PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before May 31, 2014. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11747. *Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.* Determine all $n \in \mathbb{N}$ such that $\lfloor n/k \rfloor$ divides n for $1 \le k \le n$. Similarly, determine all $n \in \mathbb{N}$ such that $\lceil n/k \rceil$ divides n for $1 \le k \le n$.

11748. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Tudorel Lupu, Decebal High School, Constanţa, Romania.* Is there a sequence $a_1, a_2, \dots$ of positive real numbers such that $\sum_{k=1}^{\infty} \frac{1}{a_k}$ converges, and $\prod_{k=1}^{n} a_k < n^n$ for all n?

11749. *Proposed by Branko Ćurgus, Western Washington University, Bellingham, WA.* For $\mathbf{x} \in \mathbb{C}^n$ and $p > 0$, let $\|\mathbf{x}\|_p$ denote the standard p-norm on $\mathbb{C}^n$. Prove that the function $p \mapsto \|\mathbf{x}\|_p$ is a strictly decreasing convex function on $(0, \infty)$ if and only if $\mathbf{x}$ is not of the form $c\mathbf{e}_k$, where $\mathbf{e}_k$ denotes the vector with 1 in the kth position and 0 elsewhere.

11750. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.* Prove or disprove that for every integral domain D and every nonzero d in D, there exist infinitely many irreducible polynomials p in the ring $D[x]$ of polynomials in one variable over D such that $p(0) = d$. (A nonzero, nonunit element f of $D[x]$ is *irreducible* if g or h is a unit of $D[x]$ whenever $gh = f$.)

11751. *Proposed by Carol Kempiak, Aliso Niguel High School, Aliso Viejo, CA, and Bogdan Suceavă, California State University, Fullerton, CA.* In a triangle with angles of radian measure A, B, and C, prove that

$$\frac{\csc A + \csc B + \csc C}{2} \ge \frac{1}{\sin B + \sin C} + \frac{1}{\sin C + \sin A} + \frac{1}{\sin A + \sin B},$$

with equality if and only if the triangle is equilateral.

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11752. *Proposed by Ádám Besenyei, Eötvös Loránd University, Budapest, Hungary.* Let $x_1, \dots, x_n$ be nonnegative numbers, where $n \geq 4$, and let $x_{n+1} = x_1$. For $p \geq 1$, prove that

$$\sum_{k=1}^{n} (x_k + x_{k+1})^p \leq \sum_{k=1}^{n} x_k^p + \left(\sum_{k=1}^{n} x_k \right)^p .$$

11753. *Proposed by Prapanpong Pongsriiam, Silpakorn University, Nakhon Pathom, Thailand.* Let f be a continuous map from $[0, 1]$ to $\mathbb{R}$ that is differentiable on $(0, 1)$, with $f(0) = 0$ and $f(1) = 1$. Show that for each positive integer n there exist distinct numbers $c_1, \dots, c_n$ in $(0, 1)$ such that $\prod_{k=1}^{n} f'(c_k) = 1$.

SOLUTIONS

Equation $x_1 + x_2 + x_3 = x_1 x_2 x_3$ Is a Disguised Triangle

11626 [2012, 162]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* Let x_1, x_2, and x_3 be positive numbers such that $x_1 + x_2 + x_3 = x_1 x_2 x_3$. Treating indices modulo 3, prove that

$$\sum_{1}^{3} \frac{1}{\sqrt{x_k^2 + 1}} \leq \sum_{1}^{3} \frac{1}{x_k^2 + 1} + \sum_{1}^{3} \frac{1}{\sqrt{(x_k^2 + 1)(x_{k+1}^2 + 1)}} \leq \frac{3}{2}. \tag{1}$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Write $A = \arctan(x_1)$, $B = \arctan(x_2)$, and $C = \arctan(x_3)$. The constraint $x_1 + x_2 + x_3 = x_1 x_2 x_3$ can be put in the form $\tan C = x_3 = (x_1 + x_2)/(x_1 x_2 - 1) = \tan[\pi - (A + B)]$. This implies that $A + B + C = \pi$, so there is a triangle $\triangle ABC$. It is an acute triangle, since x_1, x_2, x_3 are positive.

The second inequality of (1), after multiplying by 2 and rearranging, reduces to

$$(\cos A + \cos B + \cos C)^2 \leq \sin^2 A + \sin^2 B + \sin^2 C. \tag{2}$$

According to a known inequality (V. Thébault and L. Bankoff, Problem E 1272, this MONTHLY **67** (1960) 693–694): If A', B', and C' are the angles of a triangle, then

$$\left(\sin \frac{A'}{2} + \sin \frac{B'}{2} + \sin \frac{C'}{2} \right)^2 \leq \cos^2 \frac{A'}{2} + \cos^2 \frac{B'}{2} + \cos^2 \frac{C'}{2}. \tag{3}$$

Since $\triangle ABC$ is an acute triangle, (3) can be applied to it with $A' = \pi - 2A$, $B' = \pi - 2B$, $C' = \pi - 2C$ to conclude that (2) holds for $\triangle ABC$. This proves the second inequality of (1).

The first inequality of (1) is equivalent, after multiplying by 2 and rearranging, to the inequality

$$\sin^2 A + \sin^2 B + \sin^2 C \leq 2 + (\cos A + \cos B + \cos C - 1)^2. \tag{4}$$

Let a, b, and c denote the side lengths BC, CA, and AB, let R denote the circumradius, r the inradius, and s the semiperimeter of $\triangle ABC$. With this notation,

$$\begin{aligned}\cos A + \cos B + \cos C - 1 &= 4 \sin \frac{A}{2}\ \sin \frac{B}{2}\ \sin \frac{C}{2} \\ &= 4\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}} \\ &= 4\frac{(s-a)(s-b)(s-c)}{abc} = \frac{sr^2}{srR} = \frac{r}{R},\end{aligned}$$

so that inequality (4) is equivalent to

$$a^2 + b^2 + c^2 \le 8R^2 + 4r^2. \tag{5}$$

Using the fact that $ab + bc + ac = s^2 + r^2 + 4rR$, it can be seen that (5) is equivalent to $s^2 \le 3r^2 + 4rR + 4R^2$, which is the Gerretsen inequality (J. C. Gerretsen, "Ongelij Kheden in the Driehoek," *Nieuw Tijdschr. Wisk.* **41** (1953) 17—which unravels to the simple fact that the square of the distance between the incenter and the orthocenter is nonnegative). This proves the first inequality of (1).

Also solved by G. Apostolopoulos (Greece), M. Bataille (France), M. Can, C. Curtis, O. Geupel (Germany), E. A. Herman, B. Karaivanov, O. P. Lossers (Netherlands), P. Perfetti (Italy), C. R. Pranesachar (India), N. C. Singer, R. Stong, T. Viteam (Germany), Z. Vörös (Hungary), J. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposer.

A Circumradius Inequality

11630 [2012, 247]. *Proposed by Constantin Mateescu, High School 'Zinca Golescu', Pitesti, Romania.* For triangle ABC, let H be the orthocenter, I the incenter, O the circumcenter, and R the circumradius. Let b and c be the lengths of the sides opposite B and C, respectively, and let l be the length of the line segment from A to BC along the angle bisector at A. Let α be the radian measure of angle BAC. Prove that

$$\frac{bc}{l} + \max\{b, c\} \le 4R \cos\left(\frac{\alpha}{4}\right),$$

with equality if and only if rays AH, AI, and AO divide angle BAC into four equal angles.

Solution I by John G. Heuver, Grand Prairie, AB, Canada.

Let β and γ be the radian measures of the angles at B and C, respectively. Without loss of generality, suppose $b \ge c$, and recall that $l = 2bc\cos(\alpha/2)/(b+c)$. The inequality becomes

$$\frac{bc}{l} + \max\{b, c\} = \frac{b+c}{2\cos(\alpha/2)} + b \le 4R\cos\frac{\alpha}{4}.$$

Assuming that $\triangle ABC$ is nondegenerate, and using $\cos(\alpha/2) > 0$, the inequality can be rewritten as

$$\frac{1}{2R}\left(b + c + 2b\cos\frac{\alpha}{2}\right) \le 4\cos\frac{\alpha}{4}\cos\frac{\alpha}{2}.$$

Since $b = 2R\sin\beta$ and $c = 2R\sin\gamma$, and putting $d = (b + c + 2b\cos(\alpha/2))/(2R)$, we have

$$d = \sin\beta + \sin\gamma + 2\sin\beta\cos\frac{\alpha}{2} = 2\sin\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\beta-\gamma}{2}\right) + 2\sin\beta\cos\frac{\alpha}{2}$$
$$= 2\cos\frac{\alpha}{2}\left(\cos\left(\frac{\beta-\gamma}{2}\right) + \cos\left(\frac{\alpha}{2}+\frac{\gamma}{2}-\frac{\beta}{2}\right)\right)$$
$$= 4\cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\cos\left(\frac{\beta}{2}-\frac{\gamma}{2}-\frac{\alpha}{4}\right) \le 4\cos\frac{\alpha}{4}\cos\frac{\alpha}{2}.$$

Equality holds whenever $\cos(\frac{\beta}{2}-\frac{\gamma}{2}-\frac{\alpha}{4}) = 1$. The inequality yields $\frac{\beta}{2}-\frac{\gamma}{2} \le \frac{\alpha}{4}$. By assumption $\angle B \ge \angle C$, hence $\angle BAH = \frac{\alpha}{2}+\frac{\gamma}{2}-\frac{\beta}{2}$, $\angle HAI = \frac{\beta}{2}-\frac{\gamma}{2}$, $\angle IAO = \frac{\beta}{2}-\frac{\gamma}{2}$, and $\angle OAC = \frac{\alpha}{2}+\frac{\gamma}{2}-\frac{\beta}{2}$, so equality of the four angles follows.

Solution II by Prithwijit De, HBCSE, Mumbai, India. Using the known results $l = 2bc\cos(\frac{\alpha}{2})/(b+c)$ and $\max\{b,c\} = \frac{1}{2}(b+c+|b-c|)$, we have

$$\frac{bc}{l} + \max\{b,c\} = \frac{b+c}{2\cos(\alpha/2)} + \frac{b+c+|b-c|}{2} = \frac{(b+c)(1+\cos(\alpha/2))}{2\cos(\alpha/2)} + \frac{|b-c|}{2}. \tag{1}$$

Now,

$$b+c = 2R(\sin B + \sin C) = 4R\cos\frac{\alpha}{2}\cos\frac{B-C}{2},$$
$$b-c = 2R(\sin B - \sin C) = 4R\sin\frac{B-C}{2}\sin\frac{\alpha}{2}.$$

Substituting these into (1) yields

$$\frac{bc}{l} + \max\{b,c\} = \frac{(b+c)(1+\cos\frac{\alpha}{2})}{2\cos(\alpha/2)} + \frac{1}{2}|b-c|$$
$$= 2R\left(\cos\frac{B-C}{2}\left(1+\cos\frac{\alpha}{2}\right) + \left|\sin\frac{B-C}{2}\right|\sin\frac{\alpha}{2}\right)$$
$$= 4R\cos\frac{\alpha}{4}\left(\cos\frac{B-C}{2}\cos\frac{\alpha}{4} + \left|\sin\frac{B-C}{2}\right|\sin\frac{\alpha}{4}\right). \tag{2}$$

Let $\frac{1}{2}(B-C) = \theta$. Applying the Cauchy–Schwarz inequality to (2) yields

$$\cos\theta\cos\frac{\alpha}{4} + |\sin\theta|\sin\frac{\alpha}{4} \le \sqrt{\left(\cos^2\theta + \sin^2\theta\right)\left(\cos^2\frac{\alpha}{4} + \sin^2\frac{\alpha}{4}\right)} = 1.$$

Thus

$$\frac{bc}{l} + \max\{b,c\} \le 4R\cos\frac{\alpha}{4}.$$

Equality occurs if and only if equality holds in the Cauchy–Schwarz inequality. That is, it holds if and only if

$$\frac{\cos\frac{\beta-\gamma}{2}}{\cos\frac{\alpha}{4}} = \frac{|\sin\frac{\beta-\gamma}{2}|}{\sin\frac{\alpha}{4}},$$

which is equivalent to $\tan(\alpha/4) = |\tan((\beta-\gamma)/2)|$.

Now the tangent function is increasing on $[0, \pi/2)$, and $|\frac{\beta-\gamma}{2}| < \frac{\beta+\gamma}{2} < \frac{\pi-\alpha}{2}$, so $\frac{\alpha}{4} < \pi/2$ and $|\frac{\beta-\gamma}{2}| < \pi/2$. Therefore, $\frac{\alpha}{4} = \frac{\beta-\gamma}{2}$ if $\beta \geq \gamma$, and $\frac{\alpha}{4} = \frac{\gamma-\beta}{2}$ if $\beta < \gamma$. In either case, $\alpha = 2|\beta - \gamma|$. Now $\angle HAI = \angle OAI = \frac{1}{2}|\beta - \gamma|$, and since $\angle BAH = \angle CAO$, we also have $\angle BAH = \angle CAO = \frac{1}{2}(\alpha - \angle HAI - \angle OAI) = \frac{1}{2}|\beta - \gamma|$. Thus the rays AH, AI, and AO divide angle BAC into four equal angles.

Also solved by G. Apostolopoulos (Greece), M. Bataille (France), D. Beckwith, E. Braune (Austria), R. Chapman (U. K.), M. Daher (Lebanon), P. P. Dályay (Hungary), D. Fleischman, V. V. García (Spain), O. Geupel (Germany), A. Habil (Syria), B. Karaivanov, O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Minkus, C. R. Pranesachar (India), M. A. Prasad (India), F. Richman, R. Stong, Z. Vörös (Hungary), J. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposer.

A Real Inequality

11632 [2012, 248]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Dan Schwarz, Bucharest, Romania.* Let n be a positive integer, and write a vector $\mathbf{x} \in \mathbb{R}^n$ as $(x_1, \ldots, x_n)$. For $\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, let

$$[\mathbf{x}, \mathbf{y}]_{\mathbf{a},\mathbf{b}} = \sum_{1 \leq i, j \leq n} x_i y_j \min(a_i, b_j).$$

Show that for $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^n$ with nonnegative entries,

$$[\mathbf{x}, \mathbf{x}]_{\mathbf{a},\mathbf{a}} \cdot [\mathbf{y}, \mathbf{z}]_{\mathbf{b},\mathbf{c}}^2 + [\mathbf{y}, \mathbf{y}]_{\mathbf{b},\mathbf{b}} \cdot [\mathbf{z}, \mathbf{x}]_{\mathbf{c},\mathbf{a}}^2 \leq [\mathbf{x}, \mathbf{x}]_{\mathbf{a},\mathbf{a}}^{1/2} \cdot [\mathbf{y}, \mathbf{y}]_{\mathbf{b},\mathbf{b}}^{1/2} \cdot [\mathbf{z}, \mathbf{z}]_{\mathbf{c},\mathbf{c}}$$
$$\cdot \left([\mathbf{x}, \mathbf{x}]_{\mathbf{a},\mathbf{a}}^{1/2} \cdot [\mathbf{y}, \mathbf{y}]_{\mathbf{b},\mathbf{b}}^{1/2} + [\mathbf{x}, \mathbf{y}]_{\mathbf{a},\mathbf{b}} \right).$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. We will use the following lemma.

Lemma. *Let E be a real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$ and its corresponding norm $\|\cdot\|$. For any $x, y, z, u, v \in E$ with $\|u\| = |v\| = 1$,*

(i) $\langle u, z\rangle^2 + \langle v, z\rangle^2 \leq \big(1 + |\langle u, v\rangle|\big)\|z\|^2$,

(ii) $\|x\|^2 \langle y, z\rangle^2 + \|y\|^2 \langle x, z\rangle^2 \leq \|x\|\,\|y\|\,\|z\|^2 \big(\|x\|\,\|y\| + |\langle x, y\rangle|\big)$. $\quad (*)$

Proof. (i) Let $t = \langle u, v\rangle$. The case $t = \pm 1$ corresponds to $v = \pm u$ and the inequality is simply the Cauchy–Schwarz inequality $\langle u, z\rangle^2 \leq \|z\|^2$. Next, suppose that $-1 < t < 1$, so that the vectors α and β defined by $\alpha = \frac{1}{\sqrt{2(1+t)}}(u + v)$ and $\beta = \frac{1}{\sqrt{2(1-t)}}(u - v)$ are orthogonal unit vectors and

$$(1 + t)\langle \alpha, z\rangle^2 + (1 - t)\langle \beta, z\rangle^2 = \frac{1}{2}\big(\langle u + v, z\rangle^2 + \langle u - v, z\rangle^2\big) = \langle u, z\rangle^2 + \langle v, z\rangle^2.$$

Using the Bessel inequality $\langle \alpha, z\rangle^2 + \langle \beta, z\rangle^2 \leq \|z\|^2$, we see that

$$\langle u, z\rangle^2 + \langle v, z\rangle^2 \leq \big(1 + |t|\big)\big(\langle \alpha, z\rangle^2 + \langle \beta, z\rangle^2\big) \leq \big(1 + |t|\big)\|z\|^2,$$

as claimed.

(ii) The inequality $(*)$ is easy if $x = 0$ or $y = 0$. Otherwise, it follows from (i) by writing $u = x/\|x\|$ and $v = y/\|y\|$. ■

Now return to our problem. For a nonnegative real number λ, write χ_λ for the characteristic function of the interval $[0, \lambda]$. Our vector space E will consist of the finite linear combinations of the functions $\{\chi_\lambda : \lambda \geq 0\}$, with inner product $\langle f, g\rangle =$

$\int_0^\infty f(t)g(t)\,dt$. For nonnegative $\mathbf{x}, \mathbf{a} \in \mathbb{R}^n$, let $f_{\mathbf{x},\mathbf{a}} = \sum_{k=1}^n x_k \chi_{a_k}$, and note that it is in E. Now $\langle \chi_\lambda, \chi_\mu \rangle = \min(\lambda, \mu)$, so $[\mathbf{x}, \mathbf{y}]_{\mathbf{a},\mathbf{b}} = \langle f_{\mathbf{x},\mathbf{a}}, f_{\mathbf{y},\mathbf{b}} \rangle$ and similarly for the other cases. We obtain the required inequality by substitution in $(*)$ of $f_{\mathbf{x},\mathbf{a}}$, $f_{\mathbf{y},\mathbf{b}}$, and $f_{\mathbf{z},\mathbf{c}}$ for x, y, and z, respectively, and noting that $\langle f_{\mathbf{x},\mathbf{a}}, f_{\mathbf{y},\mathbf{b}} \rangle = [\mathbf{x}, \mathbf{y}]_{\mathbf{a},\mathbf{b}} \geq 0$.

Also solved by B. Karaivanov, R. Stong, and the proposers.

This Inequality Needs Adjustment

11634 [2012, 248]. *Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade, Buzău, Romania.* Let $(x_1, \dots, x_n)$ be an n-tuple of positive numbers, and let $X = \sum_{k=1}^n x_k$. Let a and m be nonnegative numbers, and let b, c, d be positive. Suppose that $p \geq 1$ and $cX^p > d \max_{1 \leq k \leq n} x_k^p$. Show that

$$\sum_{k=1}^n \frac{aX + bx_k}{cX^p - dx_k^p} \geq \frac{(an+b)n^{mp}}{(cn^p - d)^m} X^{1-mp}.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. The published inequality is false, in general (see comment below). We prove the corrected version,

$$\sum_{k=1}^n \frac{aX + bx_k}{\left(cX^p - dx_k^p\right)^m} \geq \frac{(an+b)n^{mp}}{(cn^p - d)^m} X^{1-mp}.$$

This is easy for $m = 0$, so we assume $m > 0$. Consider the function f defined for $t \in \left[0, (c/d)^{1/p}\right)$ by

$$f(t) = \frac{a + bt}{(c - dt^p)^m}.$$

We have

$$f'(t) = \frac{b}{(c - dt^p)^m} + mdp \cdot (a + bt) \cdot t^{p-1} \cdot \frac{1}{(c - dt^p)^{m+1}},$$

and thus f' is increasing, so f is convex on $\left[0, (c/d)^{1/p}\right)$. Using the assumption $cX^p > d \max_{1 \leq k \leq n} x_k^p$, we see that $t_k = x_k/X$ belongs to $\left[0, (c/d)^{1/p}\right)$ for every k in $\{1, 2, \dots, n\}$. Therefore

$$\frac{1}{n} \sum_{k=1}^n f(t_k) \geq f\left(\frac{t_1 + \cdots + t_n}{n}\right) = f\left(\frac{1}{n}\right),$$

which is equivalent to the required (corrected) inequality.

Editorial comment. The missing m is the fault of the editors, not the proposers. Richard Stong notes as follows that the version without that m is not correct: In the case where $x_k = 1$ for all k, we compute $X = n$ and the requested inequality becomes

$$\frac{n(an+b)}{cn^p - d} \geq \frac{(an+b)n}{(cn^p - d)^m}.$$

This reduces to $(cn^p - d)^{m-1} \geq 1$, which need not hold.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), B. Karaivanov, O. P. Lossers (Netherlands), P. Perfetti (Italy), M. A. Prasad (India), R. Stong, T. Viteam (Uruguay), GCHQ Problem Solving Group (U. K.), and the proposers.

When Sums of Powers Determine Their Terms

11635 [2012, 344]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bârlad, Romania, and Nicuşor Minculete, "Dimitrie Cantemir" University, Braşov, Romania.*
(**a**) Let α and β be distinct nonzero real numbers. Let a, b, c, x, y, z be real, with $0 < a < b < c$ and $a \le x < y < z \le c$. Prove that if

$$x^\alpha + y^\alpha + z^\alpha = a^\alpha + b^\alpha + c^\alpha \text{ and } x^\beta + y^\beta + z^\beta = a^\beta + b^\beta + c^\beta,$$

then $x = a$, $y = b$, and $z = c$.
(**b**) Let $\alpha_1, \alpha_2, \alpha_3$ be distinct nonzero real numbers. Let $a_1, a_2, a_3, a_4, x_1, x_2, x_3, x_4$ be real, with $0 < a_1 < a_2 < a_3 < a_4$ and $a_1 \le x_1 < x_2 < x_3 < x_4 \le a_4$. If

$$\sum_{k=1}^{4} x_k^{\alpha_j} = \sum_{k=1}^{4} a_k^{\alpha_j}$$

for $1 \le j \le 3$, must a_k then equal x_k for $1 \le k \le 4$?

Solution by Grahame Bennett, Indiana University, Bloomington, IN.
(**a**) If $\alpha > 0$, then $z^\alpha \ge y^\alpha \ge x^\alpha$ and $c^\alpha \ge b^\alpha \ge a^\alpha$. We then have

$$\begin{aligned} z^\alpha + y^\alpha + x^\alpha &= c^\alpha + b^\alpha + a^\alpha, \\ z^\alpha + y^\alpha &\ge c^\alpha + b^\alpha \qquad (\text{since } x^\alpha \ge a^\alpha), \\ z^\alpha &\le c^\alpha. \end{aligned}$$

That is, the triple $(x^\alpha, y^\alpha, z^\alpha)$ is majorized by $(a^\alpha, b^\alpha, c^\alpha)$ in the sense of the Hardy–Littlewood–Pólya book, *Inequalities*. It follows from Theorem 108 of that book that either $(x, y, z) = (a, b, c)$ or

$$\varphi(x^\alpha) + \varphi(y^\alpha) + \varphi(z^\alpha) < \varphi(a^\alpha) + \varphi(b^\alpha) + \varphi(c^\alpha),$$

whenever φ is strictly convex. The latter alternative may be ruled out by taking

$$\varphi(t) = \begin{cases} t^{\beta/\alpha} & \text{if } \beta/\alpha > 1 \text{ or } \beta/\alpha < 0, \\ -t^{\beta/\alpha} & \text{if } 0 < \beta/\alpha < 1, \end{cases}$$

to deduce $x^\beta + y^\beta + z^\beta \ne z^\beta + b^\beta + c^\beta$. Therefore, $(x, y, z) = (a, b, c)$.

The case $\alpha < 0$ is similar. We then have $x^\alpha \ge y^\alpha \ge z^\alpha$ and $a^\alpha \ge b^\alpha \ge c^\alpha$. Also

$$\begin{aligned} x^\alpha + y^\alpha + z^\alpha &= a^\alpha + b^\alpha + c^\alpha, \\ x^\alpha + y^\alpha &\le a^\alpha + b^\alpha, \\ x^\alpha &\le a^\alpha, \end{aligned}$$

so again $(x^\alpha, y^\alpha, z^\alpha)$ is majorized by $(a^\alpha, b^\alpha, c^\alpha)$.

(**b**) The answer here is *no*. A counterexample is provided by noting that

$$1^p + 5^p + 8^p + 12^p = 2^p + 3^p + 10^p + 11^p$$

for $p = 1, 2$, and 3.

Editorial comment. Problems of a similar nature are considered from a more advanced point of view in G. Bennett, "p-free ℓ^p inequalities," this MONTHLY **117** (2010), 334–351. The tool of choice there is Steinig's version of Descartes' Rule of Signs.

Also solved by P. P. Dályay (Hungary), J.-P. Grivaux (France), Y. J. Ionin, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Stong, and the proposers. Part **(a)** only by O. Geupel (Germany).

A Point on a Diagonal of a Quadrilateral

11636 [2012, 344]. *Proposed by Mowaffaq Hajja, Yarmouk University, Irbid, Jordan.* Let $ABCD$ be a convex quadrilateral, and suppose there is a point M on the diagonal BD with the property that the perimeters of ABM and CBM are equal and the perimeters of ADM and CDM are equal. Prove that $|AB| = |CB|$ and $|AD| = |CD|$.

Solution by Hugo Caerols and Rely Pellicier, Adolfo Ibáñez University, Chile. The condition that the perimeters of ABM and CBM are equal implies that A and C lie on a certain ellipse with foci B and M. The condition that the perimeters of ADM and CDM are equal implies that A and C lie on a certain ellipse with foci D and M. Thus A and C are the two intersections of these ellipses. Now, line BMD is the major axis of both ellipses, so by symmetry in that line we obtain $|AB| = |BC|$ and $|AD| = |DC|$.

Also solved by G. Apostolopoulos (Greece), M. Bataille (France), D. Beckwith, C. Blatter (Switzerland), J. Cade, R. Chapman (U. K.), C. Curtis, M. Daher (Lebanon), P. P. Dályay (Hungary), S. Durbha, A. Ercan (Turkey), D. Fleischman, O. Geupel (Germany), D. Gove, J.-P. Grivaux (France), J. W. Grossman, E. A. Herman, D. Hetzel & E. Ionascu, J. G. Heuver (Canada), E. Ionascu, Y. J. Ionin, W. Janous (Austria), B. Karaivanov, Y. H. Kim, O. Kouba (Syria), P. T. Krasopoulos (Greece), J. H. Lindsey II, G. Lord, O. P. Lossers (Netherlands), C. Martin, M. D. Meyerson, J. Minkus, J. H. Nieto (Venezuela), V. Pambuccian, I. Pinelis, M. A. Prasad (India), U. Schneider (Switzerland), M. Slattery, J. H. Steelman, R. Stong, D. B. Tyler, T. Viteam (Uruguay), Z. Vörös (Hungary), J. Zacharias, GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

A Rearrangement Inequality

11638 [2012, 345]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let a, b, c be positive real numbers. Prove that

$$a^3 + b^3 + c^3 + 3 \geq 3\left((a^2b+1)(b^2c+1)(c^2a+1)\right)^{1/3}.$$

Solution by John Zacharias, Melbourne, FL. By the Rearrangement Inequality, we have $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$. After adding 3 to each side and applying the AM–GM inequality, we get

$$\begin{aligned} a^3 + b^3 + c^3 + 3 &\geq (a^2b+1) + (b^2c+1) + (c^2a+1) \\ &\geq 3((a^2b+1)(b^2c+1)(c^2a+1))^{1/3}. \end{aligned}$$

Editorial comment. The *Rearrangement Inequality* states: $x_ny_1 + \cdots + x_1y_n \leq x_{\sigma(1)}y_1 + \cdots + x_{\sigma(n)}y_n \leq x_1y_1 + \cdots + x_ny_n$ *for every choice of real numbers* $x_1 \leq \cdots \leq x_n$ *and* $y_1 \leq \cdots \leq y_n$ *and every permutation* σ *of* $\{1, \ldots, n\}$. See: G. H. Hardy, J. E. Littlewood & G. Pólya, *Inequalities* (2nd ed., 1952, Cambridge Univ. Press), Section 10.2, Theorem 368.

Also solved by 49 other readers and the proposer.