Outer Median Triangles

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The medians are special

A *median* of a triangle is a line segment that connects a vertex of the triangle to the midpoint of the opposite side. The three medians of a triangle interact nicely with each other to yield the following properties:

(a) The medians intersect in a point interior to the triangle, called the *centroid*, which divides each of the medians in the ratio 2 : 1.

(b) The medians form a new triangle, called the *median triangle*.

(c) The area of the median triangle is 3/4 of the area of the given triangle in which the medians were constructed.

(d) The median triangle of the median triangle is similar to the given triangle with the ratio of similarity 3/4.

When we say, as in (b), that "three line segments *form a triangle*" we mean that there exists a triangle whose sides have the same lengths as the line segments.

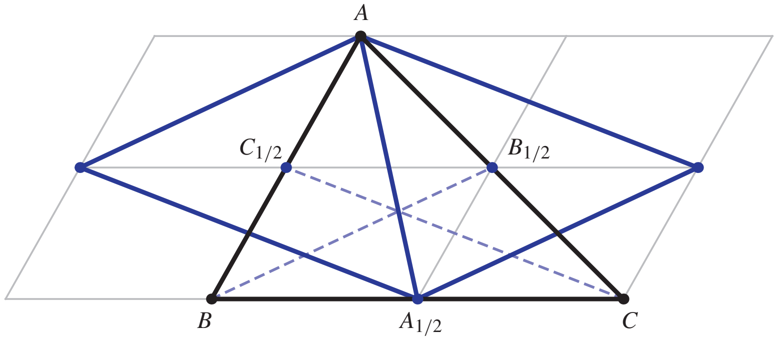

Figure 1 A "proof" of Properties (b) and (c)

Proving Property (a) is a common exercise. We provide "proofs without words" of Properties (b), (c), and (d) in FIGURES 1 and 2. Different proofs can be found in [**9**] and at [**17**]. Note that Property (b) fails for other equally important triples of cevians of a triangle; for example, as shown in [**2**], we cannot always speak about a triangle formed by bisectors or altitudes.

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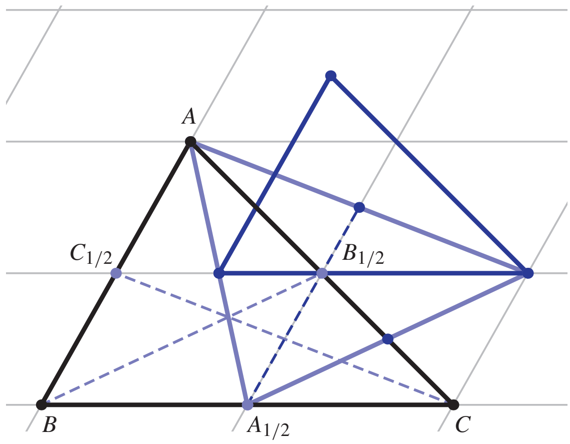

Figure 2 A "proof" of Property (d)

History Before introducing outer analogues of the medians and the median triangle, we reflect on the history of the above properties. Property (a) was proved by Archimedes of Syracuse, as Proposition 14 in Equilibrium of Planes, Book I; see [**8**], [**1**, Subsection 10.7.2], or [**16**, p. 86]. By the 19th century, Property (a) was a common proposition accompanying Euclid's Elements [**4, 14**]. Interestingly, in [**14**] and in several other books of this period, the term "median triangle" of ABC means, in our notation, the triangle $A_{1/2}B_{1/2}C_{1/2}$. Nowadays, this triangle is called the *medial triangle* of ABC.

The first usage of the current meaning of the median triangle that we found is in [**11**, Ch. XVI, §473]. However, in his 1887 paper [**13**], Mackay proves Property (b) in his §6 without explicitly stating it. He attributes his §6 to [**15**], but we could not find it there.

Furthermore, Property (c) appears as [**13**, §8(c)] and Property (d) as [**13**, §8(a)]. Mackay points out that [**13**, §8(a)] is proved in [**10**] as a solution to a problem proposed in [**12**]. Finally, Mackay believed that his [**13**, §8(c)] was new.

The medians are not alone

A median of a triangle is just a special *cevian*; a cevian is a line segment joining a vertex of a triangle to a point on the opposite side. Are there other triples of cevians from distinct vertices of a triangle that share the essential features of Properties (a), (b), (c), and (d)?

Some natural candidates for such cevians are suggested by the "median grid" already encountered in FIGURES 1 and 2. In FIGURES 3 and 4, we show more of this grid with the cevians that in some sense most resemble the medians. The labeling of the points on the line BC in FIGURE 4 originates from BC being considered as a number line with 0 at B and 1 at C. More precisely, for $\rho \in \mathbb{R}$, the point A_ρ is the point on the line BC that satisfies $\overrightarrow{BA_\rho} = \rho\,\overrightarrow{BC}$. The points on the lines AB and BC are labeled similarly.

As indicated in FIGURE 3, the cevians in the triple

$$\left(BB_{-1/2}, AA_{1/2}, CC_{3/2}\right) \tag{1}$$

are concurrent, with the understanding that three line segments are concurrent if the lines determined by them are concurrent. In addition to the triple of cevians in (1)

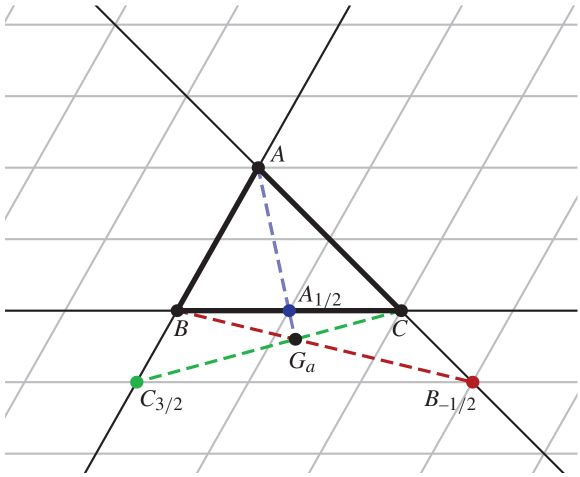

Figure 3 The "median grid"; the median from A, and one outer median from each B and C

we will consider two more triples of concurrent cevians that are symmetrically placed with respect to the other sides:

$$\left(CC_{-1/2}, BB_{1/2}, AA_{3/2}\right), \quad \left(AA_{-1/2}, CC_{1/2}, BB_{3/2}\right). \tag{2}$$

All three triples are shown in FIGURE 4.

That the triples in (1) and (2) are really concurrent follows from Ceva's theorem, which in our notation reads as:

CEVA'S THEOREM [**5**, p. 220]. *With $\rho, \sigma, \tau \in \mathbb{R}$, the cevians AA_ρ, CC_σ, BB_τ are concurrent if and only if*

$$\rho\sigma\tau - (1-\rho)(1-\sigma)(1-\tau) = 0. \tag{3}$$

Equation (3) defines a surface in $\rho\sigma\tau$-space; see FIGURE 10, below. We call it the *Ceva surface*. It will appear prominently in what follows.

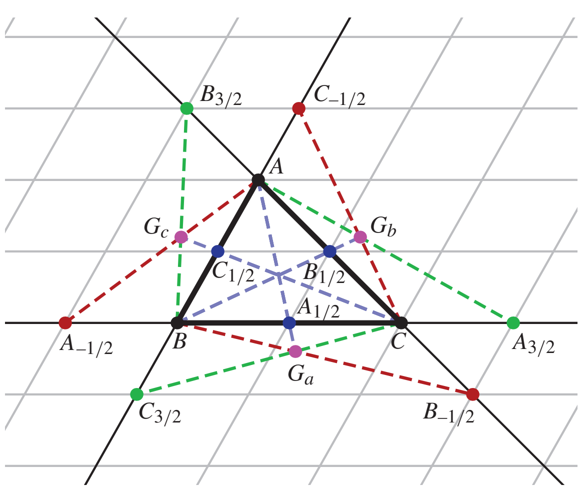

Figure 4 The "median grid"; three medians and six outer medians

Since the cevians $AA_{-1/2}$, $AA_{3/2}$, $BB_{-1/2}$, $BB_{3/2}$, $CC_{-1/2}$, $CC_{3/2}$ play the leading roles in this note and because of their proximity to the medians on the "median grid," we call them *outer medians*. Thus, for example, associated to vertex A we have one median, $AA_{1/2}$, and two outer medians, $AA_{-1/2}$ and $AA_{3/2}$. See FIGURE 4.

We find it quite remarkable that all four properties of the medians listed in the opening of this note hold for the three triples displayed in (1) and (2), each of which consists of a median and two outer medians originating from distinct vertices.

(A) The median and two outer medians in each of the triples in (1) and (2) are concurrent.

(B) The median and two outer medians in the triples in (1) and (2) form three triangles. We refer to these three triangles as *outer median triangles* of ABC; see FIGURE 5.

(C) The area of each outer median triangle of ABC is $5/4$ of the area of ABC.

(D) For each outer median triangle, one of its outer median triangles is similar to the original triangle ABC with the ratio of similarity $5/4$.

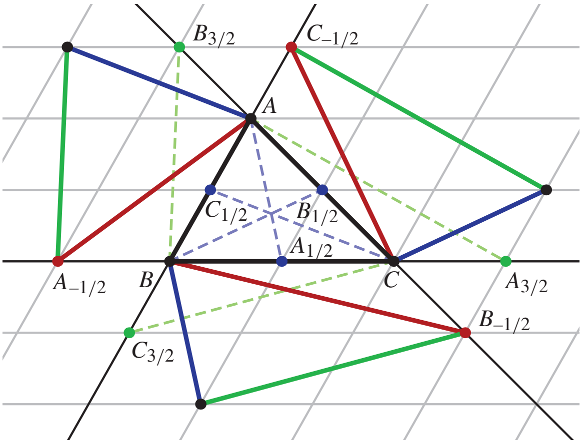

Figure 5 Three outer median triangles of ABC

As we have already mentioned, Property (A) follows from Ceva's theorem. FIGURES 6 and 7 offer "proofs without words" of Properties (B), (C), and (D).

We point out that the concurrency points G_a, G_b, G_c (see FIGURE 4) divide the corresponding outer medians in the ratio 2 : 3, that is, for example,

$$BG_a : G_aB_{-1/2} = CG_a : G_aC_{3/2} = 2 : 3.$$

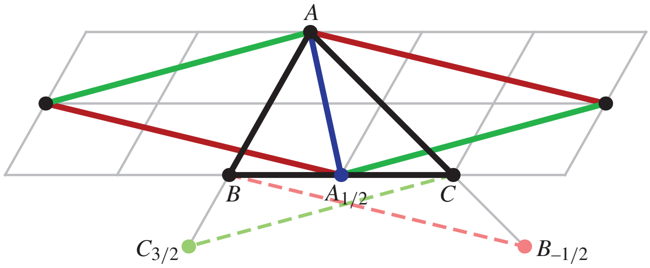

Figure 6 A "proof" of Properties (B) and (C)

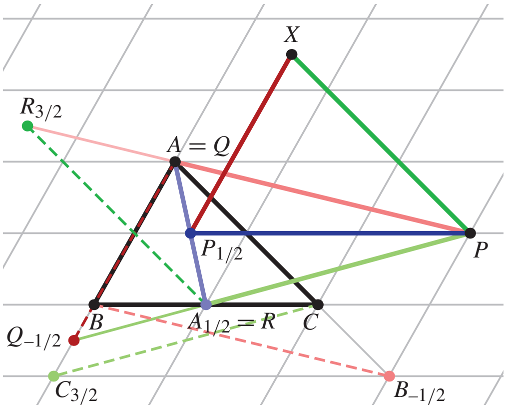

Figure 7 A "proof" of Property (D)

Similarly, it can be shown that the concurrency points divide the corresponding medians in the ratio 6 : 1, for example, $AG_a : A_{1/2}G_a = 6 : 1$. Computing these ratios is an exercise in vector algebra.

Are the medians and the outer medians alone?

We motivated our study of outer medians by their special position on the "median grid." However, the above four properties could very well hold for other triples of cevians. Is it then the case that the median triangle and the three outer median triangles are truly special?

The concurrency of three cevians is characterized by equation (3), which was used to justify the claim in Property (A). Next, akin to Property (B) and with no requirement, for the moment, that the cevians be concurrent, we look for a sufficient condition under which three cevians form a triangle.

Property (B) We first answer the following question: For which $\rho, \sigma, \tau \in \mathbb{R}$ does there exist a triangle with sides that are congruent and parallel to the cevians AA_ρ, BB_σ and CC_τ, independent of the triangle ABC in which they are constructed?

With $\mathbf{a} = \overrightarrow{BC}$, $\mathbf{b} = \overrightarrow{CA}$ and $\mathbf{c} = \overrightarrow{AB}$, we have

$$\overrightarrow{AA_\rho} = \mathbf{c} + \rho\, \mathbf{a}, \quad \overrightarrow{BB_\sigma} = \mathbf{a} + \sigma\, \mathbf{b} \quad \text{and} \quad \overrightarrow{CC_\tau} = \mathbf{b} + \tau\, \mathbf{c}.$$

Then, a necessary and sufficient condition for the existence of a triangle with sides that are congruent and parallel to the line segments AA_ρ, BB_σ, and CC_τ is that one of the following four vector equations is satisfied:

$$\overrightarrow{AA_\rho} \mathbin{\hat{\pm}} \overrightarrow{BB_\sigma} \pm \overrightarrow{CC_\tau} = \mathbf{0}. \tag{4}$$

We put a special sign $\hat{}$ above the first $\pm$ to be able to trace this sign in the calculations that follow. Substituting $\mathbf{c} = -\mathbf{a} - \mathbf{b}$ in (4), we get

$$\big(-1 \mathbin{\hat{\pm}} 1 + \rho \mp \tau\big)\mathbf{a} + \big(-1 \mathbin{\hat{\pm}} \sigma \pm 1 \mp \tau\big)\mathbf{b} = \mathbf{0}. \tag{5}$$

Using the linear independence of $\mathbf{a}$ and $\mathbf{b}$ and choosing both $+$ signs in (4), it follows from (5) that $\rho = \sigma = \tau$. Choosing the first sign in (4) to be $+$ and the second to be $-$, we get that $\rho = -\tau$, $\sigma = 2 - \tau$. Choosing the first sign in (4) to be $-$ and the second to be $+$, we get $\rho = 2 - \sigma$, $\tau = -\sigma$; and choosing both $-$ signs in (4), we get $\sigma = -\rho$, $\tau = 2 - \rho$. Thus, we have identified four sets of parameters (ρ, σ, τ) for which, independent of ABC, there exists a triangle, possibly degenerate, with sides that are congruent and parallel to the cevians AA_ρ, BB_σ, and CC_τ:

$$(\xi, \xi, \xi), \quad (2 - \xi, \xi, -\xi), \quad (-\xi, 2 - \xi, \xi), \quad (\xi, -\xi, 2 - \xi), \quad \xi \in \mathbb{R}. \tag{6}$$

The only concern here is that the cevians AA_ρ, BB_σ, and CC_τ might be parallel. However, the condition for the cevians to be parallel is easily established as follows. Since the vector $\overrightarrow{CC_\tau}$ is nonzero, we look for λ, $\mu \in \mathbb{R}$ such that

$$\mathbf{c} + \rho\,\mathbf{a} = \lambda(\mathbf{b} + \tau\,\mathbf{c}) \quad \text{and} \quad \mathbf{a} + \sigma\,\mathbf{b} = \mu(\mathbf{b} + \tau\,\mathbf{c}). \tag{7}$$

Substituting $\mathbf{c} = -\mathbf{a} - \mathbf{b}$ in (7) and using the linear independence of $\mathbf{a}$ and $\mathbf{b}$, we get from the first equation $\lambda = 1/(\tau - 1)$, $\rho = 1/(1 - \tau)$ and from the second equation $\mu = -1/\tau$, $\sigma = 1 - 1/\tau$. Hence, the line segments AA_ρ, BB_σ, and CC_τ are parallel if and only if

$$\rho = \frac{1}{1 - \xi}, \quad \sigma = 1 - \frac{1}{\xi}, \quad \tau = \xi, \quad \xi \in \mathbb{R} \setminus \{0, 1\}. \tag{8}$$

Thus, to avoid degeneracy of triangles with cevian sides corresponding to triples in (6) such as, for example, $(-\xi, 2 - \xi, \xi)$, we must exclude the values of the parameter ξ that solve $-\xi = 1/(1 - \xi)$. This, in turn, shows that the triples (ρ, σ, τ) for which there exists a non-degenerate triangle with sides that are congruent and parallel to the cevians AA_ρ, BB_σ, and CC_τ must belong to one of the following four sets:

$$\begin{aligned}
\mathbb{D} &= \left\{(\xi, \xi, \xi) : \xi \in \mathbb{R}\right\}, \\
\mathbb{E} &= \left\{(2 - \xi, \xi, -\xi) : \xi \in \mathbb{R} \backslash \{-\phi^{-1}, \phi\}\right\}, \\
\mathbb{F} &= \left\{(-\xi, 2 - \xi, \xi) : \xi \in \mathbb{R} \backslash \{-\phi^{-1}, \phi\}\right\}, \\
\mathbb{G} &= \left\{(\xi, -\xi, 2 - \xi) : \xi \in \mathbb{R} \backslash \{-\phi^{-1}, \phi\}\right\},
\end{aligned}$$

where $\phi = (1 + \sqrt{5})/2$ denotes the golden ratio.

The diagonal of the $\rho\sigma\tau$-space provides a geometric representation of the set $\mathbb{D}$. The other three sets are represented by straight lines with two points removed. All four lines are shown in FIGURE 10, together with the Ceva surface.

Generalized median and outer median triangles As we have just seen, the cevians associated with the triples in the sets $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ are guaranteed to form triangles; that is, they satisfy a property analogous to Property (B). The most prominent representatives of triangles originating from the sets $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ are the median and outer median triangles, which all correspond to the value $\xi = 1/2$. Therefore, for a fixed ξ, the triangle associated with the triple (ξ, ξ, ξ) in $\mathbb{D}$ we call *ξ-median triangle*, and the triangles associated with the corresponding triples in $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ we call *ξ-outer median triangles*. In FIGURES 8 and 9, we illustrate these triangles with $\xi = 1/\phi$, the reciprocal of the golden ratio.

Next, we explore whether the ξ-median and ξ-outer median triangles have properties analogous to (C) and (D).

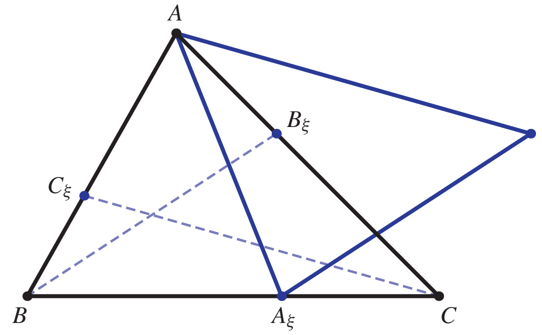

Figure 8 The ξ-median triangle with $\xi = \phi^{-1}$

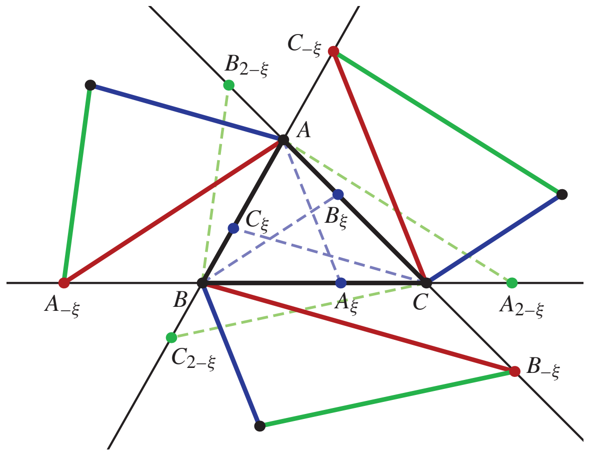

Figure 9 Three ξ-outer median triangles with $\xi = \phi^{-1}$

Property (C) First, we recall two classical formulas, which seem to be custom made for our task.

Heron's formula [**5**, 1.53], gives the square of the area of a triangle, Δ^2, in terms of its sides a, b, c:

$$\Delta^2 = s(s-a)(s-b)(s-c), \qquad \text{where} \qquad s = \frac{1}{2}(a+b+c).$$

Substituting s and simplifying yields

$$\Delta^2 = \frac{1}{16}\Big(2\big(a^2b^2 + b^2c^2 + c^2a^2\big) - \big(a^4 + b^4 + c^4\big)\Big).$$

Stewart's theorem [**6**, Section 1.2, Exercise 4], gives the square of the length of a cevian in terms of the squares of the sides of ABC:

$$(AA_\rho)^2 = \rho(\rho - 1)a^2 + \rho b^2 + (1-\rho)c^2.$$

Similar formulas hold for $\big(BB_\sigma\big)^2$ and $\big(CC_\tau\big)^2$. In matrix form, these three equations are:

$$\begin{bmatrix} (CC_\tau)^2 \\ (BB_\sigma)^2 \\ (AA_\rho)^2 \end{bmatrix} = \begin{bmatrix} \tau & 1-\tau & \tau(\tau-1) \\ 1-\sigma & \sigma(\sigma-1) & \sigma \\ \rho(\rho-1) & \rho & 1-\rho \end{bmatrix} \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix}. \tag{9}$$

We denote the 3×3 matrix in (9) by $M(\rho, \sigma, \tau)$. The idea of using this matrix is due to Griffiths [**7**]. It was further explored in [**3**, Section 3].

Now it is clear how to proceed to verify the property analogous to (C): use triples from the sets $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ to get expressions for the squares of the corresponding cevians, substitute these expressions in Heron's formula, and simplify. However, this involves simplifying an expression with 36 additive terms, quite a laborious task for a human but a perfect challenge for a computer algebra system like *Mathematica*. We first define Heron's formula as a *Mathematica* function (we call it `HeronS`) operating on the *triples* of squares of the sides of a triangle and producing the square of the area:

```
In[1]:= HeronS[ {x_, y_, z_}] := 1/16 (2 (x*y + y*z + z*x) - (x^2+y^2+z^2))
```

Next, we define in *Mathematica* the matrix function `M` as in (9):

```
In[2]:= M[{ρ_, σ_, τ_}] := { { τ, 1−τ, −(1−τ)∗τ },
                            { 1−σ, −(1−σ)∗σ, σ },
                            { −(1−ρ)∗ρ, ρ, 1−ρ } }
```

To verify the property analogous to (C) for ξ-median triangles, we put the newly defined functions in action by calculating the ratio between the squares of the area of the ξ-median triangle and the original triangle. *Mathematica*'s answer is instantaneous:

```
In[3]:= Simplify[HeronS[M[{ξ,ξ,ξ}].{x,y,z}]/HeronS[{x,y,z}]]

Out[3]= (1-ξ+ξ^2)^2
```

This "proves" that the ratio of the areas depends only on ξ, and that the ratio is exactly $1 - \xi + \xi^2$. Further, for one of the ξ-outer median triangles, we have

```
In[4]:= Simplify[HeronS[M[{2-ξ,ξ,−ξ}].{x,y,z}]/HeronS[{x,y,z}]]

Out[4]= (1+ξ-ξ^2)^2
```

"proving" that the area of the triangle formed by the cevians $AA_{2-\xi}$, BB_ξ, $CC_{-\xi}$ is $|1 + \xi - \xi^2|$ of the area of the original triangle ABC. The other two ξ-outer median triangles yield the same ratio. In summary, *Mathematica* has confirmed that the ξ-median and the three ξ-outer median triangles all have the property analogous to (C).

Property (D) The verification of the property analogous to (D) is simpler. For a ξ-median triangle, following [**7**], we just need to calculate the square of the matrix $M(\xi, \xi, \xi)$, which turns out to be $\left(1 - \xi + \xi^2\right)^2 I$. This confirms that the ξ-median triangle of the ξ-median triangle is similar to the original triangle with the ratio of similarity $1 - \xi + \xi^2$.

Similarly, for a ξ-outer median triangle corresponding to a triple in $\mathbb{E}$, we calculate the square of the matrix $M(2 - \xi, \xi, -\xi)$, which turns out to be $\left(1 + \xi - \xi^2\right)^2 I$; this confirms that one of the ξ-outer median triangles of this ξ-outer median triangle is similar to the original triangle with the ratio of similarity $|1+\xi-\xi^2|$. In contrast, for a ξ-outer median triangle corresponding to a triple in $\mathbb{F}$, to get a triangle similar to the original triangle we need to calculate its ξ-outer median triangle corresponding to a triple in $\mathbb{G}$. This amounts to multiplying the matrices

$$M(\xi, -\xi, 2-\xi)M(-\xi, 2-\xi, \xi) = \left(1+\xi-\xi^2\right)^2 I.$$

Likewise, for a ξ-outer median triangle corresponding to a triple in $\mathbb{G}$, we calculate its ξ-outer median triangle corresponding to a triple in $\mathbb{F}$ and obtain the same result.

Concurrency comes to the rescue All these calculations indicate that, after all, the median and outer median triangles are facing stiff competition from their ξ-triangles generalizations. However, property (A) comes to the rescue of the median and outer median triangles at this point. We want the triples of cevians corresponding to the triples in $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ to be concurrent as well. So which of these triples satisfy Ceva's condition (3)? Or, geometrically, what is the intersection of the lines and the Ceva surface in FIGURE 10? First, we substitute $\rho=\sigma=\tau=\xi$ in (3), which yields $\xi^3-(1-\xi)^3=0$, whose only real solution is $\xi=1/2$. The corresponding cevians are the medians. To intersect $\mathbb{E}$ with the Ceva surface, we substitute $(2-\xi, \xi, -\xi)$ in (3), obtaining $-\xi^2(2-\xi)-(1+\xi)(\xi-1)(1-\xi)=0$, which is equivalent to $(\xi-\phi)(\xi+\phi^{-1})(2\xi-1)=0$. Since $\xi \notin \{\phi, -\phi^{-1}\}$, the only solution is $\xi=1/2$,

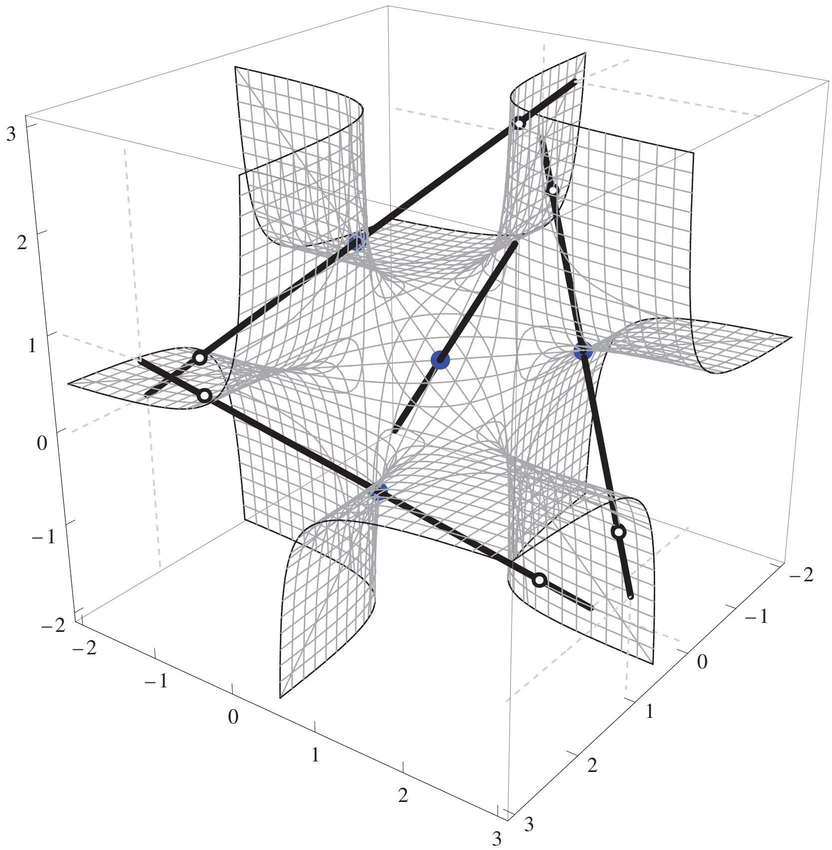

Figure 10 The sets $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$ and the Ceva surface

yielding the "outer median triple" $(3/2, 1/2, -1/2)$. Intersecting $\mathbb{F}$ with the Ceva surface gives $(-1/2, 3/2, 1/2)$ and intersecting $\mathbb{G}$ with the Ceva surface results in $(1/2, -1/2, 3/2)$. Consequently, the only triples in $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ which correspond to concurrent cevians are the "median triple" and the three "outer median triples."

There is only a slight weakness in our argument above. In identifying the sets $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$, we assumed that the triangles formed by the corresponding cevians have sides that are *parallel* to the cevians themselves. In [**3**], we proved that the only cevians AA_ρ, BB_σ, CC_τ that form triangles and with (ρ,σ,τ) not included in the sets $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ are parallel cevians, that is the cevians AA_ρ, BB_σ, and CC_τ, where ρ, σ, τ satisfy (8) with the additional restriction

$$\xi \in \left(-\phi, -\phi^{-1}\right) \cup \left(\phi^{-2}, \phi^{-1}\right) \cup \left(\phi, \phi^{2}\right).$$

As it turns out, the properties analogous to (C) and (D) do not hold for triangles formed by such cevians. In conclusion, indeed, along with the medians and the median triangle, the outer medians and their outer median triangles are unique in satisfying all four properties analogous to those from the beginning of our note.

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REFERENCES

1. A. K. T. Assis, *Archimedes, the Center of Gravity, and the First Law of Mechanics: The Law of the Lever*, 2nd ed., Apeiro, Montreal, 2010.
2. Á. Bényi, A Heron-type formula for the triangle, *Math. Gaz.* **87** (2003) 324–326.
3. Á. Bényi and B. Ćurgus, Ceva's triangle inequalities, *Math. Inequal. Appl.* **17** (2014) 591–609.
4. J. Casey, *A sequel to the first six books of the Elements of Euclid, containing an easy introduction to modern geometry, with numerous examples*, Hodges, Figgis, & Co., Dublin, 1888. https://archive.org/details/sequeltofirstsix00caserich
5. H. S. M. Coxeter, *Introduction to Geometry*, 2nd ed., John Wiley, New York, 1969.
6. H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, Washington, DC, 1967.
7. H. B. Griffiths, Iterated sequences of triangles, *Math. Gaz.* **89** (2005) 518–522.
8. T. L. Heath, *The Works of Archimedes*, Dover, New York, 2002. (For an edition from 1897, see https://archive.org/details/worksofarchimede029517mbp)
9. N. Hungerbühler, Proof without words: The triangle of medians, *Math. Mag.* **72** (1999) 142.
10. A. Jeunot, Solution to problem 1975, *Journal de Mathématiques Elémentaires*, 12^e Année, no. 3, (1887), p. 18. http://babel.hathitrust.org/cgi/pt?id=njp.32101051207957;view=1up;seq=24
11. R. A. Johnson, *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*, Houghton Mifflin, Boston, 1929. http://babel.hathitrust.org/cgi/pt?id=wu.89043163211
12. N. Lagrue, Problem 1975, Journal de Mathématiques Elémentaires, 12^e Année, no. 1, (1887), p. 8. http://babel.hathitrust.org/cgi/pt?id=njp.32101051207957;view=1up;seq=14
13. J. S. Mackay, Properties of the figure consisting of a triangle, and the squares described on its sides, *Proc. Edinburgh Math. Soc.* **6** (1887) 2–12. http://dx.doi.org/10.1017/S0013091500030030, figures at http://dx.doi.org/10.1017/S0013091500030017
14. J. S. Mackay, *The Elements of Euclid, books I. to VI., with deductions, appendices and historical notes*, Hunter, Rose & Co., Toronto, 1887. https://archive.org/details/elementsofeuwest00mack
15. J. McDowell, *Exercises on Euclid and Modern Geometry, for the use of schools, private students and junior university students*, Deighton, Bell, and Co., Cambridge, 1878. https://archive.org/details/exercisesoneucli00mcdo
16. J. Tropfke, *Geschichte der elementar-mathematik in systematischer darstellung*, Volume 1, Verlag von Veit & Comp., Leipzig, 1902. https://archive.org/details/geschichtederel00tropgoog
17. J. Wilson, EMAT 6680 Assignment 6, at http://jwilson.coe.uga.edu/emt668/Asmt6/EMT668.Assign6.html

Summary We define the notions of outer medians and outer median triangles. We show that outer median triangles enjoy similar properties to that of the median triangle.

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—Lee Sallows