

PARTIALLY FUNDAMENTALLY REDUCIBLE OPERATORS IN KREĬN SPACES

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ABSTRACT. A self-adjoint operator A in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot])$ is called partially fundamentally reducible if there exist a fundamental decomposition $\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-$ (which does not reduce A) and densely defined symmetric operators S_+ and S_- in the Hilbert spaces $(\mathcal{K}_+, [\cdot, \cdot])$ and $(\mathcal{K}_-, -[\cdot, \cdot])$, respectively, such that each S_+ and S_- has defect numbers $(1, 1)$ and the operator A is a self-adjoint extension of $S = S_+ \oplus (-S_-)$ in the Kreĭn space $(\mathcal{K}, [\cdot, \cdot])$. The operator A is interpreted as a coupling of operators S_+ and $-S_-$, relative to some boundary triples $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ and $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$. Sufficient conditions for a nonnegative partially fundamentally reducible operator A to be similar to a self-adjoint operator in a Hilbert space are given in terms of the Weyl functions m_+ and m_- of S_+ and S_- relative to the boundary triples $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ and $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$. Moreover, it is shown that under some asymptotic assumptions on m_+ and m_- all positive self-adjoint extensions of the operator S are similar to self-adjoint operators in a Hilbert space.

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1. INTRODUCTION

In this paper we study a class of self-adjoint operators in a Kreĭn space which turn out to be similar to self-adjoint operators in a Hilbert space. Recall that a Kreĭn space is a complex vector space \mathcal{K} with a sesquilinear form $[\cdot, \cdot]$ such that there exist subspaces \mathcal{K}_+ and \mathcal{K}_- of \mathcal{K} with the following three properties: $[\mathcal{K}_+, \mathcal{K}_-] = \{0\}$, $\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-$, a direct sum, and the spaces $(\mathcal{K}_+, [\cdot, \cdot])$ and $(\mathcal{K}_-, -[\cdot, \cdot])$ are Hilbert spaces. A pair of subspaces \mathcal{K}_+ and \mathcal{K}_- with the preceding three properties is

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called a *fundamental decomposition* of a Kreĭn space \mathcal{K} . The projections P_+ and P_- associated with the direct sum $\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-$ are called *fundamental projections* and the operator $J = P_+ - P_-$ is called a *fundamental symmetry* of a Kreĭn space. The space \mathcal{K} with the inner product $\langle x, y \rangle = [Jx, y]$, $x, y \in \mathcal{K}$, is a Hilbert space. Neither a fundamental decomposition nor a fundamental symmetry of a Kreĭn space is unique. However, the Hilbert space norms generated by different fundamental decompositions via the corresponding fundamental symmetries are equivalent. All topological notions in a Kreĭn space refer to the topology of the Hilbert space $(\mathcal{K}, \langle \cdot, \cdot \rangle)$. For the general theory of Kreĭn spaces and operators acting in them we refer to the monographs [5, 9].

Unlike the spectrum of a self-adjoint operator in a Hilbert space, the spectrum of a general self-adjoint operator in a Kreĭn space can be quite arbitrary. Therefore it is of interest to look for conditions that would guarantee good spectral properties of a self-adjoint operator in a Kreĭn space.

The ultimate task in this direction is to provide sufficient conditions for a self-adjoint operator in a Kreĭn space to be similar to a self-adjoint operator in a Hilbert space. A simple characterization of similarity is as follows. A self-adjoint operator A in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot])$ is similar to a self-adjoint operator in the Hilbert space $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ if and only if A is fundamentally reducible in $(\mathcal{K}, [\cdot, \cdot])$; where *fundamentally reducible* means that there exists a fundamental decomposition $\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-$ of $(\mathcal{K}, [\cdot, \cdot])$ such that A is the direct sum of its restrictions to $\mathcal{K}_+ \cap (\text{dom } A)$ and $\mathcal{K}_- \cap (\text{dom } A)$. Equivalently, an operator A is fundamentally reducible in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot])$ if and only if there exists a fundamental symmetry J on $(\mathcal{K}, [\cdot, \cdot])$ such that $J \text{dom}(A) \subseteq \text{dom}(A)$ and $JAx = AJx$ for all $x \in \text{dom}(A)$. The above characterization of similarity for bounded operators appears in [45, Theorem 1, Section 2] and in [41, Proposition 2.2], where an equivalent terminology of \mathcal{C} -symmetry is used.

Another kind of a self-adjoint operator in a Kreĭn space whose spectral properties resemble those of a self-adjoint operator in a Hilbert space is a nonnegative self-adjoint operator with a nonempty resolvent set. The spectrum of such an operator is real and, excluding arbitrary neighborhoods of 0 and ∞ , the operator A has a projector valued spectral function whose properties resemble the properties of the spectral function of a self-adjoint operator in a Hilbert space; for details see [43] and Subsection 2.8 below for a short review. If the spectrum of A accumulates on both sides of 0 (∞), then 0 (∞ , respectively) is called a *critical point* of A . If the spectral function of A is bounded in a neighborhood of a critical point, then that critical point is said to be *regular*. Otherwise, it is said to be a *singular* critical point. The set of all singular critical points of A is denoted by $c_s(A)$. Here, by definition, $c_s(A) \subseteq \{0, \infty\}$. These concepts are closely related to the similarity of A to a self-adjoint operator in a Hilbert space: A nonnegative self-adjoint operator A in a Kreĭn space such that $\rho(A) \neq \emptyset$ is similar to a self-adjoint operator in a Hilbert space if and only if $\ker(A) = \ker(A^2)$ and $0, \infty \notin c_s(A)$.

Our first step in studying the similarity question is to introduce a new concept related to the fundamental reducibility in Kreĭn spaces. For the definition of defect numbers of symmetric operators which we use in the next definition and general Hilbert space theory see [1].

Definition 1.1. We say that a self-adjoint operator A in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot])$ is *partially fundamentally reducible* if there exists a fundamental decomposition

$\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$ of $(\mathcal{K}, [\cdot, \cdot])$ such that the subspaces

$$\mathcal{D}_+ = \{f \in \mathcal{K}_+ \cap (\text{dom } A) : Af \in \mathcal{K}_+\} \quad \text{and} \quad \mathcal{D}_- = \{f \in \mathcal{K}_- \cap (\text{dom } A) : Af \in \mathcal{K}_-\}$$

are dense in \mathcal{K}_+ and \mathcal{K}_- and the restrictions

$$S_+ = A|_{\mathcal{D}_+} \quad \text{and} \quad S_- = -A|_{\mathcal{D}_-} \quad (1.1)$$

are symmetric operators with defect numbers $(1, 1)$ in the Hilbert spaces $(\mathcal{K}_+, [\cdot, \cdot])$ and $(\mathcal{K}_-, -[\cdot, \cdot])$, respectively.

Our objective in this paper is to give sufficient conditions under which a non-negative partially fundamentally reducible operator in a Kreĭn space is similar to a self-adjoint operator in a Hilbert space.

To this end we will also use a boundary triple approach to extension theory developed in [36, 28, 17]; see Subsection 2.1 below for a brief review. We will apply this theory to the symmetric operators S_+ and S_- associated via (1.1) with a partially fundamentally reducible operator A . Specifically, let $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ be a boundary triple of the operator S_+^* , the adjoint of S_+ in the Hilbert space $(\mathcal{K}_+, [\cdot, \cdot])$, and let m_+ be the corresponding Weyl function. Then there exists a unique boundary triple $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$ for S_-^* such that the operator A is a coupling of S_+ and S_- relative to the boundary triples $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ and $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$. That is, $f \in \text{dom}(A)$ if and only if there exist $f_+ \in \text{dom}(S_+^*)$ and $f_- \in \text{dom}(S_-^*)$ such that

$$f = f_+ + f_- \quad \text{and} \quad \Gamma_0^+ f_+ = \Gamma_0^- f_-, \quad \Gamma_1^+ f_+ = -\Gamma_1^- f_-$$

(see Theorems 4.1 and 4.7 below). Let m_- be the Weyl function of S_- corresponding to the boundary triple $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$. The Weyl functions m_+ and m_- completely characterize the simple (non-self-adjoint) parts of the symmetric operators S_+ and S_- . Therefore, it is natural to look for conditions for the fundamental reducibility of A in terms of the local behavior of the associated Weyl functions m_+ and m_- at 0 and ∞ .

This approach was utilized, for example, in [32, 33] where the boundedness of the function

$$y \mapsto \frac{\text{Im } m_+(iy) + \text{Im } m_-(iy)}{m_+(iy) + m_-(-iy)}, \quad y > 0, \quad (1.2)$$

on $(0, 1)$ (on $(1, \infty)$) was proved to be necessary for $0 \notin c_s(A)$ ($\infty \notin c_s(A)$, respectively). Since we use these necessary conditions in an essential way, we introduce the following terminology. A pair of functions (m_+, m_-) is said to have *D_0 -property* (*D_∞ -property*) if the function in (1.2) is bounded on $(0, 1)$ (on $(1, \infty)$, respectively).

Additionally, in Definition 3.1, for a Nevanlinna function m we introduce B_0 and B_∞ -property. With these four properties, our main results are as follows. If A is a nonnegative partially fundamentally reducible operator and if associated Weyl functions m_+ and m_- have B_∞ -property, then $\infty \notin c_s(A)$ if and only if the pair (m_+, m_-) has D_∞ -property (Theorem 5.7). Analogously, if A is a nonnegative partially fundamentally reducible operator and if the associated Weyl functions m_+ and m_- have B_0 -property, then $0 \notin c_s(A)$ if and only if the pair (m_+, m_-) has D_0 -property and $\ker(A) = \ker(A^2)$ (Theorem 5.8). Together these two results give sufficient conditions for a nonnegative partially fundamentally reducible operator in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot])$ to be similar to a self-adjoint operator in a Hilbert space.

Furthermore, in Subsection 2.7 we define the asymptotic class \mathcal{A}_∞ of Nevanlinna functions which all satisfy B_∞ -property and pairs of which have D_∞ -property and

the analogous asymptotic class \mathcal{A}_0 for B_0 -property and D_0 -property. In Theorem 5.12 we prove that if the Weyl functions m_+ and m_- associated with the partially fundamentally reducible operator A both belong to $\mathcal{A}_\infty \cap \mathcal{A}_0$, then, not only A , but all the nonnegative self-adjoint extensions of $S_+ \oplus (-S_-)$ are similar to a self-adjoint operators in a Hilbert space.

Finally, in Section 6 we apply our results to indefinite Sturm-Liouville differential operators. In some cases our results lead to a new point of view at some results from [15, 16, 25, 26, 31, 32, 33, 34]. We also get some new results for the case of nonsymmetric coefficients and the case when A is a coupling of two differential operators of different order.

Since a self-adjoint operator in a Kreĭn space is similar to a self-adjoint operator in a Hilbert space if and only if it is fundamentally reducible, throughout the rest of the paper we will use the shorter phrase fundamentally reducible to relate to this property.

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1.1. Notation. We use the standard notation \mathbb{C} for the set of complex numbers and \mathbb{R} for the set of real numbers. By \mathbb{C}_+ we denote the set of all $z \in \mathbb{C}$ with positive imaginary part. Similarly, \mathbb{R}_+ (\mathbb{R}_-) stands for the set of all positive (negative, respectively) reals. For $z \in \mathbb{C}$, z^* , $\operatorname{Re} z$, $\operatorname{Im} z$ and $\operatorname{Arg} z$ denote the complex conjugate, real, imaginary part of z and the principal value of the argument of z with $\operatorname{Arg} z \in (-\pi, \pi]$, respectively.

For a noninteger real α and $z \in \mathbb{C} \setminus [0, +\infty)$ we designate the principal branch of z^α to be $|z|^\alpha \exp(i\alpha \operatorname{Arg}(z))$, where $\operatorname{Arg}(z) \in (-\pi, \pi)$.

All operators in this paper are closed densely defined linear operators. For such an operator T , we use the common notation $\rho(T)$, $\operatorname{dom}(T)$, $\operatorname{ran}(T)$ and $\ker(T)$ for the resolvent set, the domain, the range and the null-space, respectively, of T .

We use the symbols \pm and \mp in a very specific way. Each sentence in which one or both of these symbols appear should be read twice, the first time with the top symbols and the second time with the bottom symbols. A good example of this is the sentence containing (5.8) and (5.9) in Subsection 5.2. Reading this sentence the first time defines T_{++} and T_{+-} ; reading it the second time defines T_{--} and T_{-+} . The only exception to this rule is the symbol $\operatorname{SL}^\pm(2, \mathbb{R})$ which is the common symbol for one object: the group of all real matrices with determinant equal to -1 or 1 .

2. PRELIMINARIES

2.1. Weyl functions of symmetric operators. Let S be a closed densely defined symmetric operator in the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ and let $\widehat{\rho}(S)$ denotes the set of points of regular type of S , see [1]. The subspace

$$\mathfrak{N}_z = \mathcal{H} \ominus \operatorname{ran}(S - \bar{z}) = \ker(S^* - z), \quad z \in \widehat{\rho}(S)$$

is called the defect subspace of the operator S . The dimension $\dim(\mathfrak{N}_z)$ is constant on each of the open half-planes \mathbb{C}^+ and \mathbb{C}^- and is denoted by d_+ , for $z \in \mathbb{C}^+$ and d_- , for $z \in \mathbb{C}^-$. The numbers d_+ and d_- are called the upper and lower defect numbers of S . In this paper we assume that $d_+ = d_- = 1$.

Since the space $(\text{dom } S^*)/(\text{dom } S)$ is two dimensional there exist (unbounded) non-zero linear functionals $\Gamma_j : S^* \rightarrow \mathbb{C}$, $i, j \in \{0, 1\}$, such that

$$\ker(\Gamma_0) \cap \ker(\Gamma_1) = \text{dom}(S)$$

and the abstract Green's identity

$$\langle S^* f, g \rangle_{\mathcal{H}} - \langle f, S^* g \rangle_{\mathcal{H}} = \Gamma_0(g)^* \Gamma_1(f) - \Gamma_1(g)^* \Gamma_0(f), \quad (2.1)$$

holds for all $f, g \in \text{dom}(S^*)$, see [28, Section 3.1.4] for much more general setting. In [18] the triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$ is called the *boundary triple* of a symmetric operator S .

It follows from (2.1) that the extensions S_0, S_1 of S defined as restrictions of S^* to the domains

$$\text{dom}(S_0) := \ker(\Gamma_0) \quad \text{and} \quad \text{dom}(S_1) := \ker(\Gamma_1),$$

are self-adjoint linear operators in $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$. For all $z \in \rho(S_0)$ we have

$$\text{dom}(S^*) = (\text{dom } S_0) \dot{+} \mathfrak{N}_z \quad \text{direct sum in } \mathcal{H}. \quad (2.2)$$

Since \mathfrak{N}_z is one-dimensional and $\Gamma_0 \neq 0$, it follows that the restriction $\Gamma_0|_{\mathfrak{N}_z}$ of Γ_0 to \mathfrak{N}_z is a bijection between \mathfrak{N}_z and \mathbb{C} . Then

$$(\Gamma_0|_{\mathfrak{N}_z})^{-1} : \mathbb{C} \longrightarrow \mathfrak{N}_z \subset \mathcal{H}, \quad z \in \rho(S_0),$$

and we define the following two functions:

$$\psi : \mathbb{C} \setminus \mathbb{R} \longrightarrow \mathcal{H} \quad \text{and} \quad m : \mathbb{C} \setminus \mathbb{R} \longrightarrow \mathbb{C}$$

by

$$\psi(z) := (\Gamma_0|_{\mathfrak{N}_z})^{-1}(1) \quad \text{and} \quad m(z) := \Gamma_1(\Gamma_0|_{\mathfrak{N}_z})^{-1}(1), \quad z \in \rho(S_0). \quad (2.3)$$

Clearly,

$$m(z) = \Gamma_1 \psi(z). \quad (2.4)$$

The functions ψ and m are called the *Weyl solution* and the *Weyl function*, respectively, of the symmetric operator S relative to the boundary triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$. For a fixed $z \in \rho(S_0)$ the vector $f = \psi(z)$ is the solution of the boundary value problem

$$S^* f = z f, \quad \Gamma_0 f = 1, \quad f \in \text{dom}(S^*).$$

With $z, w \in \rho(S_0)$ substituting

$$f = \psi(z), \quad S^* f = z \psi(z) \quad g = \psi(w), \quad S^* g = w \psi(w)$$

in (2.1) and using $\Gamma_0 f = \Gamma_0 g = 1$, $m(z) = \Gamma_1 f$, $m(w) = \Gamma_1 g$ we get

$$m(z) - m(w)^* = (z - w^*) \langle \psi(z), \psi(w) \rangle_{\mathcal{H}} \quad \text{for all } z, w \in \rho(S_0). \quad (2.5)$$

With $w = z^*$ the identity (2.5) yields that the Weyl function m satisfies the symmetry condition

$$m(z^*) = m(z)^* \quad \text{for all } z \in \rho(S_0). \quad (2.6)$$

The identity (2.5) was used in [40] as a definition of the Q -function. It follows from (2.5) and (2.6) that m is a Nevanlinna function; for the definition and basic properties see Subsection 2.3 below.

Remark 2.1. It turns out (see [17, page 8]) that the Weyl solution can be used to evaluate $\Gamma_1(S_0 - z)^{-1}h$ for arbitrary $h \in \mathcal{H}$ and $z \in \rho(S_0)$:

$$\Gamma_1(S_0 - z)^{-1}h = \langle h, \psi(z^*) \rangle_{\mathcal{H}}. \quad (2.7)$$

Indeed, substituting in (2.1) $f = (S_0 - z)^{-1}h$, $g = \psi(z^*)$, and using $\Gamma_0 f = 0$, $\Gamma_0 \psi(z^*) = 1$, we obtain the equality

$$\langle (S_0 - z)f, \psi(z^*) \rangle_{\mathcal{H}} = (\Gamma_1 f)(\Gamma_0 \psi(z^*))^* - (\Gamma_0 f)(\Gamma_1 \psi(z^*))^* = \Gamma_1 f,$$

which proves (2.7).

Proposition 2.2. [1, 17] *For every $z \in \rho(S_0)$ the following equivalence hold:*

$$z \in \rho(S_1) \iff m(z) \neq 0$$

and the resolvent of S_1 can be found by the formula

$$(S_1 - z)^{-1}h = (S_0 - z)^{-1}h - \frac{\langle h, \psi(z^*) \rangle_{\mathcal{H}}}{m(z)} \psi(z)$$

for all $h \in \mathcal{H}$ and all $z \in \rho(S_0) \cap \rho(S_1)$.

Remark 2.3. Any two boundary triples $(\mathbb{C}, \Gamma_0, \Gamma_1)$ and $(\mathbb{C}, \widehat{\Gamma}_0, \widehat{\Gamma}_1)$ of the same symmetric operator S are related by

$$\begin{pmatrix} \widehat{\Gamma}_0 \\ \widehat{\Gamma}_1 \end{pmatrix} = W \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix},$$

where W is a complex 2×2 matrix satisfying $W^* J W = J$, with $J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. The condition $W^* J W = J$ is equivalent to W being a unitary matrix in the Kreĭn space $(\mathbb{C}^2, [\cdot, \cdot]_2)$, where $[\mathbf{x}, \mathbf{y}]_2 = \mathbf{y}^* J \mathbf{x}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^2$. A direct calculation shows that a unitary matrix W in $(\mathbb{C}^2, [\cdot, \cdot]_2)$ allows a factorization

$$W = e^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \quad \text{and} \quad \theta \in (-\pi, \pi].$$

Here $\text{SL}(2, \mathbb{R})$ is the special linear group of all real 2×2 matrices with determinant one.

The Weyl function m relative to the boundary triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$ and the Weyl function \widehat{m} relative to the boundary triple $(\mathbb{C}, \widehat{\Gamma}_0, \widehat{\Gamma}_1)$ are related by

$$\widehat{m}(z) = \frac{dm(z) + c}{bm(z) + a}, \quad z \in \rho(S_0) \cap \rho(S_1).$$

In particular, the boundary triple $(\mathbb{C}, -\Gamma_1, \Gamma_0)$ is said to be a *transpose* of the boundary triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$. The Weyl function m^\top of S^* relative to $(\mathbb{C}, -\Gamma_1, \Gamma_0)$ is given by $m^\top(z) = -1/m(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$.

2.2. Nonnegative operators in Hilbert spaces. Recall that a symmetric operator S in a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is called nonnegative if $\langle Sf, f \rangle_{\mathcal{H}} \geq 0$ for all $f \in \text{dom}(S)$. By a result of Friedrichs, every nonnegative symmetric operator S admits a nonnegative self-adjoint extension. Moreover, as was shown by Kreĭn [39], in the set $\text{Ext}_+(S)$ of all nonnegative self-adjoint extensions of a nonnegative symmetric operator S there are two extremal extensions S_F and S_K . The extensions S_F and S_K , which are called the Friedrichs extension and the Kreĭn extension,

respectively, are maximal and minimal in the following sense: for all $\tilde{S} \in \text{Ext}_+(S)$ and all $a > 0$ we have,

$$(S_F + a)^{-1} \leq (\tilde{S} + a)^{-1} \leq (S_K + a)^{-1}.$$

The extensions S_F and S_K can be characterized in terms of boundary triples and Weyl functions. If S is a symmetric nonnegative operator and $(\mathbb{C}, \Gamma_0, \Gamma_1)$ is a boundary triple for S^* , then the extension S_0 of S is also nonnegative if and only if the corresponding Weyl function m is holomorphic on \mathbb{R}_- . Moreover (see [17, Proposition 10]),

$$\begin{aligned} S_0 = S_F &\iff \lim_{x \downarrow -\infty} m(x) = -\infty. \\ S_0 = S_K &\iff \lim_{x \uparrow 0} m(x) = +\infty. \end{aligned} \quad (2.8)$$

2.3. Nevanlinna, Stieltjes and inverse Stieltjes functions. A complex function m is called a *Nevanlinna function* if m is holomorphic at least on $\mathbb{C} \setminus \mathbb{R}$ and satisfies the following two conditions

$$m(z^*) = m(z)^* \quad \text{and} \quad \text{Im } m(z) \geq 0, \quad \text{for all } z \in \mathbb{C}_+.$$

Equivalently, m is a Nevanlinna function if and only if there exist $a, b \in \mathbb{R}$, $b \geq 0$, and a nondecreasing function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$m(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t)$$

and

$$\int_{\mathbb{R}} \frac{d\sigma(t)}{1+t^2} < +\infty. \quad (2.9)$$

If, additionally, σ is normalized by

$$\sigma(t) = \frac{\sigma(t-0) + \sigma(t+0)}{2} \quad \text{and} \quad \sigma(0) = 0, \quad (2.10)$$

then it is uniquely determined by m . For these and other facts on Nevanlinna functions see [30] and [21, Chapter II].

We consider that a Nevanlinna function is defined on its *domain of holomorphy*. That is, the domain of a Nevanlinna function m coincides with the union of $\mathbb{C} \setminus \mathbb{R}$ and the set of all those real points to which m admits a holomorphic continuation.

A Nevanlinna function m is called a *Stieltjes* (an *inverse Stieltjes*) function if it is holomorphic on $\mathbb{C} \setminus [0, +\infty)$ and it takes nonnegative (nonpositive, respectively) values on \mathbb{R}_- .

The following proposition is a direct consequence of the definitions.

Proposition 2.4. *Let m be a Nevanlinna function. The following statements are equivalent.*

- (i) m is an inverse Stieltjes function.
- (ii) The function $z \mapsto -m(1/z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, is a Stieltjes function.
- (iii) The function $z \mapsto -1/m(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, is a Stieltjes function.

A Stieltjes function m admits the integral representation

$$m(z) = \gamma + \int_0^{+\infty} \frac{d\sigma(t)}{t-z} \quad (2.11)$$

with $\gamma \geq 0$ and with a non-decreasing function $\sigma(t)$, such that

$$\int_0^{+\infty} \frac{d\sigma(t)}{1+t} < +\infty. \quad (2.12)$$

Clearly, for every Stieltjes function m

$$\operatorname{Re} m(iy) \geq 0 \quad \text{for all } y \in \mathbb{R}_+.$$

2.4. Reproducing kernel Hilbert spaces. In this section we review the basic properties of reproducing kernel Hilbert spaces associated with Nevanlinna functions. A function m defined on $\mathbb{C} \setminus \mathbb{R}$, is a Nevanlinna function if and only if the kernel

$$K_m(z, w) := \frac{m(z) - m(w)^*}{z - w^*}, \quad z \neq w^*, \quad z, w \in \mathbb{C} \setminus \mathbb{R}, \quad (2.13)$$

is non-negative.

With the kernel $K_m(z, w)$ in (2.13) we associate the reproducing kernel Hilbert space $(\mathcal{H}(m), \langle \cdot, \cdot \rangle_{\mathcal{H}(m)})$ defined as follows:

- (a) the elements of $\mathcal{H}(m)$ are holomorphic functions defined on $\mathbb{C} \setminus \mathbb{R}$,
- (b) for every $w \in \mathbb{C} \setminus \mathbb{R}$ the function $z \mapsto K_m(z, w)$, $z \in \mathbb{C} \setminus \mathbb{R}$, belongs to $\mathcal{H}(m)$,
- (c) the set of all linear combinations of the functions in (b) is dense in the reproducing kernel Hilbert space $(\mathcal{H}(m), \langle \cdot, \cdot \rangle_{\mathcal{H}(m)})$,
- (d) for every $f \in \mathcal{H}(m)$ and every $w \in \mathbb{C} \setminus \mathbb{R}$ we have

$$\langle f(\cdot), K_m(\cdot, w) \rangle_{\mathcal{H}(m)} = f(w).$$

The following theorem, proved in [18, Proposition 5.3] assures that each Nevanlinna function is a Weyl function of a closed simple symmetric operator.

Theorem 2.5. *Let m be a Nevanlinna function. The operator S_m of multiplication by the independent variable in the reproducing kernel Hilbert space $\mathcal{H}(m)$ is a closed simple symmetric operator with defect numbers $(1, 1)$. The operator S_m is densely defined if and only if*

$$\lim_{y \uparrow +\infty} \frac{\operatorname{Im} m(iy)}{y} = 0 \quad \text{and} \quad \lim_{y \uparrow +\infty} y \operatorname{Im} m(iy) = +\infty.$$

In the graph notation, the adjoint of S_m is given by

$$\begin{aligned} S_m^* &= \overline{\operatorname{span}} \{ \{ K_m(\cdot, w), w^* K_m(\cdot, w) \} : w \in \mathbb{C} \setminus \mathbb{R} \} \\ &= \{ \{ f, g \} \in \mathcal{H}(m)^2 : g(z) - zf(z) = c_0 - c_1 m(z) \text{ for some } c_0, c_1 \in \mathbb{C} \}. \end{aligned}$$

The numbers $c_0, c_1 \in \mathbb{C}$ in the last displayed equality are uniquely determined by $f \in \operatorname{dom}(S_m^*)$ and $(\mathbb{C}, \Gamma_{m,0}, \Gamma_{m,1})$ defined by

$$\Gamma_{m,0}(f) := c_0, \quad \Gamma_{m,1}(f) := c_1, \quad f \in \operatorname{dom}(S_m^*),$$

is a boundary triple for S_m^* . The Weyl function of S_m relative to the boundary triple $(\mathbb{C}, \Gamma_{m,0}, \Gamma_{m,1})$ of S_m^* is m .

The boundary triple $(\mathbb{C}, \Gamma_{m,0}, \Gamma_{m,1})$ is called the *canonical boundary triple* for the operator S_m^* .

The following theorem is in some sense a converse of Theorem 2.5.

Theorem 2.6. *Let S be a closed simple symmetric operator in a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ with defect numbers $(1, 1)$, let $(\mathbb{C}, \Gamma_0, \Gamma_1)$ be a boundary triple for S and let m be the corresponding Weyl function. Then the operator $F_m : \mathcal{H} \rightarrow \mathcal{H}(m)$ defined by*

$$(F_m f)(z) = \langle f, \psi(z^*) \rangle_{\mathcal{H}}, \quad f \in \mathcal{H}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (2.14)$$

is an isometry between $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ and $(\mathcal{H}(m), \langle \cdot, \cdot \rangle_{\mathcal{H}(m)})$ and

$$F_m S = S_m F_m.$$

The next lemma is an easy consequence of the definitions.

Lemma 2.7. *Let m be a Nevanlinna function. Define the anti-linear mapping $W : \mathcal{H}(m) \rightarrow \mathcal{H}(m)$ by*

$$(Wf)(z) = f(z^*)^*, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad f \in \mathcal{H}(m).$$

Then W is a bijection and

$$\langle Wf, Wg \rangle = \langle f, g \rangle^* \quad \text{for all } f, g \in \mathcal{H}(m). \quad (2.15)$$

In particular, $\|Wf\|_{\mathcal{H}(m)} = \|f\|_{\mathcal{H}(m)}$ for all $f \in \mathcal{H}(m)$.

2.5. Nevanlinna functions and Möbius transformations. It is clear that a composition of two Nevanlinna functions is a Nevanlinna function. However, the reproducing kernel space corresponding to the composition can be significantly different from the reproducing kernel Hilbert space of the composed functions. In this subsection we show that the situation is different when a Nevanlinna function is composed with a linear fractional (or Möbius) transformations, which are itself Nevanlinna functions. Then the corresponding reproducing kernel Hilbert spaces are isomorphic.

It is straightforward to verify that the mapping

$$\mathrm{GL}(2, \mathbb{C}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \mu(z) = \frac{az + b}{cz + d} \quad (2.16)$$

is a group homomorphism from the matrix group $\mathrm{GL}(2, \mathbb{C})$ of all 2×2 complex matrices with nonzero determinant onto the Möbius group. The kernel of this homomorphism is the subgroup of all nonzero multiples of the identity matrix I_2 .

We already encountered the subgroup $\mathrm{SL}(2, \mathbb{R})$ of $\mathrm{GL}(2, \mathbb{C})$. Below we will encounter its another subgroup $\mathrm{SL}^{\pm}(2, \mathbb{R})$ of all 2×2 real matrices with determinant 1 or -1 . The following lemma is standard. A short proof based on the homomorphism in (2.16) is included for completeness.

Lemma 2.8. *For a Möbius transformation μ the condition $\mu(z^*) = \mu(z)^*$ is equivalent to*

$$\mu(z) = \frac{az + b}{cz + d} \quad \text{for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}^{\pm}(2, \mathbb{R}). \quad (2.17)$$

A Möbius transformation μ in (2.17) is a Nevanlinna function whenever $ad - bc > 0$; its opposite $-\mu$ is a Nevanlinna function whenever $ad - bc < 0$.

Proof. Let

$$\mu(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad \text{where } \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}).$$

Assume that $\mu(z^*) = \mu(z)^*$, that is $\mu(z^*)^* = \mu(z)$. Then the homomorphism in (2.16) implies

$$\delta \begin{pmatrix} a_1^* & b_1^* \\ c_1^* & d_1^* \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \text{for some } \delta \in \mathbb{C} \setminus \{0\}. \quad (2.18)$$

Calculating determinants of both sides yields,

$$\delta^2 = e^{2i\theta} \quad \text{where } \theta = \text{Arg}(a_1 d_1 - b_1 c_1).$$

Since δ in (2.18) is uniquely determined, we have

$$\delta = e^{i\theta} \quad \text{or} \quad \delta = -e^{i\theta}. \quad (2.19)$$

It follows from (2.18) and (2.19) that

$$\frac{\sqrt{\delta}}{\sqrt{|a_1 d_1 - b_1 c_1|}} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}^\pm(2, \mathbb{R})$$

and whether $ad - bc = 1$ or $ad - bc = -1$ depends on the choice of the root in (2.19). The converse is straightforward.

The last claim follows from the identity

$$\frac{\text{Im } \mu(z)}{\text{Im } z} = \frac{ad - bc}{|cz + d|^2}. \quad \square$$

For a Möbius transformation μ given in (2.17) a direct calculation confirms the following identity

$$\mu(z) - \mu(w^*) = \frac{ad - bc}{(cz + d)(cw^* + d)}(z - w^*) \quad \text{for all } z, w \in \mathbb{C} \setminus \mathbb{R}, \quad (2.20)$$

which will be used in the next proof.

Theorem 2.9. *Let μ_1 and μ_2 be Möbius transformations given by*

$$\mu_j(z) = \frac{a_j z + b_j}{c_j z + d_j} \quad \text{where } \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \text{SL}^\pm(2, \mathbb{R}), \quad j \in \{1, 2\}. \quad (2.21)$$

For $j \in \{1, 2\}$ set $\epsilon_j := a_j d_j - b_j c_j \in \{-1, 1\}$. Let m be a complex function defined on $\mathbb{C} \setminus \mathbb{R}$ and set

$$\widehat{m}(z) = \epsilon_1 \epsilon_2 \mu_2(m(\mu_1(z))), \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.22)$$

Then m is a Nevanlinna function if and only if \widehat{m} is a Nevanlinna function. If m is a Nevanlinna function the mapping $V : \mathcal{H}(\widehat{m}) \rightarrow \mathcal{H}(m)$ defined by

$$(Vf)(z) := \frac{c_1 \mu_1(z) - a_1}{c_2 m(\mu_1(z)) + d_2} f(\mu_1(z)), \quad f \in \mathcal{H}(m), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

is an isomorphism between the Hilbert spaces $\mathcal{H}(m)$ and $\mathcal{H}(\widehat{m})$.

Proof. First notice that the Möbius transformations in (2.21) have inverses which are also Möbius transformations of the same kind. For example

$$\mu_1^{-1}(z) = \frac{d_1 z - b_1}{-c_1 z + a_1} \quad \text{and} \quad \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \in \text{SL}^\pm(2, \mathbb{R}).$$

Now the first statement follows from Lemma 2.8 and the fact that a composition of Nevanlinna functions is a Nevanlinna function.

Assume that m is a Nevanlinna function. In the following identities we assume that $z, w \in \mathbb{C} \setminus \mathbb{R}$ and we use (2.20), first for μ_2 , then for μ_1^{-1} . For convenience we set $z_1 = \mu_1(z)$ and $w_1 = \mu_1(w)$ and calculate:

$$\begin{aligned} K_{\widehat{m}}(z, w) &= \epsilon_1 \epsilon_2 \frac{\mu_2(m(\mu_1(z))) - \mu_2(m(\mu_1(w^*)))}{z - w^*} \\ &= \frac{\epsilon_1 \epsilon_2 \epsilon_2}{(c_2 m(z_1) + d_2)(c_2 m(w_1)^* + d_2)} \frac{m(z_1) - m(w_1^*)}{\mu_1^{-1}(z_1) - \mu_1^{-1}(w_1^*)} \\ &= \frac{(-c_1 z_1 + a_1)(-c_1 w_1^* + a_1)}{(c_2 m(z_1) + d_2)(c_2 m(w_1)^* + d_2)} \frac{m(z_1) - m(w_1^*)}{z_1 - w_1^*} \\ &= \frac{(c_1 \mu_1(z) - a_1)(c_1 \mu_1(w)^* - a_1)}{(c_2 m(\mu_1(z)) + d_2)(c_2 m(\mu_1(w^*)) + d_2)} K_m(\mu_1(z), \mu_1(w)). \end{aligned}$$

Consequently, substituting w with $\mu_1^{-1}(w)$, we get

$$\frac{c_1 \mu_1(z) - a_1}{c_2 m(\mu_1(z)) + d_2} K_m(\mu_1(z), w) = \frac{c_2 m(w^*) + d_2}{c_1 w^* - a_1} K_{\widehat{m}}(z, \mu_1^{-1}(w)),$$

that is

$$V(K_m(\cdot, w)) = \frac{c_2 m(w^*) + d_2}{c_1 w^* - a_1} K_{\widehat{m}}(\cdot, \mu_1^{-1}(w)) \quad \text{for all } w \in \mathbb{C} \setminus \mathbb{R}. \quad (2.23)$$

Therefore, for arbitrary $v, w \in \mathbb{C} \setminus \mathbb{R}$ we have

$$\begin{aligned} &\langle VK_m(\cdot, v), VK_m(\cdot, w) \rangle_{\mathcal{H}(\widehat{m})} \\ &= \frac{c_2 m(v^*) + d_2}{c_1 v^* - a_1} \frac{c_2 m(w) + d_2}{c_1 w - a_1} \left\langle K_{\widehat{m}}(\cdot, \mu_1^{-1}(v)), K_{\widehat{m}}(\cdot, \mu_1^{-1}(w)) \right\rangle_{\mathcal{H}(\widehat{m})} \\ &= \frac{c_2 m(v^*) + d_2}{c_1 v^* - a_1} \frac{c_2 m(w) + d_2}{c_1 w - a_1} K_{\widehat{m}}(\mu_1^{-1}(w), \mu_1^{-1}(v)) \\ &= K_m(w, v) \\ &= \langle K_m(\cdot, v), K_m(\cdot, w) \rangle_{\mathcal{H}(m)}. \end{aligned}$$

Thus, for arbitrary $v, w \in \mathbb{C} \setminus \mathbb{R}$ we have

$$\langle VK_m(\cdot, v), VK_m(\cdot, w) \rangle_{\mathcal{H}(\widehat{m})} = \langle K_m(\cdot, v), K_m(\cdot, w) \rangle_{\mathcal{H}(m)}. \quad (2.24)$$

Set

$$\begin{aligned} \mathcal{L}(m) &:= \text{span}\{K_m(\cdot, w) : w \in \mathbb{C} \setminus \mathbb{R}\}, \\ \mathcal{L}(\widehat{m}) &:= \text{span}\{K_{\widehat{m}}(\cdot, w) : w \in \mathbb{C} \setminus \mathbb{R}\}. \end{aligned}$$

As both functions m and \widehat{m} are Nevanlinna functions, $\mathcal{L}(m)$ is dense in the Hilbert space $\mathcal{H}(m)$ and $\mathcal{L}(\widehat{m})$ is dense in the Hilbert space $\mathcal{H}(\widehat{m})$.

Since V is linear, it follows from (2.23) and (2.24) that the restriction $V|_{\mathcal{L}(m)}$ is a bijection and an isomorphism of pre-Hilbert spaces $\mathcal{L}(m)$ and $\mathcal{L}(\widehat{m})$. Denote by V_1 the extension by continuity of $V|_{\mathcal{L}(m)}$ to $\mathcal{H}(m)$. Then V_1 is an isomorphism between $\mathcal{H}(m)$ and $\mathcal{H}(\widehat{m})$. Let $g \in \mathcal{H}(\widehat{m})$ be arbitrary and let $f \in \mathcal{H}(m)$ be such that $V_1 f = g$. For $z \in \mathbb{C} \setminus \mathbb{R}$ we calculate:

$$\begin{aligned} g(z) &= \langle g(\cdot), K_{\widehat{m}}(\cdot, z) \rangle_{\mathcal{H}(\widehat{m})} \\ &= \left\langle (V_1 f)(\cdot), \frac{c_1 \mu_1(z^*) - a_1}{c_2 m(\mu_1(z^*)) + d_2} VK_m(\cdot, \mu_1(z)) \right\rangle_{\mathcal{H}(\widehat{m})} \end{aligned}$$

$$\begin{aligned}
&= \frac{c_1\mu_1(z) - a_1}{c_2m(\mu_1(z)) + d_2} \left\langle f(\cdot), K_m(\cdot, \mu_1(z)) \right\rangle_{\mathcal{H}(m)} \\
&= \frac{c_1\mu_1(z) - a_1}{c_2m(\mu_1(z)) + d_2} f(\mu_1(z)).
\end{aligned}$$

Thus $V = V_1$. □

2.6. Stieltjes functions and Möbius transformations. Assigning to a meromorphic function m the value ∞ at the poles we can consider m to be defined on its whole domain of meromorphy. In particular, if the domain of meromorphy of a Nevanlinna function m includes \mathbb{R}_- , then the image $m(\mathbb{R}_-)$ of \mathbb{R}_- under m is contained in the one point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of \mathbb{R} . In the next lemma by $\overline{m(\mathbb{R}_-)}$ we denote the closure of $m(\mathbb{R}_-)$ in $\overline{\mathbb{R}}$.

Lemma 2.10. *Let m be a Nevanlinna function. The following statements are equivalent.*

- (i) *There exists a Möbius transformation μ of the form (2.17) such that $\mu \circ m$ is a Stieltjes function.*
- (ii) *There exists a Möbius transformation μ of the form (2.17) such that $\mu \circ m$ is an inverse Stieltjes function.*
- (iii) *The function m is meromorphic on $\mathbb{C} \setminus [0, +\infty)$ and $\overline{m(\mathbb{R}_-)}$ is a proper subset of $\overline{\mathbb{R}}$.*

Proof. The equivalence of (i) and (ii) follows from Proposition 2.4. The implication (ii) \Rightarrow (iii) is clear. To complete the proof, assume (iii). Then the complement of $\overline{m(\mathbb{R}_-)}$ in $\overline{\mathbb{R}}$ contains a finite open interval. By shifting m we can assume that that interval is $(0, 2/c)$ with $c > 0$. Set $\mu(z) = -z/(cz - 1)$. Then $\mu \circ m$ is a Nevanlinna function which is holomorphic on $\mathbb{C} \setminus [0, +\infty)$. Since $\mu(x) < 0$ whenever $x \in \mathbb{R} \setminus [0, 2/c]$ and $m(\mathbb{R}_-) \subset \mathbb{R} \setminus [0, 2/c]$ we conclude that $(\mu \circ m)(x) < 0$ for all $x \in \mathbb{R}_-$. Thus $\mu \circ m$ is an inverse Stieltjes function. This proves (iii) \Rightarrow (ii). □

Denote by $\mathcal{S}_{\mathcal{M}}$ the set of Nevanlinna functions that satisfy the equivalent conditions in Lemma 2.10. The set $\mathcal{S}_{\mathcal{M}}$ appears in [7] as the union of Nevanlinna functions of type I, II, III and V', see [7, Definition 2.4] and also [7, Corollary 5.6].

Notice that a function $m \in \mathcal{S}_{\mathcal{M}}$ can have at most one pole in \mathbb{R}_- and that the following limits exist

$$m(-\infty) := \lim_{x \downarrow -\infty} m(x) \in \{-\infty\} \cup \mathbb{R} \quad \text{and} \quad m(0-) := \lim_{x \uparrow 0} m(x) \in \mathbb{R} \cup \{+\infty\},$$

with at least one of them being finite. In addition, $m(-\infty) \leq m(0-)$ if and only if m is holomorphic on $\mathbb{C} \setminus [0, +\infty)$ and $m(-\infty) > m(0-)$ if and only if m has a pole in \mathbb{R}_- .

Remark 2.11. Let $m \in \mathcal{S}_{\mathcal{M}}$. It follows from Lemma S1.3.1 and Remark 5.1 in [30] that $m(-\infty) \in \mathbb{R}$ if and only if $\lim_{y \uparrow +\infty} m(iy)$ exists as a real number; in this case $\lim_{y \uparrow +\infty} m(iy) = m(-\infty)$. Also, $m(-\infty) = -\infty$ if and only if $\lim_{y \uparrow +\infty} m(iy) = \infty$.

Since $m \in \mathcal{S}_{\mathcal{M}}$ if and only if the function $z \mapsto -m(1/z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, is in $\mathcal{S}_{\mathcal{M}}$, we similarly have that $m(0-) \in \mathbb{R}$ if and only if $\lim_{y \downarrow 0} m(iy)$ exists as a real number and in this case $\lim_{y \downarrow 0} m(iy) = m(0-)$. Also, $m(0-) = +\infty$ if and only if $\lim_{y \downarrow 0} m(iy) = \infty$.

2.7. Two asymptotic classes of Nevanlinna functions. For functions f and g defined on \mathbb{R}_+ the expression

$$f(t) \sim g(t) \quad \text{as } t \rightarrow +\infty \quad \left(f(t) \sim g(t) \quad \text{as } t \rightarrow 0+ \right)$$

means

$$\lim_{t \uparrow +\infty} \frac{f(t)}{g(t)} = 1 \quad \left(\lim_{t \downarrow 0+} \frac{f(t)}{g(t)} = 1, \quad \text{respectively} \right).$$

We define the set of functions \mathcal{A}_∞ as follows: a Nevanlinna function m belongs to \mathcal{A}_∞ if and only if there exist $\alpha \in (0, 1)$, $C > 0$ and a Möbius transformation μ of the form (2.17) such that $\mu \circ m$ is a Stieltjes function and for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$(\mu \circ m)(rz) \sim \frac{C}{(-rz)^\alpha} \quad \text{as } r \rightarrow +\infty. \quad (2.25)$$

Similarly, we define the set of functions \mathcal{A}_0 as follows: a Nevanlinna function m belongs to \mathcal{A}_0 if and only if there exist $\alpha \in (0, 1)$, $C > 0$ and a Möbius transformation μ of the form (2.17) such that $\mu \circ m$ is an inverse Stieltjes function and for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$(\mu \circ m)(rz) \sim -C(-rz)^\alpha \quad \text{as } r \rightarrow 0+. \quad (2.26)$$

In the notation of Theorem 2.9, the following proposition holds.

Proposition 2.12. (i) *Assume that $\mu_1(z) = z$ and let μ_2 be a Möbius transformation as in (2.21). Then, $m \in \mathcal{A}_\infty$ ($m \in \mathcal{A}_0$) if and only if $\widehat{m} \in \mathcal{A}_\infty$ ($\widehat{m} \in \mathcal{A}_0$, respectively).*
(ii) *Let $\mu_1(z) = 1/z$ and let μ_2 be a Möbius transformation as in (2.21). Then, $m \in \mathcal{A}_\infty$ ($m \in \mathcal{A}_0$) if and only if $\widehat{m} \in \mathcal{A}_0$ ($\widehat{m} \in \mathcal{A}_\infty$, respectively).*

Proof. We will prove one implication from each (i) and (ii). The other implications are proved similarly.

Let $m \in \mathcal{A}_\infty$. Then there exists a Möbius transformation μ of the form (2.17) such that $\mu \circ m$ is a Stieltjes function and (2.25) holds for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Recall that, according to (2.22), in (i) we have $\widehat{m} = \epsilon_2 \mu_2 \circ m$. Set $\nu(z) = \epsilon_2 z$, $z \in \mathbb{C}$. Since $\mu \circ \mu_2^{-1} \circ \nu \circ \widehat{m} = \mu \circ m$ and $\mu \circ \mu_2^{-1} \circ \nu$ is a Möbius transformation of the form (2.17), we have $\widehat{m} \in \mathcal{A}_\infty$. This proves the implication $m \in \mathcal{A}_\infty \Rightarrow \widehat{m} \in \mathcal{A}_\infty$ from (i).

Again, according to (2.22), in (ii) we have $\widehat{m} = -\epsilon_2 \mu_2 \circ m \circ \mu_1$. Set $\nu(z) = -\epsilon_2 z$, $z \in \mathbb{C}$. Clearly,

$$-\mu \circ \mu_2^{-1} \circ \nu \circ \widehat{m} = -\mu \circ m \circ \mu_1.$$

Since $\mu \circ m$ is a Stieltjes function that satisfies (2.25) for all $z \in \mathbb{C} \setminus \mathbb{R}$, by Proposition 2.4 the function $-\mu \circ m \circ \mu_1$ is an inverse Stieltjes function that satisfies (2.26) for all $z \in \mathbb{C} \setminus \mathbb{R}$. As $-\mu \circ \mu_2^{-1} \circ \nu$ is a Möbius transformation of the form (2.17), we have $\widehat{m} \in \mathcal{A}_0$. This proves the implication $m \in \mathcal{A}_\infty \Rightarrow \widehat{m} \in \mathcal{A}_0$ from (ii). \square

In the next proposition we characterize the functions in \mathcal{A}_∞ and \mathcal{A}_0 by their asymptotic behavior.

Proposition 2.13. *The following equivalences hold.*

- (i) $m \in \mathcal{A}_\infty$ if and only if $m \in \mathcal{S}_\mathcal{M}$ and there exist $C_0 \in \mathbb{R} \setminus \{0\}$ and $\alpha_0 \in (-1, 0) \cup (0, 1)$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$m(rz) \sim C_0(-rz)^{\alpha_0} \quad \text{as } r \rightarrow +\infty, \quad (2.27)$$

or, corresponding to $\alpha_0 = 0$, there exist $C_0 \in \mathbb{R} \setminus \{0\}$, $C_1 > 0$ and $\alpha_1 \in (-1, 0)$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$m(rz) - C_0 \sim C_1(-rz)^{\alpha_1} \quad \text{as } r \rightarrow +\infty. \quad (2.28)$$

- (ii) $m \in \mathcal{A}_0$ if and only if $m \in \mathcal{S}_\mathcal{M}$ and there exist $C_0 \in \mathbb{R} \setminus \{0\}$ and $\alpha_0 \in (-1, 0) \cup (0, 1)$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$m(rz) \sim C_0(-rz)^{\alpha_0} \quad \text{as } r \rightarrow 0+,$$

or, corresponding to $\alpha_0 = 0$, there exist $C_0, C_1 \in \mathbb{R} \setminus \{0\}$, $C_1 > 0$ and $\alpha_1 \in (0, 1)$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$m(rz) - C_0 \sim -C_1(-rz)^{\alpha_1} \quad \text{as } r \rightarrow 0+.$$

Proof. We first prove the direct implication in (i). Clearly $\mathcal{A}_\infty \subset \mathcal{S}_\mathcal{M}$. Assume that $m \in \mathcal{A}_\infty$ is such that (2.25) holds and let $\mu^{-1}(z) = (az + b)/(cz + d)$ with $ad - bc = 1$. We distinguish two cases.

If $d = 0$, then $\mu^{-1}(z) = a/c + b/(cz)$, and therefore, for all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$m(rz) \sim \frac{b}{cC}(-rz)^\alpha \quad \text{as } r \rightarrow +\infty.$$

Thus (2.27) holds.

If $d \neq 0$, then $\mu^{-1}(z) = b/d + z/d^2 + O(z^2)$ as $z \rightarrow 0$, and therefore, for all $z \in \mathbb{C} \setminus \mathbb{R}$

$$m(rz) - \frac{b}{d} \sim \frac{C}{d^2}(-rz)^{-\alpha} \quad \text{as } r \rightarrow +\infty.$$

Consequently, (2.27) holds if $b = 0$ and (2.28) holds if $b \neq 0$. This proves the direct implication in (i).

To prove the converse let $m \in \mathcal{S}_\mathcal{M}$. First assume that m satisfies (2.27) with $\alpha_0 \in (-1, 0)$ and $C_0 > 0$. By Remark 2.11 we have $m(-\infty) = 0$. If m is holomorphic on $\mathbb{C} \setminus [0, +\infty)$, then it is a Stieltjes function; so $m \in \mathcal{A}_\infty$ in this case. If m has a pole in \mathbb{R}_- , then, since $m \in \mathcal{S}_\mathcal{M}$, $m(0-) < 0$ and $m(x) \notin (m(0-), 0)$ for all $x \in \mathbb{R}_-$. Setting $c = -m(0-)/2$ and $\mu(z) = z/(cz + 1)$, as in the proof of Lemma 2.10 we have that $\mu \circ m$ is a Stieltjes function. Since $\mu(z) = z + O(z^2)$ as $z \rightarrow 0$, we have that

$$(\mu \circ m)(rz) \sim C_0(-rz)^{\alpha_0} \quad \text{as } r \rightarrow 0.$$

Hence $m \in \mathcal{A}_\infty$.

Second, assume that m satisfies (2.27) with $\alpha_0 \in (0, 1)$ and $C_0 < 0$. Then, the already proven part implies $m^\top \in \mathcal{A}_\infty$ and Proposition 2.12 yields $m \in \mathcal{A}_\infty$.

Third, assume that m satisfies (2.28). Then, with $\mu(z) = z - C_0$, we have $\mu \circ m \in \mathcal{A}_\infty$, by the second part of this proof. Now Proposition 2.12 implies that $m \in \mathcal{A}_\infty$.

The proof of (ii) is similar. \square

2.8. Nonnegative operators in Kreĭn spaces. Let $(\mathcal{K}, [\cdot, \cdot])$ be a Kreĭn space and let A be a densely defined operator in \mathcal{K} . The *adjoint* $A^{[*]}$ of A with respect to $[\cdot, \cdot]$ is defined analogously as in a Hilbert space. In fact, if J is a fundamental symmetry on $(\mathcal{K}, [\cdot, \cdot])$ and $\langle \cdot, \cdot \rangle$ is the corresponding Hilbert space inner product, then $A^{[*]} = JA^{(*)}J$. In analogy with definitions in a Hilbert space, A is *symmetric* in $(\mathcal{K}, [\cdot, \cdot])$ if $A^{[*]} = A$ and A is *self-adjoint* in $(\mathcal{K}, [\cdot, \cdot])$ if $A = A^{[*]}$.

Since our main interest in this paper is similarity of a self-adjoint operator in a Kreĭn space to a self-adjoint operator in a Hilbert space, in the next proposition we recall a known characterization of similarity. This characterization is proved in [45, Theorem 1, Section 2] for bounded self-adjoint operators in a Kreĭn space. In [41, Proposition 2.2] an equivalent statement in terms of \mathcal{C} -stable symmetries is given and Phillips theorem [5, Chapter 2, Corollary 5.20] is cited for a proof. Below we give a simple direct proof.

Proposition 2.14. *A self-adjoint operator A in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot])$ is similar to a self-adjoint operator in the Hilbert space $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ if and only if A is fundamentally reducible in $(\mathcal{K}, [\cdot, \cdot])$.*

Proof. Let A be a self-adjoint operator in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot])$ and assume that A is similar to a self-adjoint operator in the Hilbert space $(\mathcal{K}, \langle \cdot, \cdot \rangle)$. That is, A is self-adjoint in a Hilbert space $(\mathcal{K}, (\cdot, \cdot))$ whose norm is equivalent to that of $(\mathcal{K}, \langle \cdot, \cdot \rangle)$. The equivalence of the norms of $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ and $(\mathcal{K}, (\cdot, \cdot))$ and the identity $\langle x, y \rangle = [Jx, y]$, for all $x, y \in \mathcal{K}$, imply the existence of a bounded and boundedly invertible operator $G : \mathcal{K} \rightarrow \mathcal{K}$ such that $(Gx, y) = [x, y]$ for all $x, y \in \mathcal{K}$. The last identity yields that G is self-adjoint in (\cdot, \cdot) and the operator $\text{sgn } G$ is a fundamental symmetry on $(\mathcal{K}, [\cdot, \cdot])$. As A is self-adjoint in both $[\cdot, \cdot]$ and (\cdot, \cdot) , A commutes with G . Hence, A commutes with the fundamental symmetry $\text{sgn } G$ on $(\mathcal{K}, [\cdot, \cdot])$; proving that A is fundamentally reducible. The converse is clear. \square

A densely defined operator A is *nonnegative* in $(\mathcal{K}, [\cdot, \cdot])$ if $[Af, f] \geq 0$ for all $f \in \text{dom}(A)$. A nonnegative self-adjoint operator in a Kreĭn space can have an empty resolvent set; a specific example is given in [42, 1.2] and [9, Example VII.1.5]. A modification of this example leads to a positive self-adjoint operator in a Kreĭn space with an empty resolvent set, see [38, Example 3.4].

However, if A is a nonnegative self-adjoint operator in $(\mathcal{K}, [\cdot, \cdot])$ and $\rho(A) \neq \emptyset$, then the spectrum of A is real and A has a spectral function E , see [44, Theorem II.3.1]. The domain of E is the ring \mathcal{I} which consists of finite unions of (bounded) intervals whose endpoints are nonzero real numbers and their complements in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The values of E are bounded operators on \mathcal{K} with the following properties. For all $\Delta, \Delta_1, \Delta_2 \in \mathcal{I}$ we have

- (E1) $E(\emptyset) = 0, E(\overline{\mathbb{R}}) = I,$
- (E2) $E(\Delta) = E(\Delta)^{[*]}$
- (E3) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2),$
- (E4) $E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2)$ whenever $\Delta_1 \cap \Delta_2 = \emptyset,$
- (E5) the space $(E(\Delta)\mathcal{K}, \pm[\cdot, \cdot])$ is a Hilbert space whenever $\Delta \in \mathbb{R}_{\pm},$
- (E6) $E(\Delta)$ is in the double commutant of the resolvent of $A,$
- (E7) for a bounded Δ we have $E(\Delta)\mathcal{K} \subseteq \text{dom}(A)$ and the restriction of A to $E(\Delta)\mathcal{K}$ is a bounded operator whose spectrum is contained in $\overline{\Delta}.$

For $\lambda \in \mathbb{R} \setminus \{0\}$, it follows from (E5) that in a neighborhood of λ the spectral function E behaves as a spectral function of a positive operator in a Hilbert space.

In particular, with $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ such that $\lambda_1 < \lambda < \lambda_2$ the limits

$$\lim_{t \uparrow \lambda} E([\lambda_1, t]) \quad \text{and} \quad \lim_{t \downarrow \lambda} E([t, \lambda_2]) \quad (2.29)$$

exist in the strong operator topology. The existence of these limits is a consequence of the following property of nonzero real numbers: For $\lambda \in \mathbb{R} \setminus \{0\}$ there exists a $\Delta \in \mathcal{I}$ such that λ is in the interior of Δ and $(E(\Delta)\mathcal{K}, (\text{sgn } \lambda)[\cdot, \cdot])$ is a Hilbert space.

A possible absence of the just mentioned property for 0 or ∞ is a motivation for the following definition. The point 0 (∞) is said to be a *critical point of A* if $[\cdot, \cdot]$ is indefinite on $E(\Delta)\mathcal{K}$ for every $\Delta \in \mathcal{I}$ such that $0 \in \Delta$ ($\infty \in \Delta$, respectively).

However, even if 0 or ∞ is a critical point the limits analogous to (2.29) may exist. If 0 is a critical point of A and the limits in (2.29) exist with $\lambda = 0$, then 0 is called a *regular critical point of A*. If ∞ is a critical point of A and the limits

$$\lim_{t \uparrow +\infty} E([\lambda_1, t]) \quad \text{and} \quad \lim_{t \downarrow -\infty} E([t, \lambda_2])$$

exist in the strong operator topology for some $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$, then ∞ is called *regular critical point of A*. A critical point of A which is not regular is called *singular critical point of A*. The set of all singular critical points of A is denoted by $c_s(A)$.

The following proposition is a part of folklore in this setting. For a bounded A it was proved in [6], and, in a more general form in [45]. For unbounded A with a bounded inverse it appears in [12] where [6] was cited for a proof by taking an inverse. For completeness we include a simple proof.

Proposition 2.15. *Let A be a nonnegative operator in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot])$. Then A is fundamentally reducible in $(\mathcal{K}, [\cdot, \cdot])$ if and only if $\rho(A) \neq \emptyset$, $\ker(A) = \ker(A^2)$ and $\infty, 0 \notin c_s(A)$.*

Proof. Since a fundamentally reducible operator is similar to a self-adjoint operator in a Hilbert space, the direct implication follows. For the converse, notice that $\infty, 0 \notin c_s(A)$ implies the existence of the operators $E(\mathbb{R}_-) := \lim_{t \downarrow -\infty} E([t, 1/t])$ and $E(\mathbb{R}_+) := \lim_{t \uparrow +\infty} E([1/t, t])$. It follows from (E2), (E3), (E5) and (E6) that $E(\mathbb{R}_\pm) = E(\mathbb{R}_\pm)^2 = E(\mathbb{R}_\pm)^{[*]}$, $(E(\mathbb{R}_\pm)\mathcal{K}, \pm[\cdot, \cdot])$ is a Hilbert space and $E(\mathbb{R}_\pm)$ is in the double commutant of the resolvent of A . Since $\ker(A) = \ker(A^2)$, Propositions 5.1 and 5.6 in [43] yield

$$\mathcal{K} = E(\mathbb{R}_+)\mathcal{K}[+] \ker(A)[+] E(\mathbb{R}_-)\mathcal{K},$$

direct and orthogonal sum in $(\mathcal{K}, [\cdot, \cdot])$. Since $(E(\mathbb{R}_-)\mathcal{K}[+] E(\mathbb{R}_+)\mathcal{K}, [\cdot, \cdot])$ is a Kreĭn space, [43, Theorem 5.2] implies that $(\ker(A), [\cdot, \cdot])$ is a Kreĭn space. Let $\ker(A) = \mathcal{K}_+^0[+] \mathcal{K}_-^0$ be its fundamental decomposition. Then

$$\mathcal{K} = (E(\mathbb{R}_+)\mathcal{K}[+] \mathcal{K}_+^0)[+] (\mathcal{K}_-^0[+] E(\mathbb{R}_-)\mathcal{K})$$

is a fundamental decomposition of \mathcal{K} which reduces A . □

In Subsection 5.2 we essentially use the following resolvent criterion of K. Veselić [46] for $\infty \notin c_s(A)$. We state a special case of this criterion as it has appeared in [29, Corollary 1.6].

Theorem 2.16. *Let A be a nonnegative self-adjoint operator with a nonempty resolvent set in a Kreĭn space $(\mathcal{H}, [\cdot, \cdot])$. Then:*

(a) $\infty \notin c_s(A)$ if and only if the operators

$$\int_{-i\eta}^{-i} + \int_i^{i\eta} (A - z)^{-1} dz$$

are uniformly bounded for $\eta \in (1, \infty)$.

(b) $0 \notin c_s(A)$ if and only if $\ker(A) = \ker(A^2)$ and the operators

$$\int_{-i}^{-i\varepsilon} + \int_{i\varepsilon}^i (A - z)^{-1} dz$$

are uniformly bounded for $\varepsilon \in (0, 1)$.

3. B-PROPERTIES OF NEVANLINNA FUNCTIONS.

3.1. The definition and basic properties.

Definition 3.1. Let m be a Nevanlinna function. Set

$$w_m(y) := \frac{1}{\operatorname{Im} m(iy)}, \quad y > 0.$$

Let $G_{m,\infty}$ be the mapping defined for all $h \in \mathcal{H}(m)$ by

$$(G_{m,\infty}h)(y) := h(iy), \quad y > 1.$$

A Nevanlinna function m is said to have a B_∞ -property if $\operatorname{ran}(G_{m,\infty}) \subset L_{w_m}^2(1, \infty)$ and the operator $G_{m,\infty} : \mathcal{H}(m) \rightarrow L_{w_m}^2(1, \infty)$ is bounded.

Let $G_{m,0}$ be the mapping defined for all $h \in \mathcal{H}(m)$ by

$$(G_{m,0}h)(y) := h(iy), \quad 0 < y < 1.$$

A Nevanlinna function m is said to have a B_0 -property if $\operatorname{ran}(G_{m,0}) \subset L_{w_m}^2(0, 1)$ and the operator $G_{m,0} : \mathcal{H}(m) \rightarrow L_{w_m}^2(0, 1)$ is bounded.

The following proposition is a straightforward consequence of Theorem 2.6 and the above definition.

Proposition 3.2. *Let S be a closed simple symmetric operator with defect numbers $(1, 1)$ in a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$. Let $(\mathbb{C}, \Gamma_0, \Gamma_1)$ be a boundary triple for S^* and let m be the corresponding Weyl function. Then m has B_∞ -property (B_0 -property) if and only if $G_{m,\infty}F_m$ is a bounded mapping from \mathcal{H} into $L_{w_m}^2(1, \infty)$ ($L_{w_m}^2(0, 1)$, respectively). Moreover, if m has a B_∞ -property (B_0 -property), then $\|G_{m,\infty}\| = \|G_{m,\infty}F_m\|$ ($\|G_{m,0}\| = \|G_{m,0}F_m\|$, respectively).*

In the next lemma, for a Nevanlinna function m , we introduce the operator $G_{m,\infty}^-$.

Lemma 3.3. *Let m be a Nevanlinna function. Then:*

(i) m has B_∞ -property if and only if

$$(G_{m,\infty}^-h)(y) := h(-iy), \quad y > 1.$$

is a bounded operator from $\mathcal{H}(m)$ into $L_{w_m}^2(1, \infty)$.

(ii) m has B_0 -property if and only if

$$(G_{m,0}^-h)(y) := h(-iy), \quad y \in (0, 1).$$

is a bounded operator from $\mathcal{H}(m)$ into $L_{w_m}^2(0, 1)$.

Proof. For every $\omega \in \mathbb{C} \setminus \mathbb{R}$ and $y > 0$ we have

$$\begin{aligned} (G_{m,\infty}^- K_\omega)(y) &= K_\omega(-iy) = \frac{m(-iy) - m(\omega)^*}{-iy - \omega^*} \\ &= K_{\omega^*}(iy)^* = ((G_{m,\infty} K_{\omega^*})(y))^* = ((G_{m,\infty} W K_\omega)(y))^*. \end{aligned}$$

Together with (2.15), this implies statements (i) and (ii). \square

Möbius transformations of Nevanlinna functions preserve B_∞ -property and B_0 -property. In the notation of Theorem 2.9 we have the following proposition.

- Proposition 3.4.** (a) Assume that $\mu_1(z) = z$ or $\mu_1(z) = -z$ and let μ_2 be a Möbius transformation as in (2.21). A Nevanlinna function m has B_∞ -property (B_0 -property) if and only if the Nevanlinna function \widehat{m} has B_∞ -property (B_0 -property, respectively).
- (b) Let $\mu_1(z) = 1/z$ or $\mu_1(z) = -1/z$ and let μ_2 be a Möbius transformation as in (2.21). A Nevanlinna function m has B_∞ -property (B_0 -property) if and only if the Nevanlinna function \widehat{m} has B_0 -property (B_∞ -property, respectively).

Proof. To prove (a), let $\mu_1(z) = z$ and calculate

$$w_{\widehat{m}}(y) = |c_2 m(iy) + d_2|^2 w_m(y), \quad y > 0.$$

For $f \in \mathcal{H}(m)$ we have

$$\begin{aligned} \|G_{\widehat{m},\infty} V f\|_{L_{w_{\widehat{m}}}^2(1,\infty)}^2 &= \int_1^\infty \frac{1}{|c_2 m(iy) + d_2|^2} |f(iy)|^2 w_{\widehat{m}}(y) dy \\ &= \int_1^\infty |f(iy)|^2 w_m(y) dy \\ &= \|G_{m,\infty} f\|_{L_{w_m}^2(1,\infty)}^2. \end{aligned}$$

The above equality, Theorem 2.9, and elementary arguments yield the proposition. The statement for $\mu_1(z) = -z$ follows from Lemma 3.3.

For (b), let $\mu_1(z) = -1/z$ and calculate

$$w_{\widehat{m}}(y) = |c_2 m(i/y) + d_2|^2 w_m(1/y), \quad y > 0.$$

For $f \in \mathcal{H}(m)$ we have

$$\begin{aligned} \|G_{\widehat{m},\infty} V f\|_{L_{w_{\widehat{m}}}^2(1,\infty)}^2 &= \int_1^\infty \frac{1}{y^2} \frac{1}{|c_2 m(i/y) + d_2|^2} |f(i/y)|^2 w_{\widehat{m}}(y) dy \\ &= \int_1^\infty \frac{1}{y^2} |f(i/y)|^2 w_m(1/y) dy \\ &= \int_0^1 |f(is)|^2 w_m(s) ds \\ &= \|G_{m,0} f\|_{L_{w_m}^2(0,1)}^2. \end{aligned}$$

This identity, Theorem 2.9 and elementary arguments yield claim (b). The statement for $\mu_1(z) = 1/z$ follows from Lemma 3.3. \square

Next, we characterize B -properties in terms of a bounded operator between weighted L^2 spaces.

Proposition 3.5. *Let m be a Nevanlinna function with the integral representation*

$$m(z) = a + \int_0^{+\infty} \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) d\sigma(x) \quad (3.1)$$

where a is a real number and $\sigma(x)$ is a non-decreasing function such that (2.9) holds. The following statements hold.

- (a) *The function m has B_∞ -property if and only if the mapping $H_{m,\infty}$ defined by*

$$(H_{m,\infty}f)(y) := \int_0^{+\infty} \frac{f(x)}{x+y} d\sigma(x), \quad f \in L_\sigma^2(\mathbb{R}_+), \quad y > 1,$$

is a bounded operator from $L_\sigma^2(\mathbb{R}_+)$ into $L_{w_m}^2(1, \infty)$.

- (b) *The function m has B_0 -property if and only if the mapping $H_{m,0}$ defined by*

$$(H_{m,0}f)(y) := \int_0^{+\infty} \frac{f(x)}{x+y} d\sigma(x), \quad f \in L_\sigma^2(\mathbb{R}_+), \quad 0 < y < 1,$$

is a bounded operator from $L_\sigma^2(\mathbb{R}_+)$ into $L_{w_m}^2(0, 1)$.

Proof. It follows from [3, Theorem 3.2] that the mapping

$$f \mapsto \int_0^{+\infty} \frac{f(x)}{x-z} d\sigma(x) \quad (3.2)$$

is an isomorphism between $L_\sigma^2(\mathbb{R}_+)$ and $\mathcal{H}(m)$. Therefore, by Definition 3.1, m has B_∞ -property if and only if the composition of (3.2) and $G_{m,\infty}$, that is,

$$f \mapsto \int_0^{+\infty} \frac{f(x)}{x-iy} d\sigma(x) \quad (3.3)$$

is a bounded operator from $L_\sigma^2(\mathbb{R}^+)$ into $L_{w_m}^2(1, \infty)$. The inequalities

$$\frac{1}{x+y} \leq \frac{1}{|x-iy|} \leq \frac{\sqrt{2}}{x+y}, \quad (x, y \in \mathbb{R}_+)$$

imply that the operator in (3.3) is a bounded operator from $L_\sigma^2(\mathbb{R}^+)$ into $L_{w_m}^2(1, \infty)$ if and only if $H_{m,\infty}$ is a bounded operator from $L_\sigma^2(\mathbb{R}^+)$ into $L_{w_m}^2(1, \infty)$. This proves (a).

The second statement is proved similarly. \square

3.2. Sufficient conditions for B -properties of Nevanlinna functions. We will need the following Schur test, see [27, Theorem 5.9.2], for boundedness of integral operators.

Lemma 3.6. *Let (X, Σ_1, σ_1) and (Y, Σ_2, σ_2) be σ -finite measure spaces and let $K(x, y)$ be a nonnegative measurable function on a product space $(X, \Sigma_1, \sigma_1) \times (Y, \Sigma_2, \sigma_2)$. Suppose that there exist strictly positive measurable functions q_1 on (X, Σ_1, σ_1) and q_2 on (Y, Σ_2, σ_2) such that the function*

$$y \mapsto \frac{1}{q_2(y)} \int_X K(x, y) q_1(x) d\sigma_1(x)$$

is essentially bounded on (Y, Σ_2, σ_2) and the function

$$x \mapsto \frac{1}{q_1(x)} \int_Y K(x, y) q_2(y) d\sigma_2(y)$$

is essentially bounded on (X, Σ_1, σ_1) . Then for all $f \in L^2(X, \sigma_1)$ we have that

$$(Af)(y) = \int_X K(x, y)f(x)d\sigma_1(x)$$

exists for σ_2 -almost all $y \in Y$, $Af \in L^2_{\sigma_2}(Y)$ and A is a bounded operator from $L^2_{\sigma_1}(X)$ to $L^2_{\sigma_2}(Y)$.

Combining Proposition 3.5 and Lemma 3.6 one obtains the following sufficient conditions for B_∞ and B_0 -properties.

Corollary 3.7. *Let m have the integral representation (3.1) and assume that there are strictly positive measurable functions q_1 on \mathbb{R}_+ and q_2 on $(1, \infty)$ such that the function*

$$y \mapsto \frac{1}{q_2(y)} \int_0^{+\infty} \frac{q_1(x)}{x+y} d\sigma(x) \quad (3.4)$$

is essentially bounded on $y \in (1, \infty)$ and the function

$$x \mapsto \frac{1}{q_1(x)} \int_1^{+\infty} \frac{q_2(y)}{x+y} w_m(y) dy \quad (3.5)$$

is σ -essentially bounded on \mathbb{R}_+ . Then the function m has B_∞ -property.

Proof. The assumptions of the corollary and Lemma 3.6 imply that the operator $H_{m, \infty}$ is bounded, so, by Proposition 3.5, m has B_∞ -property. \square

In Theorem 3.9 and its corollaries below we give sufficient conditions for B_∞ -property and B_0 -property in terms of the asymptotics of a Nevanlinna function m . The key step in the proof of this theorem is the following Abelian and Tauberian theorem for the Stieltjes transform at the point ∞ which is essentially contained in [8] and [11], see also [10] for a more general theorem, or [35] for the classical Karamata Tauberian theorem.

Theorem 3.8. *Let m be a Stieltjes function with the integral representation*

$$m(z) = \int_0^{+\infty} \frac{d\sigma(t)}{t-z} \quad (3.6)$$

where $\sigma(t)$ is a non-decreasing function satisfying (2.10) and (2.12). Let $C > 0$ and $\alpha \in (0, 1)$. Then

$$\text{for all } z \in \mathbb{C} \setminus \mathbb{R} \text{ we have } m(rz) \sim \frac{C}{(-rz)^\alpha} \text{ as } r \rightarrow +\infty, \quad (3.7)$$

if and only if

$$\sigma(t) \sim \frac{C \sin(\pi\alpha)}{\pi} \frac{t^{1-\alpha}}{1-\alpha} \text{ as } t \rightarrow +\infty. \quad (3.8)$$

Proof. The direct part of Theorem 3.8 is a special case of the Tauberian Theorem 7.5 from [8].

Assume now that σ satisfies (3.8). Integrating by parts in (3.6) and using (3.8) one obtains

$$m(z) = \int_0^{+\infty} \frac{\sigma(t)dt}{(t-z)^2}.$$

Now the asymptotic (3.7) is implied by the Abelian Theorem 3.2 from [11]. \square

Theorem 3.9. *Let m be a Stieltjes function. If there exist $\alpha \in (0, 1)$ and $C > 0$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have*

$$m(rz) \sim \frac{C}{(-rz)^\alpha} \quad \text{as } r \rightarrow +\infty, \quad (3.9)$$

then m has B_∞ -property.

Proof. Let m be a Stieltjes function with the integral representation (2.11) for which (2.12) and (2.10) hold. Assume (3.9). Since by [30, Lemma S1.3.1] and (3.9) for γ in (2.11) we have

$$\gamma = \lim_{y \uparrow +\infty} m(iy) = 0,$$

Theorem 3.8 applies.

Consequently, there exists $C_1 = \frac{C \sin(\pi\alpha)}{\pi(1-\alpha)} > 0$ such that

$$\sigma(t) \sim C_1 t^{1-\alpha} \quad \text{as } t \rightarrow +\infty. \quad (3.10)$$

Let $\beta > 0$ be such that $\alpha + \beta < 1$. Set

$$q_1(x) := \frac{1}{(1+x)^\beta}$$

and

$$q_2(y) := \int_0^\infty \frac{d\sigma_1(x)}{x+y}, \quad \text{where } \sigma_1(x) = \int_0^x \frac{d\sigma(t)}{(1+t)^\beta}.$$

Notice that by definitions of q_1 and q_2 we have

$$\int_0^\infty \frac{q_1(x)}{x+y} d\sigma(x) = q_2(y) \quad \text{for all } y > 1.$$

Hence, q_1 and q_2 satisfy (3.4) in Corollary 3.7. The rest of the proof is a verification of (3.5).

Integration by parts in the formula for σ_1 yields

$$\sigma_1(x) = \frac{\sigma(x)}{(1+x)^\beta} + \beta \int_0^x \frac{\sigma(t)}{(1+t)^{\beta+1}} dt. \quad (3.11)$$

It follows from (3.10) that

$$\frac{\sigma(t)}{(1+t)^{\beta+1}} \sim \frac{C_1}{t^{\alpha+\beta}} \quad \text{as } t \rightarrow +\infty,$$

and, by l'Hôpital's rule,

$$\int_0^x \frac{\sigma(t)}{(1+t)^{\beta+1}} dt \sim C_2 x^{1-\alpha-\beta} \quad \text{as } x \rightarrow +\infty.$$

Since also

$$\frac{\sigma(t)}{(1+t)^\beta} \sim C_1 t^{1-\alpha-\beta} \quad \text{as } t \rightarrow +\infty,$$

the equality (3.11) implies

$$\sigma_1(x) \sim C_3 x^{1-\alpha-\beta} \quad \text{as } x \rightarrow +\infty, \quad (3.12)$$

Now (3.12) and Theorem 3.8 yield

$$q_2(y) \sim \frac{C_4}{y^{\alpha+\beta}} \quad \text{as } y \rightarrow \infty. \quad (3.13)$$

for some $C_4 > 0$. Next consider the function

$$x \mapsto \int_1^\infty \frac{q_2(y)}{x+y} w_m(y) dy, \quad x > 0.$$

Since by (3.13) and assumption (3.9) we have

$$q_2(y)w_m(y) \sim \frac{C_5}{y^\beta} \quad \text{as } y \rightarrow \infty,$$

for some $C_5 > 0$, Theorem 3.8 yields

$$\int_1^\infty \frac{q_2(y)}{x+y} w_m(y) dy \sim \frac{C_6}{x^\beta} \quad \text{as } x \rightarrow \infty.$$

The last displayed relationship implies that the function

$$x \mapsto \frac{1}{q_1(x)} \int_1^\infty \frac{q_2(y)}{x+y} w_m(y) dy, \quad x > 0,$$

is bounded; that is (3.5) holds. Now Corollary 3.7 implies that m has B_∞ -property. \square

Corollary 3.10. *Every function in \mathcal{A}_∞ has B_∞ -property.*

Proof. Let $m \in \mathcal{A}_\infty$ be arbitrary. Then there exists a Möbius transformation μ such that $\mu \circ m$ is a Stieltjes function and (2.25) holds. By Theorem 3.9 the Nevanlinna function $\mu \circ m$ has B_∞ -property. In turn, Proposition 3.4(a) yields that $m = \mu^{-1} \circ \mu \circ m$ also has B_∞ -property. \square

Corollary 3.11. *Every function in \mathcal{A}_0 has B_0 -property.*

Proof. Let $m \in \mathcal{A}_0$ be arbitrary. Set $\mu_1(z) = 1/z$ and $\mu_2(z) = -z$. By Proposition 2.12(ii) the function $\hat{m} = \mu_2 \circ m \circ \mu_1$ belongs to \mathcal{A}_∞ , so \hat{m} has B_∞ -property by Corollary 3.10. Now, Proposition 3.4(b) implies that m has B_0 -property. \square

4. COUPLING OF SYMMETRIC OPERATORS

4.1. Coupling of symmetric operators in a Hilbert space. In this section we consider two Hilbert spaces $(\mathcal{H}_+, \langle \cdot, \cdot \rangle_{\mathcal{H}_+})$ and $(\mathcal{H}_-, \langle \cdot, \cdot \rangle_{\mathcal{H}_-})$ and their (external) direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with the natural inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. If T_+ is an operator in \mathcal{H}_+ and T_- is an operator in \mathcal{H}_- , then $T_+ \oplus T_-$ denotes their direct sum, that is

$$(T_+ \oplus T_-) \begin{pmatrix} f_+ \\ f_- \end{pmatrix} := \begin{pmatrix} T_+ f_+ \\ T_- f_- \end{pmatrix}, \quad f_+ \in \text{dom}(T_+), f_- \in \text{dom}(T_-).$$

The following assumptions apply to all four statements in this subsection. We assume that S_\pm is a closed symmetric densely defined operator with defect numbers $(1, 1)$ in the Hilbert space $(\mathcal{H}_\pm, \langle \cdot, \cdot \rangle_{\mathcal{H}_\pm})$. We let $(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm)$ be a boundary triple for S_\pm^* and m_\pm and ψ_\pm are the corresponding Weyl function and the Weyl solution. By S_0^\pm we denote the self-adjoint extension of S_\pm which is defined on $\text{dom}(S_0^\pm) = \ker(\Gamma_0^\pm)$. That is $S_0^\pm = S_\pm^*|_{\ker(\Gamma_0^\pm)}$.

In the following theorem we reformulate results from [20], (see also [13]) in a form which is convenient for further use in this paper.

Theorem 4.1. *Under the general assumptions of this subsection we have:*

(a) The linear operator S defined as the restriction of $S_+^* \oplus S_-^*$ to the domain

$$\text{dom}(S) = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : \begin{array}{l} \Gamma_0^+(f_+) = \Gamma_0^-(f_-) = 0, \\ \Gamma_1^+(f_+) + \Gamma_1^-(f_-) = 0, \end{array} f_+ \in \text{dom}(S_+^*), f_- \in \text{dom}(S_-^*) \right\}$$

is, closed, densely defined and symmetric with defect numbers $(1, 1)$ in the Hilbert space \mathcal{H} .

(b) The adjoint S^* of S is the restriction of $S_+^* \oplus S_-^*$ to the domain

$$\text{dom}(S^*) = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : \Gamma_0^+(f_+) - \Gamma_0^-(f_-) = 0, f_+ \in \text{dom}(S_+^*), f_- \in \text{dom}(S_-^*) \right\}.$$

(c) A boundary triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$ for S^* is given by

$$\Gamma_0 f = \Gamma_0^+ f_+, \quad \Gamma_1 f = \Gamma_1^+ f_+ + \Gamma_1^- f_-, \quad f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in \text{dom}(S^*). \quad (4.1)$$

(d) The Weyl function of S relative to the boundary triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$ is

$$m(z) = m_+(z) + m_-(z), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

(e) The self-adjoint extension S_1 of S such that $\text{dom}(S_1) = \ker(\Gamma_1)$ coincides with the restriction of $S_+^* \oplus S_-^*$ to the domain

$$\text{dom}(S_1) = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : \begin{array}{l} \Gamma_0^+(f_+) - \Gamma_0^-(f_-) = 0, \quad f_+ \in \text{dom}(S_+^*), \\ \Gamma_1^+(f_+) + \Gamma_1^-(f_-) = 0, \quad f_- \in \text{dom}(S_-^*) \end{array} \right\}. \quad (4.2)$$

(f) The self-adjoint extension S_0 of S such that $\text{dom}(S_0) = \ker(\Gamma_0)$ coincides with the restriction of $S_+^* \oplus S_-^*$ to the domain

$$\text{dom}(S_0) = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : \Gamma_0^+(f_+) = \Gamma_0^-(f_-) = 0, f_+ \in \text{dom}(S_+^*), f_- \in \text{dom}(S_-^*) \right\}$$

and thus $S_0 = S_0^+ \oplus S_0^-$.

The operator S_1 defined in Theorem 4.1(e) is called the *coupling* of the operators S_+ and S_- in the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ relative to the triples $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ and $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$.

In the next proposition we give a criterion for nonnegativity of the coupling S_1 of two nonnegative operators S_+ and S_- in the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$.

Proposition 4.2. *In addition to the general assumptions of this subsection assume that S_0^\pm is a nonnegative self-adjoint extension of S_\pm in \mathcal{H}_\pm . Then the coupling S_1 is a nonnegative self-adjoint operator in $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ if and only if*

$$m_+(x) + m_-(x) \neq 0 \quad \text{for all} \quad x \in \mathbb{R}_-. \quad (4.3)$$

Proof. Since the self-adjoint extensions S_0^+ and S_0^- are nonnegative, then the Weyl functions m_+ and m_- are holomorphic on \mathbb{R}_- and $(-\infty, 0) \subset \rho(S_0)$, where $S_0 = S_0^+ \oplus S_0^-$. As, by Theorem 4.1(d), $m(z) = m_+(z) + m_-(z)$ is the Weyl function of S relative to the boundary triple (4.1), it follows from Proposition 2.2 that (4.3) holds if and only if $(-\infty, 0) \subset \rho(S_1)$, or, equivalently, if and only if S_1 is nonnegative. \square

Corollary 4.3. *In addition to the general assumptions of this subsection assume that the operator S_0^+ is the Kreĭn extension of S_+ . Then the coupling S_1 is a nonnegative self-adjoint operator in the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ if and only if the following two conditions are satisfied:*

(i) $S_0^- := S_-^*|_{\ker(\Gamma_0^-)}$ is a non-negative operator in the Hilbert space \mathcal{H}_- .

$$(ii) \lim_{x \downarrow -\infty} (m_+(x) + m_-(x)) \geq 0.$$

If the coupling S_1 is a nonnegative, then there exists a constant $c \in \mathbb{R}$ such that $m_+ - c$ and $m_- + c$ are Stieltjes functions.

Proof. Since S_0^+ is the Kreĭn extension of S_+ , then by (2.8)

$$\lim_{x \uparrow 0} m_+(x) = +\infty. \quad (4.4)$$

If the coupling S_1 is nonnegative, then the operator S_- is also nonnegative and hence its extension S_0^- has at most one negative eigenvalue (see [39]). Then the Weyl function m_- has at most one pole on \mathbb{R}_- and by (4.4)

$$\lim_{x \uparrow 0} (m_+(x) + m_-(x)) = +\infty. \quad (4.5)$$

Assume that m_- has a pole at $x_0 \in (-\infty, 0)$. Then

$$\lim_{x \downarrow x_0} (m_+(x) + m_-(x)) = -\infty,$$

and, hence $m_+ + m_-$ has a zero in the interval $(x_0, 0)$. By Proposition 4.2 the operator S_1 has a negative eigenvalue which is impossible, if the operator S_1 is nonnegative. Therefore, the function m_- is holomorphic on \mathbb{R}_- and S_0^- is a non-negative operator in the Hilbert space \mathcal{H}_- .

Since the restrictions of m_+ and m_- to \mathbb{R}_- are continuous and monotonically increasing functions which satisfy the condition (4.5), the condition (4.3) of non-negativity of S_1 can be rewritten as (ii).

Conversely, if (i) and (ii) hold, then the operators S_0^+ and S_0^- are nonnegative in \mathcal{H}_+ and \mathcal{H}_- , respectively, m_- is holomorphic on \mathbb{R}_- and (4.3) holds. Therefore, the coupling S_1 is nonnegative by Proposition 4.2.

To prove the last statement, set $c = \lim_{x \downarrow -\infty} m_+(x)$. Then $m_+(x) - c \geq 0$ and by (ii)

$$m_-(x) + c \geq \lim_{x \downarrow -\infty} m_-(x) + \lim_{x \downarrow -\infty} m_+(x) \geq 0$$

for all $x \in \mathbb{R}_-$. Therefore $m_+ - c$ and $m_- + c$ are Stieltjes functions. \square

The next corollary is proved similarly.

Corollary 4.4. *In addition to the general assumptions of this subsection assume that the operator S_0^+ is the Friedrichs extension of S_+ . Then the coupling S_1 is a nonnegative self-adjoint operator in the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ if and only if the following two conditions are satisfied:*

$$(i) S_0^- := S_-^*|_{\ker(\Gamma_0^-)} \text{ is a non-negative operator in } \mathcal{H}_-.$$

$$(ii) m_+(0) + m_-(0) = \lim_{x \uparrow 0} (m_+(x) + m_-(x)) \leq 0.$$

If the coupling S_1 is nonnegative, then there exists a constant $c \in \mathbb{R}$ such that $m_+ - c$ and $m_- + c$ are inverse Stieltjes functions.

4.2. Non-separated extensions as couplings. In Theorem 4.1(e) we defined the coupling S_1 of symmetric operators S_+ and S_- relative to the boundary triples $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ for S_+^* and $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$ for S_-^* . Using different boundary triples would result in a different coupling operator. In this subsection we characterize all self-adjoint extensions of $S_+ \oplus S_-$ which can be obtained as couplings of S_+ and S_- .

Let, as before, $(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm)$ be a boundary triple for S_\pm^* . It is not difficult to see (cf. [13, Lemma 3.4], [20] for a more general setting or [2, Appendix II, 125] for

the case of differential operators) that a restriction \widehat{S} of $S_+^* \oplus S_-^*$ is a self-adjoint extension of $S_+ \oplus S_-$ in $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ if and only if

$$\text{dom}(\widehat{S}) = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in (\text{dom } S_+^*) \oplus (\text{dom } S_-^*) : M \begin{pmatrix} \Gamma_1^+ f_+ \\ \Gamma_1^- f_- \end{pmatrix} = N \begin{pmatrix} \Gamma_0^+ f_+ \\ \Gamma_0^- f_- \end{pmatrix} \right\},$$

where M and N are 2×2 matrices with complex entries such that the block matrix $\begin{pmatrix} M & N \end{pmatrix}$ has rank 2 and the matrix MN^* is self-adjoint.

If \widehat{S} is a self-adjoint extension of $S_+ \oplus S_-$ in $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ then either

$$(\text{gr } \widehat{S}) \cap (\mathcal{H}_+ \oplus \{0\})^2 = \text{gr}(S_+) \quad \text{and} \quad (\text{gr } \widehat{S}) \cap (\{0\} \oplus \mathcal{H}_-)^2 = \text{gr}(S_-) \quad (4.6)$$

or

$$(\text{gr } \widehat{S}) \cap (\mathcal{H}_+ \oplus \{0\})^2 \quad \text{and} \quad (\text{gr } \widehat{S}) \cap (\{0\} \oplus \mathcal{H}_-)^2$$

are graphs of self-adjoint operators in $(\mathcal{H}_+, \langle \cdot, \cdot \rangle_{\mathcal{H}_+})$ and $(\mathcal{H}_-, \langle \cdot, \cdot \rangle_{\mathcal{H}_-})$, respectively.

In the later case we say that the self-adjoint operator \widehat{S} is a *separated* extension of $S_+ \oplus S_-$ and the corresponding boundary conditions are called *separated* boundary conditions. If (4.6) holds, then we say that \widehat{S} is a *non-separated* extension of $S_+ \oplus S_-$ and the corresponding boundary conditions are called *non-separated* boundary conditions.

Notice that multiplying matrices M and N from the left by the same invertible matrix does not change the domain of \widehat{S} . Therefore, we can assume that the block matrix $\begin{pmatrix} M & N \end{pmatrix}$ is in reduced row echelon form. As the rank of $\begin{pmatrix} M & N \end{pmatrix}$ is 2, the reduced row echelon form of $\begin{pmatrix} M & N \end{pmatrix}$ takes one of the following six forms:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \quad \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix},$$

where $*$ stands for an arbitrary complex number.

A straightforward calculations show that the only matrices of the above 6 types for which the corresponding matrix MN^* is self-adjoint are of the following four types:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.7)$$

$$\begin{pmatrix} 1 & \rho e^{i\theta} & 0 & \sigma e^{i\theta} \\ 0 & 0 & 1 & -e^{i\theta}/\rho \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & \alpha & \omega \\ 0 & 1 & \omega^* & \beta \end{pmatrix}, \quad (4.8)$$

where $\omega \in \mathbb{C}$, $\alpha, \beta, \sigma, \theta \in \mathbb{R}$, and $\rho > 0$. Clearly, the matrices in (4.7) give rise to separated boundary conditions, while the first matrix in (4.8) leads to non-separated boundary conditions. The second matrix in (4.8) leads to separated boundary conditions if and only if $\omega = 0$. A similar classification of boundary conditions for a regular Sturm-Liouville problem was established in [14].

Theorem 4.5. *Let S_{\pm} be a closed symmetric densely defined operator with defect numbers $(1, 1)$ in the Hilbert space $(\mathcal{H}_{\pm}, \langle \cdot, \cdot \rangle_{\mathcal{H}_{\pm}})$. A self-adjoint extension \widehat{S} of $S_+ \oplus S_-$ in $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a coupling of S_+ and S_- relative to some boundary triples of S_+^* and S_-^* if and only if \widehat{S} is a non-separated extension of $S_+ \oplus S_-$.*

Proof. Assume that \widehat{S} is a coupling of S_+ and S_- relative to boundary triples $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ and $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$. Let

$$\left\{ \begin{pmatrix} f_+ \\ 0 \end{pmatrix}, \begin{pmatrix} S_+^* f_+ \\ 0 \end{pmatrix} \right\} \in \text{gr } \widehat{S}.$$

Then, by definition of \widehat{S} as a coupling, we have $\Gamma_0^+ f_+ - \Gamma_0^- 0 = 0$ and $\Gamma_1^+ f_+ + \Gamma_1^- 0 = 0$. That is, $\Gamma_0^+ f_+ = 0$ and $\Gamma_1^+ f_+ = 0$, proving that $(\text{gr } \widehat{S}) \cap (\mathcal{H}_+ \oplus \{0\})^2 = \text{gr } S_+$. Similarly, $(\text{gr } \widehat{S}) \cap (\{0\} \oplus \mathcal{H}_-)^2 = \text{gr } S_-$. Hence, \widehat{S} is a non-separated extension.

To prove the converse, assume that \widehat{S} is a non-separated self-adjoint extension of $S_+ \oplus S_-$ in $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$. Let $(\mathbb{C}, \Gamma_0^{\pm}, \Gamma_1^{\pm})$ be a boundary triple for S_{\pm}^* . Then the block matrix $\begin{pmatrix} M & N \end{pmatrix}$ corresponding to the boundary conditions that determine the domain of \widehat{S} are of two forms in (4.8) with $\omega \neq 0$.

First we consider the boundary conditions corresponding to the first matrix in (4.8)

$$\begin{aligned} \Gamma_0^+ f_+ - (e^{i\theta}/\rho) \Gamma_0^- f_- &= 0 \\ \Gamma_1^+ f_+ + (-\sigma e^{i\theta}) \Gamma_0^- f_- + \rho e^{i\theta} \Gamma_1^- f_- &= 0. \end{aligned}$$

These boundary conditions can be rewritten as:

$$\widehat{\Gamma}_0^+(f_+) - \widehat{\Gamma}_0^-(f_-) = 0, \quad \widehat{\Gamma}_1^+(f_+) + \widehat{\Gamma}_1^-(f_-) = 0, \quad (4.9)$$

where

$$\begin{pmatrix} \widehat{\Gamma}_0^+(f_+) \\ \widehat{\Gamma}_1^+(f_+) \end{pmatrix} = \begin{pmatrix} \Gamma_0^+(f_+) \\ \Gamma_1^+(f_+) \end{pmatrix}, \quad \begin{pmatrix} \widehat{\Gamma}_0^-(f_-) \\ \widehat{\Gamma}_1^-(f_-) \end{pmatrix} = e^{i\theta} \begin{pmatrix} 1/\rho & 0 \\ -\sigma & \rho \end{pmatrix} \begin{pmatrix} \Gamma_0^-(f_-) \\ \Gamma_1^-(f_-) \end{pmatrix}$$

By Remark 2.3 the triple $(\mathbb{C}, \widehat{\Gamma}_0^-, \widehat{\Gamma}_1^-)$ is a boundary triple for S_-^* . Therefore, the coupling of S_+ and S_- relative to the boundary triples $(\mathbb{C}, \widehat{\Gamma}_0^+, \widehat{\Gamma}_1^+)$ and $(\mathbb{C}, \widehat{\Gamma}_0^-, \widehat{\Gamma}_1^-)$ coincides with the operator \widehat{S} .

Next we consider the boundary conditions corresponding to the second matrix in (4.8) with $\omega \neq 0$. These boundary conditions are

$$\omega^* \Gamma_0^+ f_+ - (\Gamma_1^- f_- - \beta \Gamma_0^- f_-) = 0 \quad (4.10)$$

$$\frac{1}{\omega} \Gamma_1^+ f_+ - \frac{\alpha}{\omega} \Gamma_0^+ f_+ + (-\Gamma_0^- f_-) = 0 \quad (4.11)$$

Setting

$$\begin{pmatrix} \widehat{\Gamma}_0^+(f_+) \\ \widehat{\Gamma}_1^+(f_+) \end{pmatrix} = \begin{pmatrix} \omega^* & 0 \\ -\frac{\alpha}{\omega} & \frac{1}{\omega} \end{pmatrix} \begin{pmatrix} \Gamma_0^+(f_+) \\ \Gamma_1^+(f_+) \end{pmatrix}, \quad \begin{pmatrix} \widehat{\Gamma}_0^-(f_-) \\ \widehat{\Gamma}_1^-(f_-) \end{pmatrix} = \begin{pmatrix} -\beta & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Gamma_0^-(f_-) \\ \Gamma_1^-(f_-) \end{pmatrix}$$

one can rewrite boundary conditions (4.10), (4.11) in the form (4.9). By Remark 2.3 the triple $(\mathbb{C}, \widehat{\Gamma}_0^+, \widehat{\Gamma}_1^+)$ is a boundary triple for S_+^* and the triple $(\mathbb{C}, \widehat{\Gamma}_0^-, \widehat{\Gamma}_1^-)$ is a boundary triple for S_-^* . Therefore the coupling of S_+ and S_- relative to the boundary triple $(\mathbb{C}, \widehat{\Gamma}_0^+, \widehat{\Gamma}_1^+)$ for S_+^* and the boundary triple $(\mathbb{C}, \widehat{\Gamma}_0^-, \widehat{\Gamma}_1^-)$ for S_-^* coincides with the operator \widehat{S} . This completes the proof. \square

4.3. A partially fundamentally reducible operator as a coupling. Given two symmetric operators, S_+ in the Hilbert space \mathcal{H}_+ and S_- in the Hilbert space \mathcal{H}_- , in Subsection 4.1 we constructed a coupling S_1 of these two operators which is a self-adjoint operator in the Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Next we introduce a Kreĩn space structure on \mathcal{H} . Let J be a self-adjoint involution on $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ defined by

$$J \begin{pmatrix} f_+ \\ f_- \end{pmatrix} := \begin{pmatrix} f_+ \\ -f_- \end{pmatrix}, \quad f_+ \in \mathcal{H}_+, \quad f_- \in \mathcal{H}_-. \quad (4.12)$$

This involution induces an indefinite inner product on \mathcal{H} :

$$[f, g]_{\mathcal{H}} := \langle Jf, g \rangle_{\mathcal{H}}, \quad f, g \in \mathcal{H},$$

and with this inner product $(\mathcal{H}, [\cdot, \cdot]_{\mathcal{H}})$ is a Kreĩn space.

With S_0 and S_1 from Theorem 4.1 we define $A_0 := JS_0$ and $A_1 := JS_1$. Since S_0 and S_1 are self-adjoint in the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$, A_0 and A_1 are self-adjoint in the Kreĩn space $(\mathcal{H}, [\cdot, \cdot]_{\mathcal{H}})$. In fact, A_0 and A_1 are given by

$$A_0 \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} S_0^+ f_+ \\ -S_0^- f_- \end{pmatrix}, \quad \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in \text{dom}(A_0) = \text{dom}(S_0), \quad (4.13)$$

$$A_1 \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} S_+^* f_+ \\ -S_-^* f_- \end{pmatrix}, \quad \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in \text{dom}(A_1) = \text{dom}(S_1). \quad (4.14)$$

The operator A_1 will be called the *coupling* of the operators S_+ and $-S_-$ in the Kreĩn space $(\mathcal{H}, [\cdot, \cdot]_{\mathcal{H}})$.

In the rest of this subsection we will proceed in the opposite direction: Given a partially fundamentally reducible self-adjoint operator A in the Kreĩn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ we will prove that A is a coupling of two operators.

Recall that Definition 1.1 associates a fundamental decomposition $\mathcal{K} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-$ of $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ and symmetric operators S_+ and S_- with a partially fundamentally reducible operator A in a Kreĩn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$. As before, S_{\pm}^* denotes the adjoint of S_{\pm} in the Hilbert space $(\mathcal{K}_{\pm}, \pm[\cdot, \cdot]_{\mathcal{K}})$. By P_+ and P_- we denote the orthogonal projections and by J the fundamental symmetry corresponding to the fundamental decomposition $\mathcal{K} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-$, while $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ denotes the corresponding Hilbert space inner product and $\|\cdot\|_{\mathcal{K}}$ denotes the norm induced by $\langle \cdot, \cdot \rangle_{\mathcal{K}}$. The notation related to a partially fundamentally reducible operator A introduced in this paragraph is used throughout the rest of the paper.

Theorem 4.6. *Let A be a partially fundamentally reducible operator in a Kreĩn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$. The following statements hold.*

(a) *We have $\text{dom}(S_{\pm}^*) = P_{\pm} \text{dom}(A)$ and*

$$\pm S_{\pm}^* P_{\pm} f = P_{\pm} A f \quad \text{for all } f \in \text{dom}(A).$$

(b) *Let $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ be a boundary triple for S_+^* . Then the equalities*

$$\Gamma_0^- P_- f = \Gamma_0^+ P_+ f, \quad \Gamma_1^- P_- f = -\Gamma_1^+ P_+ f, \quad f \in \text{dom}(A), \quad (4.15)$$

define a boundary triple $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$ for S_- .

(c) *We have $A = JS_1$ where S_1 is the coupling of the operators S_+ and S_- in the Hilbert space $(\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})$ relative to the boundary triples $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ and $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$.*

- (d) If A is nonnegative in $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ and the operator $S_0^+ := S_+^*|_{\ker(\Gamma_0^+)}$ coincides with Kreĭn's or Friedrichs' extension of S_+ , then $S_0^- := S_-^*|_{\ker(\Gamma_0^-)}$ is a nonnegative operator in the Hilbert space $(\mathcal{K}_-, -[\cdot, \cdot]_{\mathcal{K}})$.

Proof. The statements (a) and (b) are proved in [20, Lemma 5.1].

(c) It follows from (4.2) and (4.15) that $\text{dom}(A) = \text{dom}(S_1)$. The equality $A = JA_1$ follows from (4.14).

The last statement is implied by Corollaries 4.3 and 4.4. \square

The results of the next theorem can be derived from [19]. However, we prefer to present a direct proof for this elementary case when defect numbers of S_{\pm} are $(1, 1)$. In the case of indefinite Sturm-Liouville operator similar construction has been used in [32, Proposition 2.5].

Theorem 4.7. *Let A be a partially fundamentally reducible operator in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ and let $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ and $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$ be the boundary triples from Theorem 4.6(b). Let m_{\pm} be the Weyl function and let ψ_{\pm} be the Weyl solution of S_{\pm} relative to the boundary triple $(\mathbb{C}, \Gamma_0^{\pm}, \Gamma_1^{\pm})$. Let the operator A_0 be given by (4.13). Then*

$$\rho(A) \setminus \mathbb{R} = \{z \in \mathbb{C} \setminus \mathbb{R} : m_+(z) + m_-(-z) \neq 0\}. \quad (4.16)$$

If $\rho(A) \setminus \mathbb{R}$ is a nonempty set, then for every $z \in \rho(A) \setminus \mathbb{R}$ and every $h \in \mathcal{K}$ the resolvent of A is given by

$$(A - z)^{-1}h = (A_0 - z)^{-1}h - \frac{[h, \psi(z^*)]_{\mathcal{K}}}{m_+(z) + m_-(-z)} \psi(z), \quad (4.17)$$

where

$$\psi(z) = \psi_+(z) + \psi_-(-z).$$

For $h \in \mathcal{K}_+$ the vector $f = P_+(A - z)^{-1}h$ is the solution of the z -dependent boundary value problem

$$(S_+^* - z)f = h, \quad \Gamma_1^+ f + m_-(-z)\Gamma_0^+ f = 0. \quad (4.18)$$

Proof. Recall that the operator A coincides with the coupling A_1 defined by (4.14).

Let z be an arbitrary non-real number. By the definition of A_1 the equation $(A_1 - z)f = h$ with $f \in \text{dom}(S_1)$ and $h \in \mathcal{K}$ is equivalent to the system

$$(S_+^* - z)f_+ = h_+, \quad -(S_-^* + z)f_- = h_-, \quad (4.19)$$

where $f_+ = P_+f$, $f_- = P_-f$, $h_+ = P_+h$ and $h_- = P_-h$. It follows from (2.2) and (4.19) that $f_+ \in \text{dom}(S_+^*)$ can be expressed as

$$f_+ = (S_0^+ - z)^{-1}h_+ + c_+\psi_+(z) \quad \text{with some } c_+ \in \mathbb{C}. \quad (4.20)$$

Similarly, $f_- \in \text{dom}(S_-^*)$ can be expressed as

$$f_- = -(S_0^- + z)^{-1}h_- + c_-\psi_-(-z) \quad \text{with some } c_- \in \mathbb{C}. \quad (4.21)$$

By (2.4) and (2.7) we have

$$\begin{aligned} \Gamma_0^+ f_+ &= c_+, \\ \Gamma_0^- f_- &= c_-, \\ \Gamma_1^+ f_+ &= \Gamma_1^+ (S_0^+ - z)^{-1}h_+ + c_+m_+(z) \\ &= [h_+, \psi_+(z^*)]_{\mathcal{K}} + c_+m_+(z), \end{aligned}$$

$$\begin{aligned}\Gamma_1^- f_- &= -\Gamma_1^- (S_0^- + z)^{-1} h_- + c_- m_-(-z) \\ &= [h_-, \psi_-(-z^*)]_{\mathcal{K}} + c_- m_-(-z).\end{aligned}$$

As f_+ and f_- satisfy the boundary conditions (4.15) in the definition of $\text{dom}(A_1) = \text{dom}(S_1)$ we get

$$c_+ - c_- = 0,$$

$$\left([h_+, \psi_+(z^*)]_{\mathcal{K}} + c_+ m_+(z) \right) + \left([h_-, \psi_-(-z^*)]_{\mathcal{K}} + c_- m_-(-z) \right) = 0.$$

For each z such that $m_+(z) + m_-(-z) \neq 0$ the above system has a unique solution for c_+, c_- :

$$c_+ = c_- = -\frac{[h_+, \psi_+(z^*)]_{\mathcal{K}} + [h_-, \psi_-(-z^*)]_{\mathcal{K}}}{m_+(z) + m_-(-z)} = \frac{-[h, \psi(z^*)]_{\mathcal{K}}}{m_+(z) + m_-(-z)}. \quad (4.22)$$

Now (4.20), (4.21) and (4.22) imply that whenever $m_+(z) + m_-(-z) \neq 0$, for arbitrary $h \in \mathcal{K}$, the system (4.19) has a unique solution $f \in \text{dom}(A_1)$ and (4.17) holds. This also proves that the right hand side of (4.16) is a subset of the left hand side.

To prove the converse inclusion in (4.16) assume that $m_+(z) + m_-(-z) = 0$ for some $z \in \mathbb{C} \setminus \mathbb{R}$. Then (2.3) and (2.4) imply that

$$\begin{aligned}\Gamma_0^+ \psi_+(z) - \Gamma_0^- \psi_-(-z) &= 1 - 1 = 0 \\ \Gamma_1^+ \psi_+(z) + \Gamma_1^- \psi_-(-z) &= m_+(z) + m_-(-z) = 0.\end{aligned}$$

In view of (4.2) this means that $\psi_+(z) + \psi_-(-z) \in \text{dom}(A_1) = \text{dom}(S_1)$. Since

$$(A_1 - z)\psi(z) = (S_+^* - z)\psi_+(z) + -(S_-^* + z)\psi_-(-z) = 0,$$

z is an eigenvalue of A_1 . That is $z \notin \rho(A_1) \setminus \mathbb{R}$.

To prove (4.18), set $h = h_+ \in \mathcal{K}_+$. Then the first equality in (4.18) is implied by (4.19). Since (4.21) takes the form $f_- = c_- \psi_-(-z) \in \ker(S_-^* + z)$, then

$$\Gamma_1^- f_- = m_-(-z) \Gamma_0^- f_- = m_-(-z) \Gamma_0^+ f_+.$$

In view of (4.15) this proves the second equality in (4.18). \square

Corollary 4.8. *Let the assumptions of Theorem 4.7 hold. Then*

$$m_+(z) + m_-(-z) = 0 \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R} \quad (4.23)$$

if and only if the resolvent set of A is empty. In this case $\mathbb{C} \setminus \mathbb{R} \subset \sigma_p(A)$.

As we pointed out in Subsection 2.8, a nonnegative self-adjoint operator in Kreĭn space can have an empty resolvent set. Next we show that this cannot happen for a nonnegative partially fundamentally reducible operator.

Corollary 4.9. *A nonnegative partially fundamentally reducible operator A in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ has a nonempty resolvent set.*

Proof. Assume that $\rho(A) = \emptyset$. Since A is nonnegative in $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$, then S_+ and S_- are nonnegative in the Hilbert spaces $(\mathcal{K}_+, [\cdot, \cdot]_{\mathcal{K}})$ and $(\mathcal{K}_-, -[\cdot, \cdot]_{\mathcal{K}})$, respectively. Hence each of the Weyl functions m_+ and m_- has at most 1 pole on \mathbb{R}_- . By Corollary 4.8 the equality (4.23) holds, which implies that m_+ is a rational function with at most three poles on \mathbb{R} . But this contradicts the assumption that S_+ is a densely defined operator. \square

The claim in the last corollary can also be deduced from [4, Corollary 2.5]. We notice that for a class of indefinite Sturm-Liouville operators the nonemptiness of the resolvent set was proved in [32, Proposition 2.5(iv)] and [41, Theorem 5.3] using the known asymptotic behavior of the Titchmarsh-Weyl coefficients.

5. REGULARITY OF CRITICAL POINTS

Throughout this section we use the notation introduced in the paragraph preceding Theorem 4.6 and in Theorem 4.7. By Corollary 4.9 a nonnegative partially fundamentally reducible operator A in a Kreĭn space has a nonempty resolvent set. Therefore A has a spectral function with critical points 0 and ∞ . In this section we study these critical points in terms of the Weyl functions m_+ and m_- .

5.1. D -properties of pairs of Nevanlinna functions. The following necessary conditions for the regularity of the critical points 0 and ∞ of an indefinite Sturm-Liouville operator in terms of the Titchmarsh-Weyl coefficients were established in [32, Corollary 5.3] (see also [33, Theorem 3.4]).

Theorem 5.1. ([32], [33]) *Let A be a nonnegative partially fundamentally reducible operator in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ and let m_+ and m_- be Weyl functions introduced in Theorem 4.7. Then the following two statements hold.*

(a) *If $\infty \notin c_s(A)$, then the functions*

$$z \mapsto \frac{\operatorname{Im} m_+(z)}{|m_+(z) + m_-(-z)|}, \quad z \mapsto \frac{\operatorname{Im} m_-(z)}{|m_+(z) + m_-(-z)|} \quad (5.1)$$

are defined on \mathbb{C}_+ and are bounded on $\Omega_R^\infty = \{z \in \mathbb{C}_+ : |z| > R\}$ for each $R > 0$;

(b) *If $0 \notin c_s(A)$, then the functions in (5.1) are defined on \mathbb{C}_+ and are bounded on $\Omega_R^0 = \{z \in \mathbb{C}_+ : |z| < R\}$ for each $R > 0$.*

Let us sketch the proof of (a) from [33] using the notation of Theorem 4.7. Since $\infty \notin c_s(A_1)$ the functions

$$z \mapsto (\operatorname{Im} z) \|(A - z)^{-1}\|, \quad z \mapsto (\operatorname{Im} z) \|(S_+ - z)^{-1}\| \quad \text{and} \quad z \mapsto (\operatorname{Im} z) \|(S_- - z)^{-1}\|$$

are bounded on Ω_R^∞ . Then it follows from (4.17) and the equalities

$$\|\psi_+(z)\|_{\mathcal{K}_+}^2 = \frac{\operatorname{Im} m_+(z)}{\operatorname{Im} z} \quad \text{and} \quad \|\psi_-(z)\|_{\mathcal{K}_-}^2 = \frac{\operatorname{Im} m_-(z)}{\operatorname{Im} z}$$

that the ratio in (5.2) is also bounded on Ω_R^∞ .

Clearly, this proof works for arbitrary nonnegative coupling A in Theorem 4.6.

Definition 5.2. We say that a pair (m_+, m_-) of two Nevanlinna functions has D_∞ -property (D_0 -property), if the function

$$y \mapsto \frac{\operatorname{Im} m_+(iy) + \operatorname{Im} m_-(iy)}{|m_+(iy) + m_-(iy)|}, \quad (5.2)$$

is bounded on the set $(1, \infty)$ ($(0, 1)$, respectively). In particular, a Nevanlinna function m_+ is said to have D_∞ -property (D_0 -property), if the pair (m_+, m_+) has D_∞ -property (D_0 -property, respectively), that is, if the function

$$y \mapsto \frac{\operatorname{Im} m_+(iy)}{\operatorname{Re} m_+(iy)},$$

is bounded on the set $(1, \infty)$ ($(0, 1)$, respectively).

It follows from Theorem 5.1 that D_∞ -property (D_0 -property) is necessary for the conditions $\infty \notin c_s(A_1)$ ($0 \notin c_s(A_1)$, respectively).

Corollary 5.3. ([32], [33]) *Let A be a nonnegative partially fundamentally reducible operator in a Krein space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ and let m_+ and m_- be Weyl functions introduced in Theorem 4.7. Then the following statements hold.*

- (i) *If $\infty \notin c_s(A)$, then the pair (m_+, m_-) has the D_∞ -property.*
- (ii) *If $0 \notin c_s(A)$, then the pair (m_+, m_-) has the D_0 -property.*

For a nonnegative densely defined symmetric operator S with defect numbers $(1, 1)$ in a Hilbert space we can always choose a boundary triple relative to which the Weyl function of S is a Stieltjes function. Therefore the following proposition is of interest.

Proposition 5.4. *If m_+ is a Stieltjes function with D_∞ -property (D_0 -property), then for every Stieltjes function m_- the pair (m_+, m_-) has D_∞ -property (D_0 -property, respectively).*

Proof. Since m_\pm is a Stieltjes function, we have $\operatorname{Re} m_\pm(iy) \geq 0$ for all $y \in \mathbb{R}_+$. Therefore,

$$\operatorname{Re}(m_+(iy)) \leq |m_+(iy) + m_-(-iy)| \quad \text{for all } y \in \mathbb{R}_+. \quad (5.3)$$

As m_+ has D_∞ -property there exists $C > 0$ such that

$$\frac{\operatorname{Im} m_+(iy)}{\operatorname{Re} m_+(iy)} \leq C \quad \text{for all } y > 1.$$

By (5.3) we have

$$\frac{\operatorname{Im} m_+(iy)}{|m_+(iy) + m_-(-iy)|} \leq \frac{\operatorname{Im} m_+(iy)}{\operatorname{Re} m_+(iy)} \leq C \quad \text{for all } y > 1.$$

Further, the inequality

$$|\operatorname{Im}(m_+(iy) - m_-(iy))| \leq |m_+(iy) + m_-(-iy)| \quad \text{for all } y \in \mathbb{R}_+$$

yields that for all $y > 1$ we have

$$\begin{aligned} \frac{\operatorname{Im} m_-(iy)}{|m_+(iy) + m_-(-iy)|} &\leq \frac{|\operatorname{Im}(m_-(iy) - m_+(iy))|}{|m_+(iy) + m_-(-iy)|} + \frac{\operatorname{Im} m_+(iy)}{|m_+(iy) + m_-(-iy)|} \\ &\leq C + 1. \end{aligned}$$

The proof of the D_0 -property is similar. \square

Asymptotic behavior of Weyl functions has been investigated for many kinds of special symmetric operators. Some specific example appear in Section 6 below. The next proposition deduces D -properties from the asymptotic behavior of Nevanlinna functions introduced in Subsection 2.7.

Proposition 5.5. (a) *If Nevanlinna functions m_+ and m_- belong to \mathcal{A}_∞ , then the pair (m_+, m_-) has D_∞ -property.*
 (b) *If Nevanlinna functions m_+ and m_- belong to \mathcal{A}_0 , then the pair (m_+, m_-) has D_0 -property.*

Proof. Let $m_{\pm} \in \mathcal{A}_{\infty}$. Proposition 2.13 applies. Let $C_0^{\pm} \in \mathbb{R} \setminus \{0\}$, $\alpha_0^{\pm} \in (-1, 1)$, and, as appropriate, $C_1^{\pm} \in \mathbb{R} \setminus \{0\}$ and $\alpha_1^{\pm} \in (-1, 0)$ be the corresponding coefficients appearing in (2.27) and (2.28). We first notice that as a consequence of the asymptotics in (2.27) or (2.28) we have

$$\lim_{y \uparrow +\infty} \frac{m_{\pm}(iy)}{y^{\alpha_0^{\pm}}} = C_0^{\pm} i^{\alpha_0^{\pm}}, \quad (5.4)$$

which, in turn, implies

$$\lim_{y \uparrow +\infty} \frac{\operatorname{Im} m_{\pm}(iy)}{y^{\alpha_0^{\pm}}} = C_0^{\pm} \sin(\pi \alpha_0^{\pm}/2). \quad (5.5)$$

Let

$$\alpha_0 = \begin{cases} \max\{\alpha_0^+, \alpha_0^-\} & \text{if } \max\{\alpha_0^+, \alpha_0^-\} \geq 0, \\ \min\{\alpha_0^+, \alpha_0^-\} & \text{if } \max\{\alpha_0^+, \alpha_0^-\} < 0. \end{cases}$$

and, further, set $\epsilon_{\pm} = 1 - \operatorname{sgn}(|\alpha_0 - \alpha_0^{\pm}|)$. By definition $\epsilon_+, \epsilon_- \in \{0, 1\}$, and at least one of ϵ_+, ϵ_- equals 1.

Next we will calculate the limit

$$\lim_{y \uparrow +\infty} \frac{\operatorname{Im} m_+(iy) + \operatorname{Im} m_-(iy)}{|m_+(iy) + m_-(-iy)|} \quad (5.6)$$

and doing so we prove (a). Using (5.4) and (5.5) this limit can be calculated as follows:

$$\lim_{y \uparrow +\infty} \frac{\frac{\operatorname{Im} m_+(iy)}{y^{\alpha_0}} + \frac{\operatorname{Im} m_-(iy)}{y^{\alpha_0}}}{\left| \frac{m_+(iy)}{y^{\alpha_0}} + \frac{m_-(-iy)}{y^{\alpha_0}} \right|} = \frac{\epsilon_+ C_0^+ \sin(\pi \alpha_0^+/2) + \epsilon_- C_0^- \sin(\pi \alpha_0^-/2)}{|\epsilon_+ C_0^+ i^{\alpha_0^+} + \epsilon_- C_0^- (-i)^{\alpha_0^-}|}. \quad (5.7)$$

If exactly one of ϵ_+, ϵ_- equals 1, then the denominator in the last fraction is clearly positive. Otherwise, that is, if $\epsilon_+ = \epsilon_- = 1$, we have $\alpha_0^+ = \alpha_0^-$ and the denominator is again clearly positive provided that $\alpha_0^+ = \alpha_0^- \neq 0$. If $\alpha_0^+ = \alpha_0^- = 0$, the denominator equals $|C_0^+ + C_0^-|$.

Thus, the limit in (5.6) is calculated to be the right-hand side of (5.7) with the exception of the case when $\alpha_0^+ = \alpha_0^- = 0$ and $C_0^+ + C_0^- = 0$.

To calculate the limit in (5.6) when $\alpha_0^+ = \alpha_0^- = 0$ and $C_0^+ + C_0^- = 0$ we use the second term of the asymptotics in (2.28). First we deduce

$$\lim_{y \uparrow +\infty} \frac{m_{\pm}(iy) - C^{\pm}}{y^{\alpha_1^{\pm}}} = C_1^{\pm} i^{\alpha_1^{\pm}},$$

which clearly yields

$$\lim_{y \uparrow +\infty} \frac{\operatorname{Im} m_{\pm}(iy)}{y^{\alpha_1^{\pm}}} = C_1^{\pm} \sin(\pi \alpha_1^{\pm}/2).$$

Similarly as before, we introduce $\alpha_1 = \min\{\alpha_1^+, \alpha_1^-\}$ and $\epsilon_1^{\pm} = 1 - \operatorname{sgn}(|\alpha_1 - \alpha_1^{\pm}|) \in \{0, 1\}$.

Since we assume that $C_0^+ + C_0^- = 0$, the limit in (5.6) equals

$$\lim_{y \uparrow +\infty} \frac{\frac{\operatorname{Im} m_+(iy)}{y^{\alpha_1}} + \frac{\operatorname{Im} m_-(iy)}{y^{\alpha_1}}}{\left| \frac{m_+(iy) - C_0^+}{y^{\alpha_1}} + \frac{m_-(-iy) - C_0^-}{y^{\alpha_1}} \right|} = \frac{\epsilon_1^+ C_1^+ \sin(\pi \alpha_1^+/2) + \epsilon_1^- C_1^- \sin(\pi \alpha_1^-/2)}{|\epsilon_1^+ C_1^+ i^{\alpha_1^+} + \epsilon_1^- C_1^- (-i)^{\alpha_1^-}|}.$$

Now recall that at least one of $\epsilon_1^-, \epsilon_1^+$ equals 1, $\alpha_1^-, \alpha_1^+ \in (-1, 0)$ and $C_1^-, C_1^+ \in \mathbb{R} \setminus \{0\}$, to deduce that the denominator of the last fraction is always positive.

Thus, in each case we calculated the limit in (5.6). This proves that the pair (m_+, m_-) has the D_∞ -property.

The proof of (b) is similar. \square

5.2. Regularity of the critical points. Let A be a nonnegative partially fundamentally reducible operator in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$. In this subsection we provide sufficient conditions in terms of m_+ and m_- for the points ∞ or 0 not to be singular critical points of the operator A .

In the next lemma we use Theorem 2.16 to obtain criteria for the regularity of the critical points 0 and ∞ for the operator A_1 defined by (4.14) and (4.2). To formulate these results we need the following family of operators. For arbitrary ϵ, η such that $0 < \epsilon < \eta$ we define the operator $T_{\pm\pm}(\epsilon, \eta) : \mathcal{K}_{\pm} \rightarrow \mathcal{K}_{\pm}$ by

$$(T_{\pm\pm}(\epsilon, \eta))f_{\pm} := \int_{\epsilon \leq |y| \leq \eta} \frac{\langle f_{\pm}, \psi_{\pm}(\mp iy) \rangle_{\mathcal{K}} \psi_{\pm}(\pm iy)}{m_+(iy) + m_-(-iy)} dy, \quad f_{\pm} \in \mathcal{K}_{\pm}, \quad (5.8)$$

and the operator $T_{\pm\mp}(\epsilon, \eta) : \mathcal{K}_{\mp} \rightarrow \mathcal{K}_{\pm}$ by

$$(T_{\pm\mp}(\epsilon, \eta))f_{\mp} := \int_{\epsilon \leq |y| \leq \eta} \frac{\langle f_{\mp}, \psi_{\mp}(\pm iy) \rangle_{\mathcal{K}} \psi_{\pm}(\pm iy)}{m_+(iy) + m_-(-iy)} dy, \quad f_{\mp} \in \mathcal{K}_{\mp}. \quad (5.9)$$

Lemma 5.6. *Let A be a nonnegative partially fundamentally reducible operator in the Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$. Then the following statements hold.*

- (i) $\infty \notin c_s(A)$ if and only if the operators $T_{++}(1, \eta)$, $T_{--}(1, \eta)$, $T_{+-}(1, \eta)$ and $T_{-+}(1, \eta)$ are uniformly bounded for $\eta \in (1, \infty)$;
- (ii) $0 \notin c_s(A)$ if and only if $\ker(A) = \ker(A^2)$ and the operators $T_{++}(\epsilon, 1)$, $T_{--}(\epsilon, 1)$, $T_{+-}(\epsilon, 1)$ and $T_{-+}(\epsilon, 1)$ are uniformly bounded for $\epsilon \in (0, 1)$.

Proof. Notice first that by Corollary 4.9 the operator A has a nonempty resolvent set. Set $A_0 = S_0^+ \oplus (-S_0^-)$, see Theorem 4.1(f) and (4.13). Since A_0 is self-adjoint in the Hilbert space $(\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})$, the family of operators

$$\int_{1 \leq |y| \leq \eta} (A_0 - iy)^{-1} dy, \quad \eta > 1,$$

is uniformly bounded. Recall from Theorem 4.7 that $\psi(z) = \psi_+(z) + \psi_-(-z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, and let $f \in \mathcal{K}$ be arbitrary. By (4.17) we have

$$\begin{aligned} & \int_{1 \leq |y| \leq \eta} \left((A - iy)^{-1} f - (A_0 - iy)^{-1} f \right) dy \\ &= \int_{1 \leq |y| \leq \eta} \frac{-[f, \psi(-iy)]_{\mathcal{K}} \psi(iy)}{m_+(iy) + m_-(-iy)} dy \\ &= \begin{pmatrix} T_{++}(1, \eta) & T_{+-}(1, \eta) \\ T_{-+}(1, \eta) & T_{--}(1, \eta) \end{pmatrix} \begin{pmatrix} -P_+ f \\ P_- f \end{pmatrix}. \end{aligned}$$

Therefore, the uniform boundedness of the family of operators

$$\int_{1 \leq |y| \leq \eta} (A_1 - iy)^{-1} dy, \quad \eta > 1,$$

is equivalent to the uniform boundedness of the families of operators $T_{++}(1, \eta)$, $T_{--}(1, \eta)$, $T_{+-}(1, \eta)$ and $T_{-+}(1, \eta)$ for $\eta \in (1, \infty)$. Now (i) follows from Theorem 2.16(a).

The statement (ii) is implied by the formula (4.17) and Theorem 2.16(b). \square

Theorem 5.7. *Let A be a nonnegative partially fundamentally reducible operator in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ and assume that Weyl functions m_+ and m_- introduced in Theorem 4.7 have B_∞ -property (see Definition 3.1). Then $\infty \notin c_s(A)$ if and only if the pair (m_+, m_-) has D_∞ -property.*

Proof. For the necessity see Corollary 5.3.

To prove the sufficiency consider first $f_+, g_+ \in \mathcal{K}_+$ and find an upper bound for

$$\left| \langle T_{++}(1, \eta) f_+, g_+ \rangle_{\mathcal{K}} \right|.$$

By the definition of the generalized Fourier transform $F_+ = F_{m_+}$ we have

$$\langle T_{++}(1, \eta) f_+, g_+ \rangle_{\mathcal{K}} = \int_{1 \leq |y| \leq \eta} \frac{(F_+ f_+)(iy) ((F_+ g_+)(-iy))^*}{m_+(iy) + m_-(-iy)} dy.$$

Since the pair (m_+, m_-) has D_∞ -property, there exists $C_1 > 0$ such that

$$|m_+(iy) + m_-(-iy)| \geq C_1^{-1} \operatorname{Im} m_+(iy) \quad \text{for all } y > 1. \quad (5.10)$$

Set $w_\pm(y) = (\operatorname{Im} m_\pm(iy))^{-1}$, $y \in \mathbb{R}_+$. Since m_+ has B_∞ -property, then in view of Definition 3.1 and Proposition 3.3 the mappings

$$\begin{aligned} G_{m_+, \infty}^+ F_+ : f_+ &\mapsto (F_+ f_+)(iy), \quad f_+ \in \mathcal{K}_+, \\ G_{m_+, \infty}^- F_+ : f_+ &\mapsto (F_+ f_+)(-iy), \quad f_+ \in \mathcal{K}_+, \end{aligned}$$

are bounded from \mathcal{K}_+ to $L_{w_+}^2(1, \infty)$ and hence there exists $C_2 > 0$, such that

$$\|G_{m_+, \infty}^+ F_+ f_+\|_{L_{w_+}^2(1, \infty)} \leq C_2 \|f_+\|_{\mathcal{K}} \quad (5.11)$$

and

$$\|G_{m_+, \infty}^- F_+ f_+\|_{L_{w_+}^2(1, \infty)} \leq C_2 \|f_+\|_{\mathcal{K}} \quad (5.12)$$

for all $f_+ \in \mathcal{K}_+$. Using (5.10), (5.11), (5.12) and (2.14) we obtain

$$\begin{aligned} \left| \langle T_{++}(1, \eta) f_+, g_+ \rangle_{\mathcal{K}} \right| &\leq C_1 \int_{1 \leq |y| \leq \eta} |(F_+ f_+)(iy)| |(F_+ g_+)(-iy)| w_+(y) dy \\ &\leq C_1 \left(\int_{1 \leq |y| \leq \eta} |(F_+ f_+)(iy)|^2 w_+(y) dy \right)^{1/2} \\ &\quad \times \left(\int_{1 \leq |y| \leq \eta} |(F_+ g_+)(iy)|^2 w_+(y) dy \right)^{1/2} \end{aligned}$$

$$\leq 2C_1 C_2^2 \|f_+\|_{\mathcal{K}} \|g_+\|_{\mathcal{K}}$$

for all $f_+, g_+ \in \mathcal{K}_+$. This proves that the family $T_{++}(1, \eta)$ is uniformly bounded for $\eta \in (1, \infty)$.

The proof for the families $T_{--}(1, \eta)$ and $T_{+-}(1, \eta)$ is similar.

Now consider $f_+ \in \mathcal{K}_+$ and $g_- \in \mathcal{K}_-$ and give an upper bound for

$$\left| \langle T_{-+}(\epsilon, \eta) f_+, g_- \rangle_{\mathcal{K}} \right|.$$

Since the function in (5.2) is bounded, then the function

$$y \mapsto \frac{\sqrt{\operatorname{Im} m_+(iy)} \sqrt{\operatorname{Im} m_-(iy)}}{|m_+(iy) + m_-(iy)|} \quad \text{for all } y > 1,$$

is bounded as well. Therefore, there exists $C_3 > 0$ such that

$$|m_+(iy) + m_-(iy)| \geq C_2 \sqrt{\operatorname{Im} m_+(iy)} \sqrt{\operatorname{Im} m_-(iy)} \quad \text{for all } y > 1,$$

Consequently, with $F_- = F_{m_-}$, we have

$$\begin{aligned} \left| \langle T_{-+}(1, \eta) f_+, g_- \rangle_{\mathcal{K}} \right| &\leq \int_{1 \leq |y| \leq \eta} \left| \frac{(F_+ f_+)(iy) ((F_- g_-)(-iy))^*}{m_+(iy) + m_-(iy)} \right| dy \\ &\leq C_3 \int_{1 \leq |y| \leq \eta} |(F_+ f_+)(iy)| |(F_- g_-)(-iy)| \sqrt{w_+(y)} \sqrt{w_-(y)} dy \\ &\leq 2C_2 C_3 \|G_{m_+} F_+ f_+\|_{L_{w_+}^2(1, \infty)} \|G_{m_-} F_- g_-\|_{L_{w_-}^2(1, \infty)} \\ &\leq 2C_2 C_3 \|f_+\|_{\mathcal{K}} \|g_-\|_{\mathcal{K}} \end{aligned}$$

for all $f_+ \in \mathcal{K}_+$ and all $g_- \in \mathcal{K}_-$. This proves that the family $T_{-+}(1, \eta)$ is uniformly bounded for $\eta \in (1, \infty)$.

The uniform boundedness of $T_{+-}(1, \eta)$ is proved similarly. Now the statement is implied by Lemma 5.6. \square

Theorem 5.8. *Let A be a nonnegative partially fundamentally reducible operator in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ and assume that Weyl functions m_+ and m_- introduced in Theorem 4.7 have B_0 -property. Then $0 \notin c_s(A)$ if and only if $\ker(A) = \ker(A^2)$ and the pair (m_+, m_-) has D_0 -property.*

Proof. We prove sufficiency. For necessity see Corollary 5.3.

Since m_+ and m_- have D_0 -property, there exists $C_4 > 0$ such that

$$|m_+(iy) + m_-(iy)| \geq C_4 \operatorname{Im} m_+(iy) \quad \text{for all } y \in (0, 1). \quad (5.13)$$

Since m_+ has B_0 -property, the mappings

$$G_{m_+, 0}^+ F_+^+ : f_+ \mapsto (F_+ f_+)(iy) \quad (f_+ \in \mathcal{K}_+)$$

$$G_{m_+, 0}^- F_+^- : f_+ \mapsto (F_+ f_+)(-iy) \quad (f_+ \in \mathcal{K}_+)$$

are bounded from \mathcal{K}_+ to $L_{w_+}^2(0, 1)$ and hence there exists $C_5 > 0$, such that

$$\|G_{m_+, 0}^+ F_+ f_+\|_{L_{w_+}^2(0, 1)} \leq C_5 \|f_+\|_{\mathcal{K}}, \quad (5.14)$$

$$\|G_{m_+, 0}^- F_+ f_+\|_{L_{w_+}^2(0, 1)} \leq C_5 \|f_+\|_{\mathcal{K}} \quad (5.15)$$

for all $f_+ \in \mathcal{K}_+$. Using (5.13), (5.14), (5.15) and (2.14) we obtain

$$\begin{aligned} \left| \langle T_{++}(\epsilon, 1)f_+, g_+ \rangle_{\mathcal{K}} \right| &\leq C_4 \int_{\epsilon \leq |y| \leq 1} |(F_+f_+)(iy)| |(F_+g_+)(-iy)| w_+(y) dy \\ &\leq C_4 \|G_{m_+, 0}^+ F_+ f_+\|_{L_{w_+}^2(0,1)} \|G_{m_+, 0}^- F_+ g_+\|_{L_{w_+}^2(0,1)} \\ &\leq 2C_4 C_5^2 \|f_+\|_{\mathcal{K}} \|g_+\|_{\mathcal{K}} \end{aligned}$$

for all $f_+, g_+ \in \mathcal{K}_+$. This proves that the family $T_{++}(\epsilon, 1)$ is uniformly bounded for $\epsilon \in (0, 1)$.

The proof for the family $T_{--}(\epsilon, 1)$ is similar. Now the statement is implied by Lemma 5.6. \square

Proposition 2.15 together with Theorems 5.7 and 5.8 yield the following statement.

Corollary 5.9. *Let A be a nonnegative partially fundamentally reducible operator in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ and let m_+ and m_- be Weyl functions introduced in Theorem 4.7. If each of the functions m_+ and m_- has both B_∞ -property and B_0 -property, then A is fundamentally reducible in the Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ if and only if the pair (m_+, m_-) has D_∞ -property and D_0 -property and $\ker(A) = \ker(A^2)$.*

Remark 5.10. The coupling A_1 in (4.14) will be called *reflexive* if $m_+(z) = m_-(z)$. In the reflexive case analogs of Theorems 5.7 and 5.8 for indefinite Sturm-Liouville operator were proven in [37]. It was shown there that the B -property for m_+ is automatically satisfied if the corresponding D -property holds.

This remark leads to the following sufficient conditions of regularity.

Corollary 5.11. *Let A be a nonnegative partially fundamentally reducible operator in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ and assume that Weyl functions m_+ and m_- introduced in Theorem 4.7 are Stieltjes functions. Then the following statements hold.*

- (i) *If each function m_+ and m_- has D_∞ -property, then $\infty \notin c_s(A)$.*
- (ii) *If each function m_+ and m_- has D_0 -property and $\ker(A) = \ker(A^2)$, then $0 \notin c_s(A)$.*
- (iii) *If each of the functions m_+ and m_- has both D_∞ -property and D_0 -property, and $\ker(A) = \ker(A^2)$, then A is fundamentally reducible in the Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$.*

Proof. (i) Since each function m_+ and m_- has D_∞ -property then also each m_+ and m_- has B_∞ -property (see Remark 5.10). By Proposition 5.4 the pair (m_+, m_-) has D_∞ -property and hence $\infty \notin c_s(A)$ by Theorem 5.7.

(ii) Similarly, if each m_+ and m_- has D_0 -property then also each m_+ and m_- has B_0 -property (see Remark 5.10). By Proposition 5.4 the pair (m_+, m_-) has D_0 -property and thus $0 \notin c_s(A)$ by Theorem 5.8. \square

In the next theorem we use the notation introduced at the beginnings of Subsections 4.1 and 4.3.

Theorem 5.12. *Let S_\pm be a closed symmetric densely defined nonnegative operator with defect numbers $(1, 1)$ in the Hilbert space $(\mathcal{K}_\pm, \langle \cdot, \cdot \rangle_{\mathcal{K}_\pm})$. Let $(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm)$ be a boundary triple for S_\pm^* and let m_\pm be the corresponding Weyl function. Let A be a nonnegative self-adjoint extension of $S_+ \oplus (-S_-)$ in the Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$. Then the following statements hold.*

- (a) If $m_+, m_- \in \mathcal{A}_\infty$, then $\infty \notin c_s(A)$.
- (b) If $m_+, m_- \in \mathcal{A}_0$ and $\ker(A) = \ker(A^2)$, then $0 \notin c_s(A)$.
- (c) If $m_+, m_- \in \mathcal{A}_0 \cap \mathcal{A}_\infty$ and $\ker(A) = \ker(A^2)$, then A is fundamentally reducible.

Proof. Let A be an arbitrary nonnegative self-adjoint extension of $S_+ \oplus (-S_-)$ in the Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$. Then A is a nonnegative fundamentally reducible operator in $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$, so $\rho(A) \neq \emptyset$ by Corollary 4.9.

The operator $S = JA$ is a self-adjoint extension of $S_+ \oplus S_-$ in the Hilbert space $(\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})$. Assume first that S is a separated extension of $S_+ \oplus S_-$. By the definition of a separated extension, there exists a self-adjoint extension T_+ of S_+ in $(\mathcal{K}_+, [\cdot, \cdot]_{\mathcal{K}_+})$ and a self-adjoint extension T_- of S_- in $(\mathcal{K}_-, [\cdot, \cdot]_{\mathcal{K}_-})$ such that $S = T_+ \oplus T_-$. Then, S commutes with the fundamental symmetry J introduced in (4.12). Therefore, $A = JS$ is fundamentally reducible, so all three claims are trivial in this case.

Next assume that S is a non-separated extension of $S_+ \oplus S_-$. By Theorem 4.5 there exist a boundary triple $(\mathbb{C}, \widehat{\Gamma}_0^+, \widehat{\Gamma}_1^+)$ for S_+^* and a boundary triple $(\mathbb{C}, \widehat{\Gamma}_0^-, \widehat{\Gamma}_1^-)$ for S_-^* , such that S is a coupling of S_+ and S_- relative to the triples $(\mathbb{C}, \widehat{\Gamma}_0^+, \widehat{\Gamma}_1^+)$ and $(\mathbb{C}, \widehat{\Gamma}_0^-, \widehat{\Gamma}_1^-)$. By Remark 2.3 there exists a Möbius transformation μ_{\pm} such that the Weyl function \widehat{m}_{\pm} of S_{\pm} relative to $(\mathbb{C}, \widehat{\Gamma}_0^{\pm}, \widehat{\Gamma}_1^{\pm})$ is given by

$$\widehat{m}_{\pm} = \mu_{\pm} \circ m_{\pm}.$$

We proceed with a proof of (a). Assume that $m_+, m_- \in \mathcal{A}_\infty$. Proposition 2.12(i) implies that $\widehat{m}_+, \widehat{m}_- \in \mathcal{A}_\infty$ and, in turn, Corollary 3.10 yields that \widehat{m}_+ and \widehat{m}_- have B_∞ -property. Since by Proposition 5.5(a) the pair $(\widehat{m}_-, \widehat{m}_+)$ has D_∞ -property, the claim follows from Theorem 5.7.

Next assume that $m_+, m_- \in \mathcal{A}_0$. As before, Proposition 2.12(i) implies that $\widehat{m}_+, \widehat{m}_- \in \mathcal{A}_\infty$. Now Corollary 3.11 yields that \widehat{m}_+ and \widehat{m}_- have B_0 -property and by Proposition 5.5(b) the pair $(\widehat{m}_-, \widehat{m}_+)$ has D_0 -property. The claim now follows from Theorem 5.8.

Statement (c) is a consequence of Corollary 5.9. \square

6. EXAMPLES

Example 6.1. Consider the singular differential expression

$$\ell(f)(x) := -\frac{\operatorname{sgn} x}{w(x)} \left((p(x) f'(x))' + q(x) f(x) \right) \quad \text{for a.a. } x \in \mathbb{R}, \quad (6.1)$$

where the coefficients p, q and w are real functions on \mathbb{R} satisfying the conditions

- (C1) $1/p, q, w \in L_{\text{loc}}^1(\mathbb{R})$ and $p, w > 0$ a.e. on \mathbb{R} ,
- (C2) the expression ℓ is in the limit point case at $-\infty$ and at $+\infty$.

It is natural to consider the expression in (6.1) in the Kreĭn space $(L_w^2(\mathbb{R}), [\cdot, \cdot])$. Here $(L_w^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$ is the standard weighted L^2 -space with the positive definite inner product $\langle \cdot, \cdot \rangle$ and the indefinite inner product is given by $[f, g] = \langle Jf, g \rangle$, $f, g \in L_w^2(\mathbb{R})$ where

$$(Jf)(x) = (\operatorname{sgn} x) f(x), \quad f \in L_w^2(\mathbb{R}),$$

is a fundamental symmetry on $(L_w^2(\mathbb{R}), [\cdot, \cdot])$. Set

$$\mathcal{K}_{\pm} = \{f \in L_w^2(\mathbb{R}) : f = 0 \text{ a.e. on } \mathbb{R}_{\pm}\}. \quad (6.2)$$

Then $L_w^2(\mathbb{R}) = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$ is the fundamental decomposition corresponding to J .

Let A be the operator associated with the expression in (6.1) in the Hilbert space $L_w^2(\mathbb{R})$; that is the operator defined by $Af = \ell(f)$ for all

$$f \in \text{dom}(A) = \{f \in L_w^2(\mathbb{R}) : f, pf' \in AC_{\text{loc}}(\mathbb{R}), \ell(f) \in L_w^2(\mathbb{R})\}.$$

The differential operator A is partially fundamentally reducible in the Kreĭn space $(L_w^2(\mathbb{R}), [\cdot, \cdot])$. To see this consider the range restriction S_{\pm} of $\pm A$ to \mathcal{K}_{\pm} which are defined on

$$\{f \in \text{dom}(A) : Af \in \mathcal{K}_{\pm}\} = \mathcal{K}_{\pm} \cap \text{dom}(A).$$

Then S_{\pm} is the minimal operator associated in $L_{w_{\pm}}^2(\mathbb{R}_{\pm})$ with the restriction of $\pm \ell$ to \mathbb{R}_{\pm} ; here w_{\pm} denotes the restriction of w to \mathbb{R}_{\pm} . In fact we have,

$$\begin{aligned} \text{dom}(S_{\pm}^*) &= \{f \in L_{w_{\pm}}^2(\mathbb{R}_{\pm}) : f, pf' \in AC_{\text{loc}}[0, \pm\infty), \ell(f) \in L_{w_{\pm}}^2(\mathbb{R}_{\pm})\}, \\ \text{dom}(S_{\pm}) &= \{f \in \text{dom}(S_{\pm}^*) : f(0) = f'(0) = 0\}, \end{aligned}$$

and

$$S_{\pm}f := \pm \ell(f), \quad f \in \text{dom}(S_{\pm}). \quad (6.3)$$

Since we assume that ℓ is in the limit point case at $\pm\infty$, the operator S_{\pm} is a densely defined symmetric operator with defect numbers $(1, 1)$ in the Hilbert space $L_{w_{\pm}}^2(\mathbb{R}_{\pm})$.

Let $z \in \mathbb{C} \setminus \mathbb{R}$ and denote by $\vartheta(\cdot, z)$ and $\varphi(\cdot, z)$ the unique solutions of the equation

$$-(pf')' + qf = zwf$$

satisfying the boundary conditions

$$\vartheta(0, z) = 1, \quad (p\vartheta')(0, z) = 0, \quad \text{and} \quad \varphi(0, z) = 0, \quad (p\varphi')(0, z) = 1, \quad \text{respectively.}$$

Since we assume that ℓ is in the limit point case at $\pm\infty$, for each $z \in \mathbb{C} \setminus \mathbb{R}$ there is a unique (up to a constant multiple) solution

$$\psi_{\pm}(t, z) = \varphi(t, z) \mp m_{\pm}(z)\vartheta(t, z), \quad t \in \mathbb{R}_{\pm}, \quad (6.4)$$

of the restriction of $\pm \ell(f) = zf$ to \mathbb{R}_{\pm} which belongs to $L_{w_{\pm}}^2(\mathbb{R}_{\pm})$. Relation (6.4) defines the function $m_{\pm} : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ uniquely. The function m_{\pm} is called *Titchmarsh-Weyl coefficient* of the restriction of the expression $\pm \ell$ to \mathbb{R}_{\pm} .

A boundary triple for S_{\pm}^* is $(\mathbb{C}, \Gamma_0^{\pm}, \Gamma_1^{\pm})$, where

$$\Gamma_0^{\pm}f := (pf')(0_{\pm}), \quad \Gamma_1^{\pm}(f) = \mp f(0_{\pm}), \quad f \in \text{dom}(S_{\pm}^*). \quad (6.5)$$

It follows from (2.3) and (6.5) that the Titchmarsh-Weyl coefficient defined by (6.4) coincide with the Weyl function of the operator S_{\pm} in (6.3) relative to the boundary triple in (6.5).

Assume additionally, that the coefficients p, q and w satisfy the conditions:

(C3) $p > 0$ and $p' \in AC[-b, b]$ for all $b > 0$;

(C4) $w = 1$.

The following asymptotic for the Titchmarsh-Weyl coefficient m_{\pm} has been established in [22] (see also [44], where the asymptotic of the spectral function was found). For all $z \in \mathbb{C} \setminus \mathbb{R}$

$$m_{\pm}(rz) = \frac{1}{\sqrt{p(0)}} \frac{1}{\sqrt{-zr}} + O\left(\frac{1}{r}\right) \quad \text{as} \quad r \rightarrow +\infty. \quad (6.6)$$

It is clear that this asymptotic implies (3.9).

By Theorem 4.6, the coupling A_1 of S_+ and S_- relative to the boundary triples given in (6.5) coincides with the differential operator A associated with the expression ℓ in $L_w^2(\mathbb{R})$.

In addition, assume that the operator S_\pm is nonnegative in the Hilbert space $L_{w_\pm}^2(\mathbb{R}_\pm)$. Then the Friedrichs extension S_F^\pm of S_\pm defined on

$$\text{dom}(S_F^\pm) = \{f \in \text{dom}(S_\pm^*) : f(0) = 0\}$$

is also nonnegative in the Hilbert space $L_{w_\pm}^2(\mathbb{R}_\pm)$. Since $\text{dom}(S_F^\pm) = \ker(\Gamma_1^\pm)$, the function $m_\pm^\top = -1/m_\pm$ is holomorphic on $\mathbb{R}_- \subset \rho(S_F^\pm)$ and satisfies

$$\lim_{x \downarrow -\infty} m_\pm^\top(x) = -\infty.$$

By Corollary 4.3 the self-adjoint operator $A = A_1$ is nonnegative in the Kreĩn space $(L_w^2(\mathbb{R}), [\cdot, \cdot])$ if and only if

$$\lim_{x \uparrow 0} (m_+^\top(x) + m_-^\top(x)) \leq 0. \quad (6.7)$$

Assuming that $A = A_1$ is nonnegative, (6.7) implies that $m_+^\top(0-), m_-^\top(0-) \in \mathbb{R}$. Therefore, if A_1 is nonnegative, then $m_+^\top, m_-^\top \in \mathcal{S}_M$, and consequently, $m_+, m_- \in \mathcal{S}_M$. Now, (6.6) and Proposition 2.13 imply that $m_+, m_- \in \mathcal{A}_\infty$.

Further, by Corollary 3.10 the functions m_+ and m_- have B_∞ -property and by Proposition 5.4 m_+ and m_- have D_∞ -property. Therefore, by Theorem 5.7, we have $\infty \notin c_s(A)$. This result also follows from [15, Theorem 3.6], which was proved by completely different methods.

Moreover, Theorem 5.12 yields that not only A , but an arbitrary nonnegative self-adjoint extension of $S_+ \oplus (-S_-)$ in the Kreĩn space $(L_w^2(\mathbb{R}), [\cdot, \cdot])$ has a nonempty resolvent set and ∞ is not its singular critical point.

Example 6.2. Consider the differential expression (6.1) with $p = w = 1$ and assume that the potential q satisfies

$$\int_{\mathbb{R}} (1 + |x|)|q(x)|dx < +\infty.$$

Let the operator S_\pm and the corresponding Titchmarsh-Weyl coefficient m_\pm be as in Example 6.1. Assume that S_\pm is a nonnegative operator in the Hilbert space $L^2(\mathbb{R}_\pm)$ and that (6.7) holds. Then the coupling A_1 , which coincides with the differential operator A associated with the expression (6.1), is a nonnegative self-adjoint operator in the Kreĩn space $(L^2(\mathbb{R}), [\cdot, \cdot])$.

The asymptotic behavior of m_\pm at 0 has been established in [34] as follows. It was shown that either there exists $k_\pm > 0$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$m_\pm(z) \sim -k_\pm \sqrt{-rz} \quad \text{as } r \rightarrow 0, \quad (6.8)$$

or there exist $a_\pm > 0$ and $b_\pm \in \mathbb{R}$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$m_\pm(rz) \sim \frac{a_\pm}{b_\pm + \sqrt{-rz}} \quad \text{as } r \rightarrow 0. \quad (6.9)$$

Assumption (6.7) implies that the case (6.8) is not possible. Thus (6.9) holds. The asymptotic in (6.9) implies that for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$m_\pm(rz) - \frac{a_\pm}{b_\pm} \sim -\frac{a_\pm}{b_\pm^2} \sqrt{-rz} \quad \text{as } r \rightarrow 0. \quad (6.10)$$

Since in Example 6.1 we proved that $m_+, m_- \in \mathcal{S}_M$, (6.10) and Proposition 2.13 yield $m_+, m_- \in \mathcal{A}_0$. Recall that in Example 6.1 we proved that $m_+, m_- \in \mathcal{A}_\infty$ and in [34, Proposition 4.3] it was proved that $\ker(A) = \{0\}$. Now Theorem 5.12 implies that A is fundamentally reducible. This has been proved in [34] and Theorem 5.12 provides an alternative proof of this result. Under a stronger assumption on q the fundamental reducibility of A was proved in [24] using a different approach.

However, Theorem 5.12 implies more. Let \tilde{A} be an arbitrary nonnegative self-adjoint extension of $S_+ \oplus (-S_-)$ in the Kreĭn space $(L^2(\mathbb{R}), [\cdot, \cdot])$. As in [34, Proposition 4.3], it follows that $\ker(\tilde{A}) = \{0\}$. Since $m_+, m_- \in \mathcal{A}_0 \cap \mathcal{A}_\infty$, Theorem 5.12 implies that \tilde{A} is fundamentally reducible.

Example 6.3. Consider the differential expression (6.1) with $q = 0$ and

$$w(x) = \begin{cases} (-x)^{\alpha_-}, & \text{if } x < 0, \\ x^{\alpha_+}, & \text{if } x > 0, \end{cases} \quad \text{and} \quad p(x) = \begin{cases} (-x)^{\beta_-}, & \text{if } x < 0, \\ x^{\beta_+}, & \text{if } x > 0, \end{cases}$$

with $\alpha_+, \alpha_- > -1$ and $\beta_+, \beta_- < 1$. In this example we use the notation introduced in Example 6.1. It was proved in [23] that the function m_\pm has the form

$$m_\pm(z) = C_\pm(-z)^{-\nu_\pm},$$

where

$$\nu_\pm = \frac{1 - \beta_\pm}{\alpha_\pm - \beta_\pm + 2}, \quad C_\pm = \frac{(2k_\pm)^{2\nu_\pm} \Gamma(1 + \nu_\pm)}{(1 - \beta_\pm) \Gamma(1 - \nu_\pm)}, \quad k_\pm = \frac{\alpha_\pm - \beta_\pm + 2}{2}.$$

As in Example 6.1 the coupling A_1 of S_+ and S_- relative to the boundary triples $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ and $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$ given in (6.5) coincides with the differential operator associated in $L_w^2(\mathbb{R})$ with the expression (6.1) with $q = 0$ and above p and w .

It follows from Proposition 4.2 that A_1 is a nonnegative self-adjoint operator in the Kreĭn space $(L_w^2(\mathbb{R}), [\cdot, \cdot])$. Moreover, since the equation $(py)^\prime = 0$ does not have nontrivial solutions in $L_w^2(\mathbb{R})$, we have $\ker(A_1) = \{0\}$.

As $\nu_\pm \in (0, 1)$ we have $m_+, m_- \in \mathcal{A}_\infty \cap \mathcal{A}_0$. Now, Theorem 5.12 implies that A_1 is fundamentally reducible in the Kreĭn space $L_w^2(\mathbb{R})$. In the case $\alpha_\pm = \beta_\pm = 0$ this result was proved in [16] and for $\alpha_+ = \alpha_-$ and $\beta_\pm = 0$ in [26].

However, Theorem 5.12 implies more. Let \tilde{A} be an arbitrary nonnegative self-adjoint extension of $S_+ \oplus (-S_-)$ in the Kreĭn space $(L_w^2(\mathbb{R}), [\cdot, \cdot])$. Then, clearly, $\ker(\tilde{A}) = \{0\}$ and Theorem 5.12 yields that \tilde{A} is fundamentally reducible in the Kreĭn space $(L_w^2(\mathbb{R}), [\cdot, \cdot])$.

Example 6.4. In this example we present a nonnegative coupling in a Kreĭn space which has a singular critical point at 0. This example is modeled after the example in [33, Section 5]. It is proved in [33] that for the expression in (6.1) with $p = 1$, $q = 0$ and $w(x) = (3|x| + 1)^{-4/3}$ the corresponding Titchmarsh-Weyl coefficients m_+ and m_- are $m_+(z) = m_-(z) = 1/\sqrt{-z} - 1/z$. From this it was deduced in [33] that 0 is a singular critical point of the associated nonnegative differential operator in $(L_w^2(\mathbb{R}), [\cdot, \cdot])$. We think it will be instructive to present a similar example as a coupling.

First define the Hilbert space $\mathcal{K}_\pm = L^2(\mathbb{R}_\pm) \oplus \mathbb{C}$ with the inner product

$$\left\langle \begin{pmatrix} f_\pm \\ u_\pm \end{pmatrix}, \begin{pmatrix} g_\pm \\ v_\pm \end{pmatrix} \right\rangle_\pm := \int_{\mathbb{R}_\pm} f_\pm g_\pm^* + u_\pm v_\pm^*, \quad f_\pm, g_\pm \in L^2(\mathbb{R}_\pm), \quad u_\pm, v_\pm \in \mathbb{C}.$$

In this space we consider the operator S_{\pm} given by its graph

$$\text{gr}(S_{\pm}) = \left\{ \left\{ \begin{pmatrix} f_{\pm} \\ \mp f(0_{\pm}) \end{pmatrix}, \begin{pmatrix} -f''_{\pm} \\ 0 \end{pmatrix} \right\} : \begin{array}{l} f_{\pm} \in W^{2,2}(\mathbb{R}_{\pm}) \\ f'(0_{\pm}) = 0 \end{array} \right\}.$$

It is easy to see that S_{\pm} is densely defined and positive. The graph of its adjoint is

$$\text{gr}(S_{\pm}^*) = \left\{ \left\{ \begin{pmatrix} f_{\pm} \\ u_{\pm} \end{pmatrix}, \begin{pmatrix} -f''_{\pm}(t) \\ f'_{\pm}(0_{\pm}) \end{pmatrix} \right\} : f_{\pm} \in W^{2,2}(\mathbb{R}_{\pm}) \right\}.$$

A boundary triple $(\mathbb{C}, \Gamma_0^{\pm}, \Gamma_1^{\pm})$ for S_{\pm} is

$$\Gamma_0^{\pm} \begin{pmatrix} f_{\pm} \\ u_{\pm} \end{pmatrix} = f'_{\pm}(0_{\pm}), \quad \Gamma_1^{\pm} \begin{pmatrix} f_{\pm} \\ u_{\pm} \end{pmatrix} = \mp f_{\pm}(0_{\pm}) - u_{\pm}.$$

The corresponding Weyl solution is

$$\psi_{\pm}(t, z) = \begin{pmatrix} \mp \exp(\mp t \sqrt{-z}) / \sqrt{-z} \\ 1/z \end{pmatrix}, \quad z \in \mathbb{C}^+.$$

That is,

$$\Gamma_0^{\pm} \psi_{\pm}(\cdot, z) = 1 \quad \text{and} \quad S_{\pm}^* \psi_{\pm}(\cdot, z) = z \psi_{\pm}(z), \quad z \in \mathbb{C}^+.$$

Therefore the corresponding Weil function is

$$m_{\pm}(z) = \frac{1}{\sqrt{-z}} - \frac{1}{z}, \quad z \in \mathbb{C}^+.$$

The graph of the coupling S_1 of the symmetric operators S_- and S_+ in the direct sum Hilbert space

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- = L^2(\mathbb{R}) \oplus \mathbb{C}^2$$

is

$$\left\{ \left\{ \begin{pmatrix} f \\ u_+ \\ u_- \end{pmatrix}, \begin{pmatrix} -f'' \\ f'(0_+) \\ f'(0_-) \end{pmatrix} \right\} : \begin{array}{l} f \in W^{2,2}(\mathbb{R}_-) \oplus W^{2,2}(\mathbb{R}_+) \\ f'(0_+) = f'(0_-), \\ f(0_-) - f(0_+) = u_- + u_+, \end{array} \right\}.$$

By Corollary 4.3 S_1 is a nonnegative operator in $(\mathcal{K}, \langle \cdot, \cdot \rangle)$.

The following indefinite inner product turns the Hilbert space $\mathcal{K} = L^2(\mathbb{R}) \oplus \mathbb{C}^2$ into a Kreĭn space:

$$\left[\begin{pmatrix} f \\ u_+ \\ u_- \end{pmatrix}, \begin{pmatrix} g \\ v_+ \\ v_- \end{pmatrix} \right] := \int_{\mathbb{R}} (\text{sgn } t) f(t) g(t)^* dt + u_+ v_+^* - u_- v_-^*$$

for all $f, g \in L^2(\mathbb{R})$ and all $u_+, u_-, v_+, v_- \in \mathbb{C}$. The direct sum $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ is a corresponding fundamental decomposition of $(\mathcal{K}, [\cdot, \cdot])$.

The graph of the coupling A_1 of the operators S_- and S_+ is

$$\left\{ \left\{ \begin{pmatrix} f(t) \\ u_+ \\ u_- \end{pmatrix}, \begin{pmatrix} -(\text{sgn } t) f''(t) \\ f'(0_+) \\ -f'(0_-) \end{pmatrix} \right\} : \begin{array}{l} f \in W^{2,2}(\mathbb{R}_-) \oplus W^{2,2}(\mathbb{R}_+) \\ f'(0_+) = f'(0_-), \\ f(0_-) - f(0_+) = u_- + u_+, \end{array} \right\}. \quad (6.11)$$

It follows from (6.11) that $\ker(A_1) = \ker(A_1^2)$ and $\ker(A_1)$ is spanned by $(0 \ 1 \ -1)^{\top}$.

Since the pair of Weyl functions $m_+(z) = m_-(z) = 1/\sqrt{-z} - 1/z$ does not have D_0 -property, it follows from Corollary 5.3 that 0 is a singular critical point for A_1 .

Example 6.5. Consider the following boundary value problem: For $g \in L^2(\mathbb{R}_+)$ find $f \in L^2(\mathbb{R}_+)$ such that

$$-f'' - zf = g, \quad f'(0) - \sqrt{2}\sqrt[4]{z}f(0) = 0. \quad (6.12)$$

In this example we will show that a linearization of this boundary value problem in the Kreĭn space $(L^2(\mathbb{R}), [\cdot, \cdot])$ is fundamentally reducible.

Here the Kreĭn space $(L^2(\mathbb{R}), [\cdot, \cdot])$ is the L^2 space introduced in Example 6.1 with $w = 1$. We also use the fundamental decomposition $L^2(\mathbb{R}) = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$ with \mathcal{K}_+ and \mathcal{K}_- from (6.2).

Let S_+ be the minimal operator associated with the differential expression $-\frac{d^2}{dt^2}$ in the Hilbert space \mathcal{K}_+ and let $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ be the boundary triple for S_+^* given by

$$\Gamma_0^+ f_+ = f_+(0), \quad \Gamma_1^+ f_+ = f_+'(0).$$

The Weyl function of S_+ relative to this boundary triple is $m_+(z) = -\sqrt{-z}$.

Let S_- be the differential operator in the Hilbert space $(\mathcal{K}_-, -[\cdot, \cdot])$ defined on

$$\text{dom}(S_-) = \{f_- \in W^{4,2}(\mathbb{R}_-) : f_-(0) = f_-'(0) = f_-'(0) = 0\}$$

by $S_- f_- = f_-^{(4)}$. Then the adjoint operator S_-^* is defined on

$$\text{dom}(S_-^*) = \{f_- \in W^{4,2}(\mathbb{R}_-) : f_-(0) = 0\}$$

by the same expression $S_-^* f_- = f_-^{(4)}$. With

$$\Gamma_0^- f_- = f_-'(0), \quad \Gamma_1^- f_- = -f_-'(0), \quad f_- \in \text{dom}(S_-^*),$$

the triple $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$ is a boundary triple for S_-^* . The corresponding Titchmarsh-Weyl coefficient m_- is

$$m_-(z) = -\sqrt{2}\sqrt[4]{-z}.$$

By Theorem 4.7 the coupling A_1 of the operators S_+ and S_- in the graph notation takes the form

$$A_1 = \left\{ \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix}, \begin{pmatrix} -f_+' \\ f_-^{(4)} \end{pmatrix} \right\} : \begin{array}{l} f_+ \in W^{2,2}(\mathbb{R}_+), f_+(0) - f_-'(0) = 0 \\ f_- \in W^{4,2}(\mathbb{R}_-), f_+'(0) - f_-'(0) = f_-(0) = 0 \end{array} \right\}$$

and the compressed resolvent $P_+(A_1 - z)^{-1}|_{\mathcal{K}_+}$ of A_1 gives a solution of the boundary value problem (6.12) by the formula

$$f = P_+(A_1 - z)^{-1}g.$$

The operator A_1 is called a linearization of the boundary value problem (6.12).

By Corollary 4.4 the operator A_1 is nonnegative. Since clearly $\ker(A_1) = \{0\}$ and the functions m_+ and m_- belong $\mathcal{A}_\infty \cap \mathcal{A}_0$, Theorem 5.12 yields that the linearization A_1 of (6.12) is fundamentally reducible.

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