 & 0 \\ z & z^2 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad z \in \mathbb{C}$$

satisfies $(\mathscr{U}1)$–$(\mathscr{U}4)$ and

$$K_{\mathscr{U}}(z, w) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -w^* \\ -1 & -z & 0 \end{bmatrix}.$$

This kernel has 2 positive and 2 negative squares and therefore the dimension of $\mathscr{H}(K_{\mathscr{U}})$ is 4. Since the determinant of the matrix

$$\begin{bmatrix} \mathscr{U}(z) \\ \mathscr{U}(w) \end{bmatrix}$$

evaluates to 0 for each $z, w \in \mathbb{C}$, this matrix is degenerate. The row reduction yields that for any three distinct numbers $z, w, v \in \mathbb{C}$ the matrix

$$\begin{bmatrix} \mathscr{U}(z) \\ \mathscr{U}(w) \\ \mathscr{U}(v) \end{bmatrix}$$

has the maximal rank 6. Thus $\mathscr{U}(z)$ is a minimal Q-boundary condition. The reproducing kernel space $\mathscr{H}(K_{\mathscr{U}})$ is

$$\left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 + z\alpha_4 \end{bmatrix} : \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C} \right\}.$$

A basis of this space is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = K_{\mathscr{U}}(z, 0) \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = K_{\mathscr{U}}(z,0)\begin{bmatrix} 0 \\ 0 \\ i \end{bmatrix} - K_{\mathscr{U}}(z,i)\begin{bmatrix} 0 \\ 0 \\ i \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = K_{\mathscr{U}}(z,0)\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = K_{\mathscr{U}}(z,0)\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Applying definition (6.2) to these basis vectors we conclude that the space $\mathscr{H}(K_{\mathscr{U}})$ is isomorphic to the space $\mathbb{C}^4_{\Delta}$ with the inner product

$$[x,y] := y^*\Delta x, \quad x,y\in\mathbb{C}^4, \quad \Delta = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Under this isomorphism the operator of multiplication by z is isomorphic to the operator

$$S = \left\{ \left\{ \begin{bmatrix} 0 \\ 0 \\ \alpha \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha \end{bmatrix} \right\} : \alpha\in\mathbb{C} \right\},$$

with the adjoint

$$S^{[*]} = \left\{ \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \right\} : x_j, y_j\in\mathbb{C} \right\}.$$

Using (6.4) and the proof of Theorem 6.2(b), we find that the boundary mapping

$$\mathsf{b} : \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \right\} \mapsto \begin{bmatrix} y_4 - x_3 \\ -x_4 \\ -x_1 \\ -x_2 \\ -y_2 \\ -y_3 \end{bmatrix}$$

of S has Gram matrix $-\mathsf{Q}$. Since

$$S^{[*]} \cap zI = \left\{ \left\{ \begin{bmatrix} x_1 \\ zx_1 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} zx_1 \\ z^2x_1 \\ zx_3 \\ zx_4 \end{bmatrix} \right\} : x_j \in \mathbb{C} \right\},$$

we conclude that

$$\Phi(z) = \left(\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -z \\ 0 \\ 0 \end{bmatrix} \right)$$

is a holomorphic basis for $\ker(S^{[*]} - z)$. Therefore

$$(\mathsf{Q}[\mathsf{b}(\Phi(z^*))])^* = \begin{bmatrix} 0 & 0 & z & -1 & 0 & 0 \\ 0 & 0 & 0 & z & -1 & 0 \\ z & z^2 & 0 & 0 & 0 & -1 \end{bmatrix},$$

that is $\mathbb{C}^4_\Delta, S, \mathsf{b}$ and Φ provide a relation model for the minimal Q-boundary coefficient $\mathscr{U}(z)$.

Example 6.5. A meromorphic $m \times m$ matrix valued function N defined on $\mathbb{C}\backslash\mathbb{R}$ is called a generalized Nevanlinna function with $\varkappa$ negative squares if $N(z)^* = N(z^*)$ for $z \in \mathrm{hol}(N)$ and the kernel

$$\frac{N(z) - N(w)^*}{z - w^*}$$

has $\varkappa$ negative squares. We denote the class of such functions by $\mathbf{N}_\varkappa^{m\times m}$. It is easily checked that

$$\mathscr{U}(z) := [\, N(z) \quad I \,]$$

is a Q-boundary coefficient with

$$\mathsf{Q} = i \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

The $2m \times 2m$ matrix

$$\begin{bmatrix} \mathscr{U}(z) \\ \mathscr{U}(z^*) \end{bmatrix} = \begin{bmatrix} N(z) & I \\ N(z)^* & I \end{bmatrix} \tag{6.12}$$

is invertible if and only if $\operatorname{Im} N(z)$ is invertible. Indeed,

$$\det \begin{bmatrix} \mathscr{U}(z) \\ \mathscr{U}(z^*) \end{bmatrix} = \det \begin{bmatrix} 2i \operatorname{Im} N(z) & 0 \\ N(z)^* & I \end{bmatrix} = (2i)^m \det \operatorname{Im} N(z).$$

We shall assume that $\operatorname{Im} N(v)$ is invertible for some $v \in \operatorname{hol}(N)$ and relate the representation of $\mathscr{U}(z)$ given in Theorem 6.2 to the operator representation of $N(z)$.

For this we use results from [1,11,15], where also earlier references can be found. As $N \in \mathbf{N}_\varkappa^{m\times m}$, it admits a representation of the form

$$N(z) = N(\mu^*) + (z-\mu^*)\Gamma^*(I + (z-\mu)(A-z)^{-1})\Gamma,$$

where A is a self-adjoint relation in a Pontryagin space $(\mathscr{H}, [\,\cdot\,,\,\cdot\,])$ with non-empty resolvent set $\rho(A)$, μ is a point in $\rho(A)\cap\mathbb{C}^+$, and Γ is a linear mapping from $\mathbb{C}^m$ in $\mathscr{H}$. Evidently, $\rho(A)\subset\operatorname{hol}(N)$. We set $\Gamma_z = I + (z-\mu)(A-z)^{-1}\Gamma$, $z\in\rho(A)$. Then for $z,\ w\in\rho(A)$,

$$\Gamma_z = (I + (z-w)(A-z)^{-1})\Gamma_w \tag{6.13}$$

and

$$\frac{N(z)-N(w)^*}{z-w^*} = \Gamma_w^*\Gamma_z. \tag{6.14}$$

Define the relation S in $\mathscr{H}$ by

$$\begin{aligned} S &= \{\{f,g\}\in A : \Gamma_z^*(g-z^*f) = 0\} \\ &= \{\{f,g\}\in A : [\![\, \{f,g\}, \{\Gamma_z x, z\Gamma_z x\}\,]\!] = 0 \text{ for all } x\in\mathbb{C}^m\}. \end{aligned}$$

It is closed and since it is a restriction of a self-adjoint relation it is symmetric. The sets on the right-hand side are independent of $z\in\rho(A)$, because by (6.13), for $\{f,g\}\in A$ we have

$$\Gamma_z^*(g-z^*f) = \Gamma_w^*(I + (z^*-w^*)(A-z^*)^{-1})(g-z^*f) = \Gamma_w^*(g-w^*f).$$

From the definition of S it follows that $\operatorname{ran}(S-z^*) = (\operatorname{ran}\Gamma_z)^{[\perp]}$ and hence $\ker(S^{[*]}-z) = \operatorname{ran}\Gamma_z$, $z\in\rho(A)$. Hence the defect indices of S coincide and are equal to $m - \dim(\ker\Gamma_z)$. This number is independent of $z\in\rho(A)$, because by (6.13), $\ker\Gamma_z = \ker\Gamma_w$, $z,w\in\rho(A)$.

The definition of S implies that for all $z\in\rho(A)$,

$$S^{[*]} = A \dotplus \{\{\Gamma_z x, z\Gamma_z x\} : x\in\mathbb{C}^m\}, \quad \text{direct sum.}$$

The map Γ_z maps $\mathbb{C}^m$ onto the kernel $\ker(S^{[*]}-z)$.

The model consisting of $\mathscr{H}$, A, and Γ can always be constructed such that

$$\mathscr{H} = \overline{\operatorname{span}}\{\Gamma_z x : x \in \mathbb{C}^m,\ z \in \rho(A)\}.$$

We shall assume that the model satisfies this closely connectedness condition. Then

(a) $\mathscr{H}$, A, and Γ are uniquely determined by N up to unitary equivalence,
(b) the negative index of $\mathscr{H}$ equals $\varkappa$, the number of negative squares of N,
(c) $\operatorname{hol}(N) = \rho(A)$,
(d) S is a simple symmetric operator.

We now assume that for some $v \in \operatorname{hol}(N)$, $\operatorname{Im} N(v)$ is invertible. Since by (6.14) and (6.13), we have

$$N(z) = N(v^*) + (z - v^*)\Gamma_v^*(I + (z - v)(A - z)^{-1})\Gamma_v,$$

without loss of generality we may assume that $v = \mu$. Then by (6.14), Γ_μ is injective and consequently, for all $z \in \rho(A)$, Γ_z is injective, that is, $\Gamma_z : \mathbb{C}^m \to \ker(S^{[*]} - z)$ is a bijection. Also, the defect indices of S are equal to m and, because $\ker(S^{[*]} - z)$ is a non-degenerate subspace, von Neumann's formula holds:

$$S^{[*]} = S + S^{[*]} \cap zI + S^{[*]} \cap z^* I, \quad z \in \rho(A) \backslash \mathbb{R}.$$

Let $e_1, e_2, \ldots, e_m$ be the standard orthonormal basis in $\mathbb{C}^m$ and define the boundary operator for S by

$$\mathsf{b}(f, g) = \mathsf{A}^{-1} \begin{bmatrix} [\![\{f, g\}, \{\Gamma_\mu e_1, \mu\Gamma_\mu e_1\}]\!] \\ \vdots \\ [\![\{f, g\}, \{\Gamma_\mu e_m, \mu\Gamma_\mu e_m\}]\!] \\ [\![\{f, g\}, \{\Gamma_{\mu^*} e_1, \mu^*\Gamma_{\mu^*} e_1\}]\!] \\ \vdots \\ [\![\{f, g\}, \{\Gamma_{\mu^*} e_m, \mu^*\Gamma_{\mu^*} e_m\}]\!] \end{bmatrix},$$

where

$$\mathsf{A} = \begin{bmatrix} N(\mu)^* & I \\ N(\mu) & I \end{bmatrix} = \frac{i}{2} \begin{bmatrix} (\operatorname{Im} N(\mu))^{-1} & -(\operatorname{Im} N(\mu))^{-1} \\ -N(\mu)(\operatorname{Im} N(\mu))^{-1} & N(\mu)^*(\operatorname{Im} N(\mu))^{-1} \end{bmatrix}^{-1}.$$

If we set

$$\Phi(z) = (\Gamma_z e_1, \Gamma_z e_2, \ldots, \Gamma_z e_m)$$

then after some calculations we find that b is a boundary mapping for S with Gram matrix $-\mathsf{Q}$ and $\mathscr{U}(z) = (\mathsf{Q}\mathsf{b}(\hat{\Phi}(z^*)))^*$.

The operator $U:\mathscr{H}\to\mathscr{H}(K_{\mathscr{U}})$ defined by $U:f\mapsto f(z):=\Gamma_{z*}^{*}f$ is unitary; U^{-1} maps the function $K(z,w)x$ to $\Gamma_{w}^{*}x$, $x\in\mathbb{C}^{m}$. The symmetric operator S and the self-adjoint relation A in $\mathscr{H}$ in the operator representation of N are isomorphic under U to the operator $S_{\mathscr{U}}$ of multiplication by the independent variable in the space $\mathscr{H}(K_{\mathscr{U}})$ and

$$A_{\mathscr{U}}:=\{\{f,g\}\in\mathscr{H}(K_{\mathscr{U}})^{2}:\exists c\in\mathbb{C}^{m}\ \text{s.t.}\ g(z)-zf(z)=c,\ \forall z\in\mathbb{C}\backslash\mathbb{R}\};$$

for details, see [1,11]. Finally, note that $\mathscr{U}(z)$ in Example 6.4 is of the form

$$\mathscr{U}(z)=A(z)[\,N(z)\quad I\,],$$

where

$$A(z)=\begin{bmatrix}-1 & 0 & 0\\ z & -1 & 0\\ 0 & 0 & -1\end{bmatrix},\quad N(z)=\begin{bmatrix}0 & 0 & -z\\ 0 & 0 & -z^{2}\\ -z & -z^{2} & 0\end{bmatrix},$$

$A(z)$ is invertible and $N(z)\in\mathbf{N}_{\varkappa}^{2\times 2}$. Thus the reproducing kernel space $\mathscr{H}(K_{\mathscr{U}})$ is isomorphic to the space associated with $N(z)$.

Example 6.6. As remarked in the Introduction the case where $d_{-}=0$ or $d_{+}=0$ is included in the theory. We consider the first case in more detail; the other case where $d_{+}=0$ can be treated similarly. If $d_{-}=0$, then $d=d_{+}$ and Q is assumed to be a positive $d\times d$ matrix. According to the definition Q-boundary coefficient $\mathscr{U}(z)$ is a meromorphic $d\times d$ matrix valued function on the upper half plane $\mathbb{C}^{+}$ which has maximal rank, that is, invertible on its domain $\mathrm{dom}(\mathscr{U})$ of holomorphy in the upper half plane $\mathbb{C}^{+}$. The minimality condition is now superfluous. Since Q is positive, the kernel $K_{\mathscr{U}}(z,w)$, now only defined for $z,w\in\mathrm{dom}(\mathscr{U})$, is non-negative. Indeed, using complex contour integration and the residue theorem we obtain that for points $z_1,z_2,\ldots,z_n\in\mathbb{C}^{+}$ and vectors $x_1,x_2,\ldots,x_n\in\mathbb{C}^{d}$

$$\begin{aligned}\sum_{j,k=1}^{n}x_j^{*}K(z_j,z_k)x_k&=i\sum_{j,k=1}^{n}\frac{x_j^{*}\mathscr{U}(z_j)Q^{-1}\mathscr{U}(z_k)^{*}x_k}{z_j-z_k^{*}}\\&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\sum_{j=1}^{n}\frac{\mathscr{U}(z_j)^{*}x_j}{t-z_j^{*}}\right)^{*}Q^{-1}\left(\sum_{k=1}^{n}\frac{\mathscr{U}(z_k)^{*}x_k}{t-z_k^{*}}\right)\\&\geqslant 0.\end{aligned}$$

Since $\mathscr{U}(z)\mathsf{Q}^{-1/2}$ is invertible, on account of, for example, [2, Theorem 1.5.7], the reproducing Hilbert space $\mathscr{H}(K_{\mathscr{U}})$ is isomorphic to the (infinite dimensional) Hardy space $\mathbb{H}$ associated with the kernel $\frac{iI}{z-w^{*}}$ of functions defined on the upper half-plane $\mathbb{C}^{+}$. The isomorphism is given by the map $f(z)\in\mathbb{H}\mapsto\mathscr{U}(z)\mathsf{Q}^{-1/2}f(z)$. Thus we may as well assume that $\mathscr{U}(z)=\mathsf{Q}=I$. Theorem 6.2 holds true provided, in for example,

(6.4) we consider only $w\in\mathbb{C}^+$ and $z\in\mathbb{C}^+$. The boundary operator $\mathsf{b}_1(\{f,g\})$ is simply the vector c for which $g(z)-zf(z)\equiv c$. With

$$\Phi_1(z)=\left(\frac{e_1}{z-w^*},\frac{e_2}{z-w^*},\ldots,\frac{e_m}{z-w^*}\right),$$

where $e_1,e_2,\ldots,e_m$, is the usual orthonormal basis for $\mathbb{C}^m$, we see that part (b) of Theorem 6.2 holds.

These results are consistent with the facts that (i) a standard maximal symmetric operator in a Pontryagin space $\mathscr{H}$ has a unique (up to unitary equivalence) minimal self-adjoint extension in Pontryagin space $\tilde{\mathscr{H}}$ and (ii) the exit space $\tilde{\mathscr{H}}\ominus\mathscr{H}$ is necessarily an infinite dimensional Hilbert space; see [3].

Example 6.7. For $j=1,2$, let $\mathscr{U}_j(z)$ be a Q_j-boundary coefficient. Then

$$\mathscr{U}(z)=\mathscr{A}(z)\begin{bmatrix}\mathscr{U}_1(z) & 0\\ 0 & \mathscr{U}_2(z)\end{bmatrix}\mathsf{A},$$

where $\mathscr{A}(z)$ is an invertible holomorphic matrix function on $\mathrm{dom}(\mathscr{U}_1)\cap\mathrm{dom}(\mathscr{U}_2)$ and A is an invertible matrix, is a Q-boundary coefficient with $\mathsf{Q}=\mathsf{A}^*\mathrm{diagonal}(\mathsf{Q}_1,\mathsf{Q}_2)\mathsf{A}$. This includes the case that for example $\mathscr{U}_1(z)$ is only defined for $z\in\mathbb{C}^+$, because $d_{1-}=0$. Then $\mathscr{U}(z)=\mathscr{U}_2(z)$ for $z\in\mathbb{C}^-$.

Appendix A. A Krein space version of the maximum principle for generalized Schur functions

In this appendix we give a geometric interpretation of the maximum modulus principle [9, Proposition 8.1] for generalized Schur functions with $\varkappa$ negative squares in terms of subspaces of a Krein space. This is used in the proof of Theorem 5.1, see Lemmas 5.2(e),(f) and 5.3.

As in Section 5 $(\mathscr{F},\langle\cdot,\cdot\rangle_{\mathscr{F}})$ and $(\mathscr{G},\langle\cdot,\cdot\rangle_{\mathscr{G}})$ are Hilbert spaces and $T:\mathbb{C}^+\to\mathscr{L}(\mathscr{F},\mathscr{G})$ is a meromorphic operator function such that the kernel

$$K_T(z,w)=i\frac{I-T(z)T(w)^*}{z-w^*},\quad z,w\in\mathrm{hol}(T)$$

has $\varkappa$ negative squares. Here if T is a meromorphic operator valued function $\mathrm{hol}(T)$ stands for its domain of holomorphy. By $T^*:\mathbb{C}^+\to\mathscr{L}(\mathscr{G},\mathscr{F})$ we denote the function defined by $T^*(z):=T(z)^*$, $z\in\mathbb{C}^+$. In Theorem A.5, the main theorem in this appendix, we consider the graphs of $T(z)$,

$$\mathscr{L}(z):=G[T(z)]=\{\{f,T(z)f\}:f\in\mathscr{F}\}\subset\mathscr{K},\quad z\in\mathbb{C}^+,$$

as subspaces of the Krein space

$$(\mathscr{K}, [\,\cdot\,,\,\cdot\,]) \coloneqq (\mathscr{F}, \langle\cdot,\cdot\rangle_{\mathscr{F}}) \oplus (\mathscr{G}, -\langle\cdot,\cdot\rangle_{\mathscr{G}}).$$

The maximum modulus principle [9, Proposition 8.1] for generalized Schur functions reads as follows

Theorem A.1 (Maximum principle). *Let $S : \mathbb{D} \mapsto \mathscr{L}(\mathscr{F}, \mathscr{G})$ be a meromorphic operator function with $0 \in \mathrm{hol}(S)$ such that the kernel*

$$K_S(z, w) = \frac{I - S(z)S(w)^*}{1 - zw^*}, \quad z, w \in \mathrm{hol}(S), \tag{A.1}$$

has $\varkappa$ negative squares.

(a) *Let $f \in \mathscr{F}, g \in \mathscr{G}$ and assume $g = S(z)f$ for more then $\varkappa$ points $z \in \mathrm{hol}(S)$. Then we have $||g|| \leqslant ||f||$, and the equality $||g|| = ||f||$ implies that $g = S(z)f$ and $f = S(z)^* g$ for all $z \in \mathrm{hol}(S)$.*

(b) *Let $f \in \mathscr{F}$, $g \in \mathscr{G}$ and assume $f = S(z)^* g$ for more then $\varkappa$ points $z \in \mathrm{hol}(S)$. Then we have $||f|| \leqslant ||g||$, and the equality $||f|| = ||g||$ implies that $f = S(z)^* g$ and $g = S(z)f$ for all $z \in \mathrm{hol}(S)$.*

The statement (a) follows from [9, Proposition 8.1] since by Alpay et al. [2, Theorem 2.5.2], the kernel

$$(z, w) \mapsto \frac{I - S(w)^* S(z)}{1 - zw^*}, \quad z, w \in \mathrm{hol}(S),$$

has $\varkappa$ negative squares on $\mathrm{hol}(S)$. In addition to the original statement of [9, Proposition 8.1], statement (a) claims the equality $f = S(z)^* g$. That this equality holds true is clear from the proof of [9, Proposition 8.1]. As to (b) we consider the meromorphic function $S_1(z) \coloneqq S(z^*)^*$, $z \in \mathbb{D}$. Since S is holomorphic at 0, S_1 is holomorphic at 0 and, since kernel (A.1) has $\varkappa$ negative squares, the kernel

$$(z, w) \mapsto \frac{I - S_1(w)^* S_1(z)}{1 - zw^*}, \quad z, w \in \mathrm{hol}(S)^*,$$

has $\varkappa$ negative squares on $\mathrm{hol}(S_1) = \mathrm{hol}(S)^*$. Now statement (b) follows from [9, Proposition 8.1] applied to S_1.

In the following corollary we prove that the family of operators $T(z)$, $z \in \mathrm{hol}(T)$, $\varkappa + 1$ of the operators can coincide on a subspace of $\mathscr{F}$ only as contractions. If $\varkappa + 1$ of the operators $T(z)$ coincide on a subspace of $\mathscr{F}$ as isometries, then $T(z)$ is independent of $z \in \mathrm{hol}(T)$ on this subspace. We recall from

Section 5 that

$$\mathcal{N}(T)=\left(\bigcap_{z,v\in\mathrm{hol}(T)}\ker(T(z)-T(v))\right)\bigcap\left(\bigcap_{w\in\mathrm{hol}(T)}\ker(I-T(w)^*T(w))\right). \tag{A.2}$$

Corollary A.2. *Let $z_0,z_1,\ldots,z_\varkappa\in\mathrm{hol}(T)$ be distinct complex numbers and put*

$$\mathcal{M}(T)=\mathcal{M}(T;z_0,\ldots,z_\varkappa):=\bigcap_{j=0}^{\varkappa}\ker(T(z_j)-T(z_0)).$$

Then for $j=0,1,\ldots,\varkappa$, $T(z_j)|_{\mathcal{M}(T)}:\mathcal{M}(T)\to\mathcal{G}$ and $T(z_j)^*|_{\mathcal{M}(T^*)}:\mathcal{M}(T^*)\to\mathcal{F}$ are contractions and

$$\mathcal{M}(T)\bigcap\ker(I-T(z_j)^*T(z_j))=\mathcal{N}(T), \tag{A.3}$$

$$\mathcal{M}(T^*)\bigcap\ker(I-T(z_j)T(z_j)^*)=\mathcal{N}(T^*). \tag{A.4}$$

That is, the sets on the left-hand sides of (A.3) *and* (A.4) *are independent of the choice of the distinct points $z_0,z_1,\ldots,z_\varkappa\in\mathrm{hol}(T)$.*

Proof. Let $u_0\in\mathbb{C}^+$ be a point at which T is holomorphic. The holomorphic transformation $\phi:z\mapsto\frac{z-u_0}{z-u_0^*}=\lambda$ maps $\mathbb{R}$ onto $\mathbb{T}$ and $\mathbb{C}^+$ onto $\mathbb{D}$. Its inverse is the holomorphic mapping $\psi:\lambda\mapsto\frac{u_0-\lambda u_0^*}{1-\lambda}=z$ which maps $\mathbb{T}$ onto $\mathbb{R}$ and $\mathbb{D}$ onto $\mathbb{C}^+$. The composition $S=T\circ\psi$ is a meromorphic function on $\mathbb{D}$ which is holomorphic at 0. The equality

$$\frac{I-S(\lambda)S(\mu)^*}{1-\lambda\mu^*}=\left(\frac{z-u_0^*}{\sqrt{2\,\mathrm{Im}\,u_0}}\right)i\frac{I-T(z)T(w)^*}{z-w^*}\left(\frac{w-u_0^*}{\sqrt{2\,\mathrm{Im}\,u_0}}\right)^*$$

implies that the kernel

$$\frac{I-S(\lambda)S(\mu)^*}{1-\lambda\mu^*},\quad \lambda,v\in\mathrm{hol}(S),$$

has $\varkappa$ negative squares. Hence we may apply Theorem A.1(a) to S.

Let $z_0,z_1,\ldots,z_\varkappa\in\mathrm{hol}(T)$ be distinct complex numbers. Then $\lambda_j:=\psi(z_j)$, $j=0,1,\ldots,\varkappa$, are distinct numbers in $\mathrm{hol}(S)$. Let $f\in\mathcal{M}(T)$ be arbitrary. Then, by the definition of $S(\lambda_j)$ and $\mathcal{M}(T)$, we have $S(\lambda_j)f=S(\lambda_0)f=:g$. Theorem A.1(a) implies that $\|g\|=\|S(\lambda_j)f\|\leqslant\|f\|$ and consequently $\|T(z_j)f\|\leqslant\|f\|$ for all $j=0,1,\ldots,\varkappa$.

Since $f\in\mathscr{M}(T)$ was arbitrary this proves that $T(z_j)|_{\mathscr{M}(T)}:\mathscr{M}(T)\to\mathscr{G}$ is a contraction for each $j=0,1,\dots,\varkappa$.

To prove equality (A.3), let $j\in\{0,1,\dots,\varkappa\}$ and let $f\in\mathscr{M}(T)$ be such that $f=T(z_j)^*T(z_j)f$. Then $T(z_j)f=S(\lambda_j)f=S(\lambda_0)f=g$ for all $j=0,1,\dots,\varkappa$, and $||g||=||f||$. By Theorem A.1 it follows that $g=S(\mu)f$ and $f=S(\mu)^*g$ for all $\mu\in\mathrm{hol}(S)$. Consequently $S(\mu)f=S(v)f$ and $f=S(v)^*S(v)f$ for all $\mu,v\in\mathrm{hol}(S)$, or equivalently, with $v=\psi(w)$ and $\mu=\psi(z)$, $f\in\ker(T(z)-T(w))$ and $f\in\ker(I-T(z)^*T(z))$ for all $z,w\in\mathrm{hol}(T)$. Thus, the left-hand side of (A.3) contained in $\mathscr{N}(T)$. The opposite inclusion is trivial, and hence (A.3) is proved. The statements about T^* are proved in a similar way using Theorem A.1(b). □

Remark A.3. It follows from the definition of $\mathscr{N}(T)$ that the condition $\mathscr{N}(T)=\{0\}$ is equivalent to the condition that for one (and equivalently for each) set of $\varkappa+1$ distinct complex numbers $z_0,\dots,z_\varkappa\in\mathrm{hol}(T)$ the operator $T(z_0)|_{\mathscr{M}(T;z_0,\dots,z_\varkappa)}$ is a strict contraction. For $\varkappa=0$ the condition $\mathscr{N}(T)=\{0\}$ is equivalent to the condition that at least one (and, equivalently all) of the operators $T(z)$, $z\in\mathrm{hol}(T)$, are strict contractions.

A part of the following corollary is a restatement of [10, Corollary, p. 356] in terms of the function T from Corollary A.2. In the proof of this corollary we use the first part of Lemma 5.2.

Corollary A.4. *Let $z_0,\dots,z_\varkappa\in\mathrm{hol}(T)$ and $w_0,\dots,w_\varkappa\in\mathrm{hol}(T)$ be two sets of $\varkappa+1$ distinct complex numbers. Then for $\mathscr{N}(T)$ defined in* (A.2) *we have*

$$\mathscr{N}(T)=\mathscr{M}(T;z_0,\dots,z_\varkappa)\bigcap\left(\bigcap_{j=0}^{\varkappa}\ker(I-T(w_j)^*T(z_j))\right)\tag{A.5}$$

and

$$\mathscr{N}(T)=\left(\bigcap_{z,v\in\mathrm{hol}(T)}\ker(T(z)-T(v))\right)\bigcap\left(\bigcap_{u,w\in\mathrm{hol}(T)}\ker(I-T(u)^*T(w))\right).\tag{A.6}$$

Proof. The matrix representations (5.4) and (5.6) imply that the set on the left-hand side of (A.5) is contained in the set on the right-hand side. To prove the opposite inclusion, let f be an arbitrary element of the intersection in (A.5). Then $f\in\mathscr{M}(T;z_0,\dots,z_\varkappa)$ and $T(w_j)^*T(z_0)f=f$ for all $j=0,\dots,\varkappa$. Therefore

$$T(z_0)f\in\mathscr{M}(T^*;w_0,\dots,w_\varkappa).$$

Corollary A.2, applied to both T^* and T, yields

$$||f|| = ||T(w_0)^* T(z_0)f|| \leqslant ||T(z_0)f|| \leqslant ||f||.$$

This implies $||T(z_0)f|| = ||f||$ and $||T(w_0)^* T(z_0)f|| = ||T(z_0)f||$. Using again Corollary A.2, we conclude that $T(z)f$ is independent of z and that $T(w)^* T(z_0)f$ is independent of w. Therefore

$$T(w)T(z)f = T(w_0)^* T(z_0)f = f \quad \text{for all } w, z \in \mathbb{C}^+.$$

Thus

$$f \in \mathscr{M}(T; z_0, \ldots, z_\varkappa) \cap \left(\bigcap_{u,w \in \mathbb{C}^+} \ker(I - T(u)^* T(w)) \right) \subset \mathscr{N}(T).$$

This proves (A.5). Equality (A.6) can be proved in the same way. □

The next theorem concerns the geometric interpretation of the maximum modulus principle. The isotropic part of a subspace $\mathscr{L}$ of a Krein space is denoted by $\mathscr{L}^\circ$.

Theorem A.5. *For arbitrary distinct complex numbers $z_0, \ldots, z_\varkappa$ in the set* $\mathrm{hol}(T)$ $(\subset \mathbb{C}^+)$, *the intersection $\bigcap_{j=0}^{\varkappa} \mathscr{L}(z_j)$ is a non-negative subspace of $\mathscr{K}$ and*

$$\mathscr{L}(z_0)^\circ \cap \left(\bigcap_{j=1}^{\varkappa} \mathscr{L}(z_j) \right) = \bigcap_{z \in \mathrm{hol}(T)} \mathscr{L}(z)^\circ \tag{A.7}$$

holds. Moreover, for any two sets of $\varkappa + 1$ distinct complex numbers $z_0, \ldots, z_\varkappa \in \mathrm{hol}(T)$ and $w_0, \ldots, w_\varkappa \in \mathrm{hol}(T)$ we have

$$\begin{aligned} \bigcap_{z \in \mathrm{hol}(T)} \mathscr{L}(z)^\circ &= \bigcap_{j=0}^{\varkappa} (\mathscr{L}(z_j) \cap \mathscr{L}(w_j)^{[\perp]}) \\ &= \bigcap_{u,v \in \mathrm{hol}(T)} (\mathscr{L}(u) \cap \mathscr{L}(v)^{[\perp]}). \end{aligned} \tag{A.8}$$

Proof. First note that

$$\bigcap_{j=0}^{\varkappa} \mathscr{L}(z_j) = G[T(z_0)|_{\mathscr{M}(T; z_0, \ldots, z_\varkappa)}]. \tag{A.9}$$

Indeed, if $\{f, g\}$ belongs to the intersection in (A.9), then $g = T(z_j)f$ for $j = 0, \ldots, \varkappa$. This clearly means that $\{f, g\}$ belongs to the graph in (A.9). Conversely, if $\{f, g\}$

belongs to the graph in (A.9), then $f\in\mathscr{M}(T;z_0,\dots,z_\varkappa)$ and $g=T(z_0)f$. Since, by the definition of the subspace $\mathscr{M}(T;z_0,\dots,z_\varkappa)$, the operators $T(z_j)$, $j=0,\dots,\varkappa$, coincide on $\mathscr{M}(T;z_0,\dots,z_\varkappa)$, we have $g=T(z_j)f$, that is, $\{f,g\}$ belongs to $\mathscr{L}(z_j)$ for each $j=0,\dots,\varkappa$. Corollary A.2 implies that $T(z_0)|_{\mathscr{M}(T;z_0,\dots,z_\varkappa)}$ is a contraction, and consequently $\bigcap_{j=0}^{\varkappa}\mathscr{L}(z_j)$ is a non-negative subspace of $\mathscr{K}$. With the unitary operator $V=T(z_0)|_{\mathscr{N}(T)}$ from Lemma 5.2(a) and (b) we have

$$\mathscr{L}(z_0)^\circ\bigcap\left(\bigcap_{j=1}^{\varkappa}\mathscr{L}(z_j)\right)=G[T(z_0)|_{\mathscr{N}(T)}]=G[V]. \tag{A.10}$$

Indeed, if $\{f,g\}$ belongs to the left-hand side of (A.10) then, by (A.9), $g=T(z_0)f$ with $f\in\mathscr{M}(T;z_0,\dots,z_\varkappa)$ and $\{f,T(z_0)f\}[\perp]\{u,T(z_0)u\}$ for all $u\in\mathscr{F}$, that is,

$$0=\langle f,u\rangle_{\mathscr{F}}-\langle T(z_0)f,T(z_0)u\rangle_{\mathscr{G}}=\langle (I-T(z_0)^*T(z_0))f,u\rangle_{\mathscr{F}},$$

and consequently $f\in\ker(I-T(z_0)^*T(z_0))$. By (A.3), $f\in\mathscr{N}(T)$. Thus the left-hand side of (A.10) is contained in the right-hand side. The proof of the opposite inclusion is similar.

Analogous to (A.9), we have

$$\bigcap_{j=0}^{\varkappa}\mathscr{L}(z_j)^{[\perp]}=G[T(z_0)^*|_{\mathscr{M}(T^*;z_0,\dots,z_\varkappa)}]. \tag{A.11}$$

To justify (A.7) and (A.8) it suffices to show

$$\bigcap_{j=0}^{\varkappa}\mathscr{L}(z_j)^\circ\subset G[V]\subset\bigcap_{z,v\in\mathrm{hol}(T)}(\mathscr{L}(z)\cap\mathscr{L}(v)^{[\perp]}). \tag{A.12}$$

Let $\{f,g\}$ belong to the first intersection in (A.12). Then, by (A.9) and (A.11), $\{f,g\}$ belongs to both

$$G[T(z_0)|_{\mathscr{M}(T;z_0,\dots,z_\varkappa)}]\quad\text{and}\quad G[T(z_0)^*|_{\mathscr{M}(T^*;z_0,\dots,z_\varkappa)}].$$

This means that $g=T(z_0)f$ and $f=T(z_0)^*g$. Consequently, $f\in\ker(I-T(z_0)^*T(z_0))$ and therefore, on account of (A.3), $f\in\mathscr{N}(T)$. According to Lemma 5.2, $g=T(z_0)f=Vf$, that is, $\{f,g\}\in G[V]$. Further, if $\{f,g\}\in G[V]$, then $g=Vf$. Therefore $g=T(z)f$ and $f=T(v)^*g$ for arbitrary $z,v\in\mathrm{hol}(T)$. Consequently $\{f,g\}\in\mathscr{L}(z)\cap\mathscr{L}(v)^{[\perp]}$, that is, $\{f,g\}$ belongs to the last intersection in (A.12). □

From the proof of Theorems 5.1 and A.5 we obtain the following list of equivalent formulations of ($\mathscr{U}6$). Note that items (a) and (d) contain each 4 statements.

Corollary A.6. *Let $\mathscr{U}$ be a (Q)-boundary coefficient satisfying $(\mathscr{U}1)$–$(\mathscr{U}5)$. The following statements are equivalent*:

(a) *For some* (*and then for any*) *set of distinct complex numbers $z_0, \dots, z_\varkappa$ in $\mathbb{C}^{\pm} \cap \mathrm{dom}(\mathscr{U})$, the matrix*

$$[\mathscr{U}(z_0)^* \quad \mathscr{U}(z_0^*)^* \quad \mathscr{U}(z_1^*)^* \quad \cdots \quad \mathscr{U}(z_\varkappa^*)^*]$$

has the maximal rank d

(b) $\mathrm{span}\{\mathrm{ran}\,\mathscr{U}(z)^* : z \in \mathrm{dom}(\mathscr{U})\} = \mathbb{C}^d$.

(c) $\bigcap_{z\in\mathrm{dom}(\mathscr{U})} \mathrm{ran}\,\mathscr{U}(z)^* = \{0\}$.

(d) *For some* (*and then for any*) *set of distinct numbers $z_0, \dots, z_\varkappa \in \mathbb{C}^{\pm} \cap \mathrm{dom}(\mathscr{U})$,*

$$\mathrm{ran}\,\mathscr{U}(z_0^*)^* \bigcap \left(\bigcap_{j=0}^{\varkappa} \mathrm{ran}\,\mathscr{U}(z_j)^* \right) = \{0\}.$$

Proof. These equivalences follow from the construction of the subspaces $\mathscr{R}(z)$ and $\mathscr{R}(z^*) = \mathscr{R}(z)^*$ in the proof of Theorem 5.1, the equivalences between equalities (5.13)–(5.15) and Theorem A.5. We leave the details to the reader. □

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