# A COUNTEREXAMPLE IN STURM-LIOUVILLE COMPLETENESS THEORY

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Abstract. We give an example of an indefinite weight Sturm-Liouville problem whose eigenfunctions form a Riesz basis under Dirichlet boundary conditions but not under anti-periodic boundary conditions.

## 1. Introduction

Completeness of eigenfunctions for Sturm-Liouville equations of the form

$$\ell(y) := -(py')' + qy = \lambda ry \tag{1.1}$$

on $[-1, 1]$ was already investigated for the case of positive weight functions $r$ in the 1830s. In a modern setting one seeks complete orthonormal bases in the weighted Hilbert space $L_{2,r}(-1, 1)$ with inner product given by

$$(y, z)_r = \int_{-1}^{1} ry\overline{z}. \tag{1.2}$$

For the purposes of this note, it will suffice to consider regular positive definite $\ell$ with self-adjoint boundary conditions and $r \in L_\infty(-1, 1)$.

Problems with indefinite $r$ were studied around the turn of the 20th century, and orthonormal bases of eigenfunctions were examined in a Hilbert space with the Dirichlet inner product generated by $\ell$. We remark that early work on all the above problems usually assumed Dirichlet boundary conditions $y(1) = y(-1) = 0$ for (1.1), but the theory now encompasses arbitrary self-adjoint boundary conditions.

Around 1970, "half-range" completeness for certain forward-backward equations led to renewed interest in the $L_{2,r}(-1, 1)$ setting, but now for equations (1.1) with indefinite $r$. An important step in the corresponding analysis of completeness is the production of a Riesz basis of eigenfunctions. In this case $L_{2,r}(-1, 1)$ is a Krein space, but the Riesz basis is taken with respect to the underlying Hilbert space inner product (with $r$ replaced by $|r|$ in (1.2)).

A standard reference in this area is [2] where Beals produces a basis (which turns out to be a Riesz basis) of eigenfunctions for a certain class of weight functions $r$. In particular, certain "extra" conditions on $r$ are imposed at the single turning point, i.e., where $r$ changes sign. As Beals points out, the analysis extends to finitely many turning points, but there is an example of Pyatkov [9] with infinitely many turning points and no Riesz basis of eigenfunctions. Also Volkmer [11] has shown that some "extra" condition is necessary for the Beals result to hold. Explicit counterexamples, based on Volkmer's ideas, can be found in [1] and [6]. We remark that all these works assume Dirichlet boundary conditions at 1 and $-1$.

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We are aware of two works on completeness in $L_{2,r}(-1,1)$ for problems of the form (1.1) with indefinite $r$ and general self-adjoint boundary conditions. One is [5] which involves "extra" conditions on $r$ at the turning points and at the boundary. The other [10] assumes "extra" conditions only at the turning point. We shall show by example that some "extra" condition is indeed required in general at the boundary so [10, Theorems 3.1, 3.2] are incorrect as stated. An intuitive explanation for our example can be given as follows. There is a turning point at $x = 0$ where $r$ satisfies an "extra" condition, and the Dirichlet problem has a Riesz basis of eigenfunctions. When the boundary conditions are not separated, however, the points 1 and $-1$ are linked and (1.1) can be thought of as living on a circle, with a second turning point at $\pm 1$. In our example, there is no "extra" condition on $r$ at $\pm 1$ and there is no Riesz basis of eigenfunctions for the antiperiodic problem.

We shall adapt Volkmer's strategy, which is to establish an inequality, to be denoted by $I$, see (2.3), satisfied by a class $\mathcal{V}$, see (2.4), of absolutely continuous functions. While such inequalities have been examined by various authors (and they are sometimes called HELP inequalities, cf. [3]), Volkmer emphasizes the role of the boundary conditions on the functions in $\mathcal{V}$. Indeed he shows that $I$ is false in the absence of boundary conditions, true independently of (1.1) if two Neumann type conditions are imposed, and true if one such condition is imposed and there is a Riesz basis of Dirichlet eigenfunctions.

In Section 2 we shall show that $I$ also holds if there is a Riesz basis of antiperiodic eigenfunctions and another boundary condition is imposed (in addition to Volkmer's, giving a smaller class $\mathcal{V}_0$). We then show that an example of (1.1) given in [1] generates functions in $\mathcal{V}_0$ and failing $I$, so it does not have a Riesz basis of antiperiodic eigenfunctions. In Section 3 we modify this example so as to have the properties in the abstract. This shows incidentally that $I$, even on the larger class $\mathcal{V}$, is not sufficient for a Riesz basis of antiperiodic eigenfunctions.

Finally, we note that mixed boundary conditions raise the possibility of double eigenvalues. If there are infinitely many such eigenvalues, then there could be some sets of (normalized) eigenfunctions which form Riesz basis and other sets which do not. In Section 4 we prove that this difficulty cannot occur. In fact all the eigenproblems considered here have only simple eigenvalues.

## 2. A necessary condition for a Riesz basis

Throughout, $w$ will denote an odd function in $L_\infty(-1,1)$. Here odd will mean that $w(-x) = -w(x)$ for all $x \neq 0$. We use $\|\cdot\|$ to denote the norm in $L_2(-1,1)$.

We define an operator $A$ in $L_2(-1,1)$ on

$$\operatorname{dom}(A) = \left\{ f \in W_2^2(-1,1) \ : \ f(1) + f(-1) = 0,\ f'(1) + f'(-1) = 0 \right\}$$

by

$$Af = -f''.$$

It is well known, see [12], that $A$ is self-adjoint with compact resolvent, and an easy calculation shows that the eigenvalues are all positive.

We shall consider the weighted eigenvalue problem

$$Af = \lambda w f, \tag{2.1}$$

which can be studied in a natural way as the standard eigenvalue problem for the positive self-adjoint operator $w^{-1}A$ in the weighted space $L_{2,w}(-1,1)$.

The first step in our analysis is the following modification of [11, Section 4]. We use the notation

$$K(h) = \frac{\left(\int_0^1 \frac{|h'|^2}{w}\right)^2}{\int_0^1 |h|^2 \int_0^1 \left|\left(\frac{h'}{w}\right)'\right|^2} \tag{2.2}$$

whenever the right hand side makes sense.

**Lemma 2.1.** *If $w$ is an odd essentially bounded function on $[-1,1]$ such that $xw(x) > 0$ for all nonzero $x$ and if there is a Riesz basis of eigenfunctions in $L_{2,w}(-1,1)$ for the eigenvalue problem (2.1), then there exists $C > 0$ such that*

$$K(h) \le C^2 \tag{2.3}$$

*for all nonzero $h$ such that*

$$h \in AC(0,1), \quad h'/w \in W_2^1(0,1), \quad (h'/w)(1) = 0, \tag{2.4}$$

*and*

$$h(1) = 0. \tag{2.5}$$

*Proof.* Since $A$ has a positive compact inverse in $L_2(-1,1)$, [11, Theorem 2.2] applies and gives $C$ such that

$$\int_{-1}^1 |w||f|^2 \le C\|A^{1/2}g\|\|A^{1/2}f\| \tag{2.6}$$

for all $f \in \mathrm{dom}(A^{1/2})$ and $g = A^{-1}(wf)$.

Let $h$ satisfy (2.4) and (2.5). Extend $h$ to an even function on $[-1,1]$ and put $f := -h'/w$. By (2.4) we have $f(-1) = f(1) = 0$ and therefore $f$ belongs to

$$\mathrm{dom}(A^{1/2}) = \big\{y \in W_2^1(-1,1) \,:\, y(-1) + y(1) = 0\big\},$$

see [7, §7]. Moreover,

$$\|A^{1/2}f\|^2 = \int_{-1}^1 |f'|^2. \tag{2.7}$$

Solving $-g'' = wf = -h'$ for $g \in \mathrm{dom}(A)$ we obtain

$$g(x) = \int_0^x h(t)dt, \quad x \in [-1,1]. \tag{2.8}$$

Since $g \in \mathrm{dom}(A)$ and $h(-1) = h(1) = 0$, integration by parts yields

$$\|A^{1/2}g\|^2 = \int_{-1}^1 Ag\overline{g} = \int_{-1}^1 wf\overline{g} = -\int_{-1}^1 h'\overline{g} = \int_{-1}^1 |h|^2. \tag{2.9}$$

Substituting (2.7) and (2.9) in (2.6) we obtain

$$\left(\int_{-1}^1 |w||f|^2\right)^2 \le C^2 \int_{-1}^1 |h|^2 \int_{-1}^1 |f'|^2,$$

and since all the integrands are even functions it follows that $K(h) \le C^2$ for all $h$ satisfying (2.4) and (2.5). □

The next step is to show that functions $w$ exist which fail the conclusion of Lemma 2.1. An analogous approach has already been used for Dirichlet boundary conditions in [1] and [6] via Volkmer's version of Lemma 2.1, see [11, Section 4], which includes conditions (2.4), but not (2.5).

**Theorem 2.2.** *There exists an odd essentially bounded function $w$ defined on $[-1,1]$ with $xw(x) > 0$ for all nonzero $x$, continuously differentiable near 1, and such that no set of eigenfunctions of the problem $Af = \lambda wf$ is a Riesz basis of $L_{2,w}(-1,1)$.*

*Proof.* In [1, Theorem 1] a condition (see [1, equation (5)]) for $w$ is given under which Lemma 2.1 fails for the set of functions $h$ that satisfy (2.4) (but not (2.5)). The proof of [1, Theorem 1] involves functions $f_n$ defined by $f_n(x) = -\int_x^1 g_n$, $x \in [0,1]$, and these evidently vanish at $x = 1$. The functions $f_n$ play the role of our $h$ and so our additional condition (2.5) is automatically incorporated in the construction in [1, Theorem 1]. Thus, [1, Theorem 1] gives a sufficient condition for $w$ under which no set of eigenfunctions of the problem $Af = \lambda wf$ is a Riesz basis of $L_{2,w}(-1,1)$. Therefore [1, Example 2] applies to our situation. □

**Remarks.** It follows from the construction in [1] that the function $w$ in Theorem 2.2 can in fact be selected to be arbitrary (essentially bounded) function on any interval $[\delta,1]$, $0 < \delta < 1$. In view of this, the set of functions $w$ as above is dense in $L_{2,w}$ (and indeed in other spaces; see [1, Theorem 2] for the Dirichlet case). In [11, Section 5] Volkmer gave a similar Dirichlet density result using (2.4) alone.

## 3. Boundary conditions and Riesz bases

In this section we modify the construction in Section 2 to produce the example promised in the abstract.

Let $w$ be the essentially bounded function whose existence is established in Theorem 2.2. Define $S$ on $L_{2,w}(-1,1)$ by

$$(Sf)(x) = f(x - \operatorname{sgn} x), \quad x \in [-1,1],$$

where for definiteness we take $\operatorname{sgn} 0 = 1$. Then $r := Sw$ is essentially bounded and odd. Since $S^2 f = f$ for all $f \in L_{2,w}(-1,1)$, we have $Sr = w$. We also define $Q$ on $L_{2,w}(-1,1)$ by

$$(Qf)(x) = (Sf)(x) \operatorname{sgn} x.$$

The operator $Q$ will be our main tool below and it enjoys the following properties.

**Lemma 3.1.** (a) *The operator $Q$ is a unitary isomorphism between the Hilbert spaces $L_{2,|w|}(-1,1)$ and $L_{2,|r|}(-1,1)$.*
(b) *If $f \in \operatorname{dom}(A)$, then $Qf \in \operatorname{dom}(A)$ and $(Qf)'' = Q(f'')$ a.e.*
(c) *If $f \in \operatorname{dom}(A)$, then $Q(rf) = wQf$ a.e.*

*Proof.* (a) If $f, g \in L_{2,|w|}(-1,1)$ we calculate

$$\begin{aligned}\int_{-1}^{1} |r| Qf \overline{Qg} &= \int_{-1}^{1} S(|w| f\overline{g}) \\ &= \int_0^1 |w| f\overline{g} + \int_{-1}^{0} |w| f\overline{g} \\ &= \int_{-1}^{1} |w| f\overline{g}\,.\end{aligned}$$

Thus $Q$ is an isometry. Now suppose that $g \in L_{2,|r|}(-1,1)$ and redefine $g$ on the null set $\{-1,0,1\}$ by $g(-1) = g(0) = g(1) = 0$. If we define $f = Qg$, then it is easily checked that $g = Qf$ and $f \in L_{2,|w|}(-1,1)$ so $Q$ maps $L_{2,|w|}(-1,1)$ onto $L_{2,|r|}(-1,1)$ and hence is unitary.

(b) The continuity of $Qf$ and $(Qf)'$ at zero follow from the boundary conditions on $f \in \mathrm{dom}(A)$, and conversely (interchanging the roles of $f$ and $Qf$). The remaining contentions are straightforward.

(c) Since $Sr = S^2 w = w$, the result follows from

$$Q(rf)(x) = S(rf)(x)\operatorname{sgn} x = (Sr)(x)(Sf)(x)\operatorname{sgn} x = w(x)(Qf)(x).$$

□

Now we are ready for the main result.

**Theorem 3.2.** *There exists odd essentially bounded $r$, with $x\,r(x) > 0$ for all nonzero $x \in (-1,1)$, continuously differentiable near 0, and such that*

(a) *appropriately normalized eigenfunctions of the Dirichlet problem*

$$-f'' = \lambda r f \tag{3.1}$$

*with $f(-1) = f(1) = 0$ form a Riesz basis of $L_{2,|r|}(-1,1)$, but*

(b) *no set of eigenfunctions of the corresponding antiperiodic problem*

$$Af = \lambda r f \tag{3.2}$$

*forms such a basis.*

*Proof.* (a) Since the function $w$ is continuously differentiable in the interval $[\delta, 1], 0 < \delta < 1$, it follows that the function $r$ is continuously differentiable on $[-1+\delta, 0) \cup (0, 1-\delta]$. Thus [5, Theorem 3.6] shows that the normalized eigenfunctions of the Dirichlet problem form a Riesz basis in $L_{2,|r|}(-1,1)$.

(b) Suppose there is a Riesz basis of eigenfunctions $f_n$ for (3.2) in $L_{2,|r|}(-1,1)$. By Lemma 3.1(a), the functions $Qf_n$ form a Riesz basis of $L_{2,|w|}(-1,1)$, and by Lemma 3.1(b) and (c), they satisfy

$$AQf_n = \lambda w Q f_n.$$

This evidently contradicts Theorem 2.2, and the proof is complete. □

**Remarks.** As in Section 2, various modifications are possible. By virtue of [5, Theorem 3.6], Dirichlet conditions can be replaced by any separated boundary conditions. One can also restate the results in the Naimark framework [8] as follows. Denote by $W$ the operator in $L_{2,w}(-1,1)$ defined on

$$\mathrm{dom}(W) = \Big\{ f \in L_{2,w}(-1,1) \,:\, f, f' \in AC[-1,1],\ (1/w)f'' \in L_{2,w}(-1,1),\\ f(1) + f(-1) = 0,\ f'(1) + f'(-1) = 0 \Big\}$$

by $Wf = -(1/w)f''$, and denote by $R$ the analogous operator with $w$ replaced by $r$. Then $QW = RQ$ so $W$ and $R$ are unitarily similar and therefore if either operator has a Riesz basis of eigenfunctions, so must the other.

## 4. Simplicity of the eigenvalues

As mentioned in Section 1, if infinitely many double eigenvalues $\lambda_n$ exist, then the question of whether the (normalized) eigenfunctions form a Riesz basis can be ambiguous. Specifically, if the angles $\alpha_n$ in $L_{2,|r|}(-1,1)$ between the two eigenfunctions at $\lambda_n$ tend to 0 as $\lambda_n \to \infty$, then the eigenfunctions do not form a Riesz basis, even though other choices (for which $\alpha_n$ do not tend to 0) could form a Riesz basis. We mention the related (abstract) example in [4] where angles between pairs of eigenvectors must tend to 0, preventing existence of a Riesz basis.

Below we prove that all the eigenvalues of the problems considered earlier are in fact simple.

**Lemma 4.1.** *Let $v : [0,1] \to \mathbb{R}$ be a positive essentially bounded measurable function. Then the unique solutions of the initial value problems*

$$-y'' = vy, \qquad y(0) = 0, \quad y'(0) = 1, \tag{4.1}$$

$$z'' = vz, \qquad z(0) = 0, \quad z'(0) = 1, \tag{4.2}$$

*satisfy $z(t) > y(t)$ for all $t \in (0,1]$.*

*Proof.* We write

$$s = z + y \quad \text{and} \quad d = z - y$$

for the sum and difference of $z$ and $y$. We note that

$$s(0) = d(0) = d'(0) = 0 \quad \text{and} \quad s'(0) = 2.$$

Thus $s(t) > 0$ for sufficiently small $t > 0$.

Arguing by contradiction we assume that there is $b \leq 1$ so that $s(t) > 0$ for $0 < t < b$, but $s(b) = 0$. Then

$$d'(t) = \int_0^t vs > 0,$$

so

$$d(t) = \int_0^t d' > 0, \tag{4.3}$$

for all $t \in (0,b]$. Moreover

$$s'(t) = 2 + \int_0^t vd > 0,$$

so

$$s(b) = \int_0^b s' > 0.$$

This the desired contradiction. Thus $s > 0$, and then from (4.3) also $d > 0$, on $(0,1]$. □

**Proposition 4.2.** *Let $r$ be an odd, essentially bounded, and measurable function defined on $[-1,1]$ such that $xr(x) > 0$ for all nonzero $x \in [-1,1]$. Then the eigenvalues of the problems (3.1) and (3.2) in Theorem 3.2 are real, symmetric with respect to 0 and simple.*

*Proof.* It is standard that the eigenvalues of the Dirichlet problem (3.1) are real and simple. The fact that all the eigenvalues of (3.2) are real follows from [5, Propositions 1.1, 2.3 and 2.6]. To see that the eigenvalues of both problems are symmetric with respect to 0, replace $f(t)$ by $f(-t), t \in [-1,1]$, and $\lambda$ by $-\lambda$.

It remains to prove simplicity of the eigenvalues of (3.2). By the symmetry expressed above, it is sufficient to consider the positive eigenvalues, so let $\lambda$ be a positive double eigenvalue of (3.2). Then the space of all $f$ such that

$$-f'' = \lambda r f, \tag{4.4}$$

$$f(1) + f(-1) = 0, \tag{4.5}$$

$$f'(1) + f'(-1) = 0, \tag{4.6}$$

is two dimensional. Equivalently, all solutions of (4.4) satisfy (4.5) and (4.6).

Let $f = z$ be the unique solution of (4.4) such that $z(0) = 0$ and $z'(0) = 1$, and define $y(x) := -z(-x)$, $x \in [0,1]$. Then with $v = \lambda r$, $z$ is a solution of (4.1) and $y$ is a solution of (4.2). By Lemma 4.1 it follows that $z(1) > -z(-1)$, that is $z(1) + z(-1) > 0$. Thus $f = z$ satisfies (4.4) but not (4.5), and we have a contradiction. □

**Remark.** With the aid of different initial conditions in (4.1) and (4.2) of Lemma 4.1, similar reasoning shows that the eigenvalues of the periodic problem

$$-f'' = \lambda r f, \quad f(1) - f(-1) = 0, \quad f'(1) - f'(-1) = 0, \tag{4.7}$$

are also simple. In particular, all the eigenvalues of (3.2) and (4.7) are simple for the function $r(x) = \operatorname{sgn}(x)$, $x \in [-1,1]$. It is interesting to observe that all the nonzero eigenvalues of the corresponding definite problems (with $|r| = 1$ instead of $r$), viz.,

$$-f'' = \lambda f, \quad f(1) + f(-1) = 0, \quad f'(1) + f'(-1) = 0,$$
$$-f'' = \lambda f, \quad f(1) - f(-1) = 0, \quad f'(1) - f'(-1) = 0,$$

are double.

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